A PROPERTY OF AN ITERATION PROCESS

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Abstract

The familiar variation-iteration method for solving the eigenvalue equation $C\psi = \lambda B\psi$ (C and B are Hermitean operators), is applied to a case in which the operator C, and hence also the eigenvalues λ, depend on a continuous parameter a. It is shown that certain *exact* properties of the functions $\lambda = \lambda(a)$ can be deduced from *low-order* results in the variation-iteration scheme.

1. Introduction

In a previous paper (1) a detailed discussion was presented of the Mathieu equation in the form

$$\left(\frac{d^2}{dt^2} + a\right) \xi_r(a, t) = - bA^2 \sin^2 \omega t \xi_r(a, t) \tag{1}$$

where ξ_r is restricted to being of period $2\pi/\omega$ in t. In this case, for a given value of the constant a in (1) there exists a discrete, infinite set of values $b_1(a), b_2(a), \cdots, b_r(a), \cdots$ such that $\xi_r(a, t)$ is periodic with period $2\pi/\omega$ in t. We may plot the functions $b_r(a)$ in the $a - b$ plane, thereby getting a "stability diagram" for certain particular motions of a non-linear mechanical system. The curves $b_1(a), b_2(a), \cdots$ etc. separate "stable" from "unstable" regions in the $a—b$ plane.

In reference [1] an iteration scheme was developed, which leads to a series of successive approximations $\xi_r^{[n]}(a, t)$ to the true eigenfunctions $\xi_r(a, t)$. Correspondingly, we obtained approximations $b_r^{[n]}(a)$ to the true stability lines $b_r(a)$.

In this paper, we show that certain *exact* properties of these stability lines can be inferred from *low-order* approximations $b_r^{[n]}(a)$, i.e., it is not always necessary to go to the limit $n \to \infty$ in order to obtain exact information.

In order to keep the discussion general, we replace the Hermitean operator $d^2/dt^2 + a$ in (1) by a general Hermitean operator $C(a, t)$, and we replace the Hermitean operator $-A^2 \sin^2(\omega t)$ in (1) by a general Hermitean operator $B(t)$. Thus our basic equation reads

$$C(a, t)\xi_r(a, t) = b_r(a)B(t)\xi_r(a, t) \tag{2}$$

If C is a differential operator (as is usual), the boundary conditions on the eigenfunctions $\xi_r(a, t)$ must be such as to ensure that C is Hermitean. In (1), the boundary conditions are that ξ_r is periodic with period $2\pi/\omega$ in t; this indeed makes both C and B Hermitean operators.

2. The Iteration Procedure

The iteration procedure is based on the two equations

$$C(a, t)\xi^{[n]}(a, t) = b^{[n-1]}(a)B(t)\xi^{[n-1]}(a, t); \qquad n = 1, 2, \cdots \tag{3}$$

$$b^{[n]}(a) = \frac{(\xi^{[n]}, C\xi^{[n]})}{(\xi^{[n]}, B\xi^{[n]})}; \qquad n = 0, 1, 2, \cdots \tag{4}$$

The notation (χ, ψ) is used for the Hermitean inner product of χ and ψ. $\xi^{[0]}(a, t)$ can be chosen in any way, so long as it satisfies the boundary conditions and any constants of integration are chosen to ensure that $\xi^{[n]}(a, t)$ satisfies these boundary conditions. (3) and (4) may be manipulated to obtain

$$b^{[n]}(a) = b^{[n-1]}(a)\frac{(\xi^{[n]}, B\xi^{[n-1]})}{(\xi^{[n]}, B\xi^{[n]})}; \qquad n = 1, 2 \cdots \tag{5}$$

3. A Particular Choice of $\xi^{[0]}$

If $\xi^{[0]}$ is chosen in the particular manner described in this section, the results proved in section 5 follow. First of all we classify the eigenvalues $b_0(a)$, $b_1(a) \cdots$ of equation (2). This is done using the value of a for which $b_r(a)$ is zero; we term this value a_r. We restrict our attention to those equations (2) for which this classification is possible. That is to say, we assume that the equation $b_r(a) = 0$ has one and only one real root a_r. Equation (1) satisfies this assumption. The possible values of a_r are clearly solutions of the equation

$$C(a, t)\xi(a, t) = 0 \tag{6}$$

The function $b_r(a)$ is termed degenerate if there are two or more distinct functions $b_r(a)$ which are solutions of equation (2) and for which $b_r(a_r) = 0$. If $b_r(a)$ is not degenerate, the choice of a_r results in a unique eigenvalue $b_r(a)$ and a unique eigenfunction $\xi_r(a, t)$ of equation (2). In this non-degenerate case, we define the zero order function $\xi_r^{[0]}(a, t)$ as the solution of the equation

$$C(a_r, t)\xi_r^{[0]}(a, t) = 0 \tag{7}$$

If $b_r(a)$ is degenerate, further analysis is required. For example, in the case of the Mathieu equation (1) there are two distinct functions $b_{r+}(a)$ and $b_{r-}(a)$ each of which is zero when $a = a_r = \omega^2 r^2$; $(r = 1, 2, 3, \cdots)$. In general an approach based on degenerate perturbation theory is required to determine what combinations of the two linearly independent solutions $\xi_r^{[0]}(a, t)$ are suitable zero order approximations to $\xi_{r+}(a, t)$ and $\xi_{r-}(a, t)$. However, if these solutions can be distinguished on a group theoretic basis, there is no need for further analysis. For the purposes of the iteration procedure we need merely restrict the approximate functions $\xi_{r+}^{[n]}(a, t)$ to belong to the same irreducible representation to which the true function $\xi_{r+}(a, t)$ belongs; this automatically ensures that the correct combination is maintained throughout the iteration. This classification in the case of the Hill equation is carried out in section 5 of reference [1]. In the notation there we require that (θ_1, θ_2) is chosen before the iteration is carried out.

In this paper we restrict ourselves to non-degenerate eigenvalues $b(a)$ and to degenerate eigenvalues whose degeneracy can be removed using group theory. In either case the zero order eigenfunction $\xi_r^{[0]}(a, t)$ is determined uniquely by equation (7).

It follows from (6) and (7) that

$$\xi_r^{[0]}(a_r, t) = \xi_r(a_r, t) \tag{8}$$

that is to say the zero order function is the same as the true function at $a = a_r$. In view of this, any number of iterations give functions which are also the true function at $a = a_r$, and the iterated eigenvalues are exact at $a = a_r$. Hence

$$\xi_r^{[n]}(a_r, t) = \xi_r(a_r, t) \qquad n = 0, 1, \cdots \tag{9}$$

and

$$b_r^{[n]}(a_r) = b_r(a_r) = 0 \qquad n = 0, 1, \cdots \tag{10}$$

4. Theorem

We now choose $\xi_r^{[0]}(a, t)$, for $a \neq a_r$, in the particular way described in Section 3, and we carry out the iteration defined by equations (3) and (4). The successive approximations $b_r^{[0]}(a)$, $b_r^{[1]}(a) \cdots$ and the true eigenvalue $b_r(a)$ are plotted on a graph with a as ordinate and b as abscissa. When this is done *all the curves $b^{[n]}(a)$ touch the exact curve $b(a)$ at the point $(a_r, 0)$ and the curve $b^{[n]}(a)$ has $(n+2)$ point contact with $b(a)$ at this point.* That is to say, the values of $b^{[n]}(a)$ and its first $n+1$ derivatives evaluated at $a = a_r$

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are the same as the corresponding values of $b(a)$ and its first $n+1$ derivatives evaluated at the same point. We prove this theorem in the following section.

5. Proof of Theorem

In this proof we use the symbol δ to denote $\delta a \partial/\partial a$. Thus to establish that $b^{[n]}(a)$ has $n+2$ point contact with $b(a)$ where $a = a_r$ we have to show that

$$(\delta^{t+1} b^{[n]})_{a=a_r} = (\delta^{t+1} b)_{a=a_r}; \qquad t = -1, 0, 1, \cdots, n \tag{11}$$

This result for $t = -1$ has been established in equation (9). We start by evaluating $\delta b^{[n]}$ and δb to see whether or not they become identical when we substitute $a = a_r$.

Using expression (4) for $b^{[n]}$ and applying the Hermitean property of C in the form

$$(\delta \xi^{[n]}, C\xi^{[n]}) = (\xi^{[n]}, C\delta \xi^{[n]}) \tag{12}$$

straightforward differentiation yields

$$\delta b^{[n]} = \frac{2(\delta \xi^{[n]}, \{C - b^{[n]} B\} \xi^{[n]}) + (\xi^{[n]}, \delta C \xi^{[n]})}{(\xi^{[n]}, B\xi^{[n]})} \qquad n = 0, 1, 2 \cdots \tag{13}$$

Using (3), this can be written (if $n \neq 0$)

$$\delta b^{[n]} = \frac{2(\delta \xi^{[n]}, B\{b^{[n-1]} \xi^{[n-1]} - b^{[n]} \xi^{[n]}\}) + (\xi^{[n]}, \delta C \xi^{[n]})}{(\xi^{[n]}, B\xi^{[n]})} \tag{14}$$

$$n = 1, 2 \cdots$$

If we adopt the convention

$$\delta b^{[-1]} = b^{[-1]} = 0 \tag{15}$$

then (14) is also valied for $n = 0$.

We now suppose that, instead of starting with expression (4) for $b^{[n]}$, we differentiate the corresponding expression for b:

$$b(a) = \frac{(\xi, C\xi)}{(\xi, B\xi)} \tag{16}$$

Carrying out this differentiation we arrive at an expression for δb, identical with (14) except that the superscripts $[n]$ and $[n-1]$ are omitted.

In view of (9) and (10) and using definition (15), it is clear that

$$(\delta b^{[n]})_{a=a_r} = (\delta b)_{a=a_r} \qquad n = 0, 1, 2 \cdots \tag{17}$$

This establishes that all the curves $b^{[n]}(a)$ touch the curve $b(a)$ at $a = a_r$.

To obtain $\delta^2 b^{[n]}$ we differentiate (14) and of the lengthy expression obtained we consider only the term

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$$\frac{2(\delta\xi^{[n]}, B\{\delta b^{[n-1]}\xi^{[n-1]} + b^{[n-1]}\delta\xi^{[n-1]} - \delta b^{[n]}\xi^{[n]} - b^{[n]}\delta\xi^{[n]}\})}{(\xi^{[n]}, B\xi^{[n]})} \tag{18}$$

A corresponding term exists in $\delta^2 b$, and this term is the same as (18) with the superscripts removed. This term is clearly zero. If we insert $a = a_r$ into (18) if $n > 1$, the expression in curly brackets is zero because of (9) and (10). But if $n = 0$, the definition of $\delta b^{[-1]}$ in (15) implies that the term in curly brackets is $-(\delta b)_{a=a_r}\xi(a_r, t)$ and not necessarily zero. It is clear from inspection that all the other terms in the expansions of $\delta^2 b^{[n]}$ and $\delta^2 b$ coincide when $a = a_r$. Thus we have the result

$$(\delta^2 b^{[n]})_{a=a_r} = (\delta^2 b)_{a=a_r} \qquad n = 1, 2 \cdots \tag{19}$$

The proof of the general result follows in much the same way as the proof of (19). We use (17) and (19) as the basis of an inductive proof and we assume that

$$\begin{aligned} (\delta^t b^{[n]})_{a=a_r} &= (\delta^t b)_{a=a_r} \\ t &= 1, 2, \cdots T \\ n &= t-1, t, t+1, \cdots \end{aligned} \tag{20}$$

Now $\delta^{T+1} b^{[n]}$ contains a term

$$\frac{2(\delta\xi^{[n]}, B\delta^T\{b^{[n-1]}\xi^{[n-1]} - b^{[n]}\xi^{[n]}\})}{(\xi^{[n]}, B\xi^{[n]})} \tag{21}$$

and $\delta^{T+1} b$ contains this term, with the superscripts omitted. On substitution of a_r for a in (21) these two corresponding terms are identical so long as

$$(\delta^t b^{[n-1]}) = (\delta^t b)_{a=a_r} \qquad t = 1, 2, \cdots T \tag{22}$$

The other corresponding terms are seen to be identical. But using (20) this equality is seen to hold so long as $n - 1 > T$, that is $n > T + 1$. This establishes that

$$\begin{aligned} (\delta^t b^{[n]})_{a=a_r} &= (\delta^t b)_{a=a_r} \\ t &= T+1 \\ n &= T, T+1, \cdots \end{aligned} \tag{23}$$

This concludes the proof of (20) for all t and we rewrite it as

$$\begin{aligned} (\delta^{t+1} b^{[n]})_{a=a_r} &= (\delta^{t+1} b)_{a=a_r} \\ t &\leqq n \end{aligned} \tag{24}$$

(24) is a statement of the fact that the curve $b^{[n]}(a)$ has $(n + 2)$ point contact with the curve $b(a)$ at $a = a_r$.

In reference [1] this property of the iterations $b_r^{[n]}(a)$ is used to prove that certain of the 'stability lines' of the Hill equation touch each other, and to determine the degree of this contact.

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7. Reference

[1] J. N. Lyness and J. M. Blatt, J. Aust. Math. Soc.

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