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AN INTERNAL CHARACTERIZATION OF REALCOMPACTNESS

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1. Introduction. A space is realcompact if it is a homeomorph of a closed subspace of a product of real lines. Many external characterizations of realcompactness have appeared, but there seems to be no simple internal characterization. We provide such a characterization in terms of the existence of a collection of covers of a certain type and use it to examine realcompact extensions of a space and to characterize the Q-closure of a space in a compactification.

2. Structures. A *structure* on X is a collection of covers of X that forms a filter under refinement ordering; the members of a structure are called *gauges*. A *balanced refinement* of a gauge α is a gauge β with cardinal not greater than that of α such that for each $B \in \beta$ there is $A \in \alpha$ such that $\{A, X - B\}$ is also a gauge; thus a balanced refinement is certainly a refinement. A structure is *balanced* if every gauge has a balanced refinement. A structure is *distinguishing* if when x and y are distinct points of X the cover $\{X - \{x\}, X - \{y\}\}$ is a gauge.

A filter on X is *Cauchy* if it contains a member of every gauge. A structure is *complete* if for every Cauchy filter, there is a point $x \in X$ such that if $A \subset X$ and $\{A, X - \{x\}\}$ is a gauge, then A is a member of the filter. Equivalently, every Cauchy filter converges in the topology induced by the structure (see § 3).

3. Proximity and topology. A structure induces a *proximity relation* $<$ between subsets of X given by $A < B$ if $\{B, X - A\}$ is a gauge.

THEOREM A. *The $<$-relation induced by a balanced structure is a completely regular proximity relation.*

Proof. It must be shown that $A \subset X$ implies $\emptyset < A$, $A < B$ implies $A \subset B$, $C \subset A < B \subset D$ implies $C < D$, $A < B$ and $A < C$ implies $A < B \cap C$, $A < B$ implies $X - B < X - A$, and that if $A < B$ then there is C with $A < C < B$. Of these six conditions the first five are readily shown, and in fact they hold for the $<$-relation of any structure (see [**1**, § 25A ff.]). To show the sixth, observe that if $A < B$, then the gauge $\{B, X - A\}$ has a balanced refinement $\{E, D\}$ with $E < B$ and $D < X - A$; on setting $C = X - D$ and using the third and fifth conditions, it follows that $A < C < B$.

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The converse of Theorem A also holds as will be seen in § 7.

In view of Theorem A there is a completely regular topology induced by a balanced structure via the operator $\operatorname{int} A = \{x: \{A, X - \{x\}\} \text{ is a gauge}\}$; this topology is Hausdorff if and only if the structure is distinguishing.

An *admissible* balanced structure on a space is a balanced structure that induces the topology of the space. There may of course be many admissible balanced structures on a space.

4. Realcompact structures.

A *realcompact structure* is a structure that is balanced, distinguishing, and has a base of countable gauges.

THEOREM B. *A space is realcompact if and only if it admits a complete realcompact structure.*

Proof. Suppose that X is realcompact, so that it can be embedded as a closed subspace of a product of lines. It is easy to see that the collection of all open covers of the line is an admissible complete realcompact structure for the line. Now the product structure formed on the product of lines, using inverse images of open covers of the factors under the projection maps, is readily seen to be an admissible complete realcompact structure on the product. Finally it is straightforward to show that the subspace structure induced on the closed subspace X, using inverse images of product gauges under the inclusion map, is a complete admissible realcompact structure on X.

Conversely, suppose that there is an admissible complete realcompact structure for X. Then, in view of Theorem A and the remarks following that theorem, the space X is completely regular and Hausdorff: thus X can be considered to be a subspace of the product of lines $R^{C(X)}$ under the parametric embedding. Now let p be in the closure of X but not in X and let γ be the trace on X of the neighbourhood filter of p. Since the structure is complete and γ does not converge, the filter γ is not Cauchy; thus there is a gauge α such that γ contains no member of α. The gauge α has a countable balanced refinement β; thus for each $B_n \in \beta$, there is $A_n \in \alpha$ with $B_n < A_n$, and in view of Theorem A, and the properties of completely regular proximities, there is, for each n, a function $f_n \in C(X)$ that has the value 1 on B_n and the value 0 on $X - A_n$. Since $A_n \notin \gamma$ for each n it follows that for each $F \in \gamma$ there is $x \in F$ with $f_n(x) = 0$, and therefore the f_n-th coordinate of p must be 0. From this it follows that $X - B_n \in \gamma$ for each n. Also, since $\{B_n\}$ is a cover of X, there is for each $x \in X$ an n such that $f_n(x) = 1$.

Consider the function $f(x) = \sum_{n=1}^{\infty} \min(f_n(x), 1/2^n)$. Then $f \in C(X)$ and $f(x) > 0$ for each $x \in X$, therefore there is the function $g = 1/f \in C(X)$. Now since each $X - B_n \in \gamma$, it follows that no finite union of A_n's belongs to γ, because $\bigcup A_n$ is disjoint from the corresponding $\bigcap (X - B_n)$. Thus f has arbitrarily small values on members of γ and so the g-th coordinate of p is arbitrarily large, which is a contradiction.

5. Compact structures. A structure is *compact* if it is balanced, distinguishing, and has a base of finite gauges.

The following result is readily established.

THEOREM C. *A space is compact Hausdorff if and only if it admits a complete compact structure.*

It is in fact the case that the compact structures are precisely the totally bounded uniform structures; this gives a simple characterization of totally bounded uniform structures. This topic will again be considered in § 7.

6. Completions. Just as uniform spaces have completions, so there is a completion of a space in a balanced structure. The following result is the key to the construction, as it is in uniform space theory.

6.1. *In a balanced structure every Cauchy filter contains a minimal Cauchy filter.*

Proof. Let λ be a Cauchy filter and define

$$\gamma = \{E \in \lambda: \text{for some } F \in \lambda,\ F < E\};$$

then γ is a filter and $\gamma \subset \lambda$. If α is a gauge and β is a balanced refinement, then λ contains some member of β, and therefore γ contains some member of α; thus γ is Cauchy. Suppose that ζ is Cauchy and $\zeta \subset \gamma$. If $E \in \gamma$, there is $F \in \lambda$ with $F < E$; since $\zeta \subset \lambda$ and $F \in \lambda$, the member $X - F$ of the gauge $\{E, X - F\}$ is not in the Cauchy filter ζ, and thus $E \in \zeta$. Clearly, if λ is itself minimal Cauchy, then $\lambda = \gamma$ and it follows that for each $A \in \lambda$ there is $C \in \lambda$ with $A > C$.

The proof of the following result is straightforward.

6.2. *In a balanced structure, the filter* $\{A: A > \{x\}\}$ *is minimal Cauchy for each* $x \in X$.

The *completion* cX *of* X in the balanced structure $\mathscr{S}$ can now be defined. Its points are an index set for the family $\{O^p: p \in cX\}$ of minimal Cauchy filters on X. For each $A \subset X$ there is the subset A' of cX defined by $A' = \{p \in cX: A \in O^p\}$, and for each gauge $\alpha \in \mathscr{S}$ there is the cover $\alpha' = \{A': A \in \alpha\}$ of cX. The collection $\{\alpha': \alpha \in \mathscr{S}\}$ of covers of cX generates a structure on cX, called the *structure of the completion* cX and written as $c\mathscr{S}$.

The *canonical map* c of X into cX is given by letting $c(x)$ be the index of the minimal Cauchy filter $\{A: A > \{x\}\}$.

THEOREM D. *Let* $\mathscr{S}$ *be an admissible balanced distinguishing structure on the space* X. *Then the completion structure* $c\mathscr{S}$ *is a complete balanced distinguishing structure on* cX *and the map* c *is an embedding of* X *into* cX *with the structure topology.*

Proof. The proof of the theorem is quite similar to the usual proofs used in proximity theory and uniform space theory; thus only an outline need be given here.

It is clear from the fact that $\mathscr{S}$ is distinguishing that the map c is one-to-one, and since $\mathscr{S}$ is admissible, the filter $O^{c(x)}$ is the neighbourhood filter of $x \in X$. Given $A \subset X$ define $A^* = c(A) \cup A' \subset cX$. Then $A^* \cap c(X) = c(A)$. Given a filter γ on X, there is an associated filter $\gamma^* = \{A^*: A \in \gamma\}$ on cX. Writing $\leqq$ for the proximity relation of the structure $c\mathscr{S}$ one can establish (as in [**3**, Lemma 4]) the following results:

6.3. *$A < B$ if and only if $A^* \leqq B^*$.*

6.4. *The structure $c\mathscr{S}$ is balanced.*

6.5. *The function $\gamma \mapsto \gamma^*$ is a bijection from the set of minimal Cauchy filters on X onto the set of minimal Cauchy filters on cX.*

6.6. *The structure $c\mathscr{S}$ is complete.*

6.7 *The structure $c\mathscr{S}$ is distinguishing.*

Proof of 6.7. Suppose that $p, q \in cX$ and that $\{cX - \{p\}, cX - \{q\}\}$ is not a gauge of $c\mathscr{S}$. Let $A, C \in O^q$ with $A > C$; then $\{A', (X - C)'\} \in c\mathscr{S}$ and $(X - C)' \subset cX - \{q\}$, so that $A' \not\subset cX - \{p\}$ and therefore $A \in 0^p$. It follows that $O^p = O^q$; thus $p = q$.

It is now clear that the map c is an embedding; thus the proof of Theorem D is complete.

Clearly if the structure $\mathscr{S}$ has a base of countable (finite) gauges, then the structure $c\mathscr{S}$ also has a base of countable (finite) gauges. Applying Theorems B, C, and D, the following result is obtained.

THEOREM E. *The completion of a space in an admissible realcompact* (*compact*) *structure is realcompact* (*compact*).

The completion is unique among complete balanced distinguishing structures in the obvious sense; the exact statement will not be formulated here, since it is a special instance of the more general theory developed in [**2**].

7. Compactifications, proximities, and Q-closures. Given a compactification Z of X (where X and Z are completely regular spaces), there is the *associated completely regular proximity* defined by $A < B$ if $cl_Z A \subset Z - cl_Z(X - B)$. Conversely, given a completely regular proximity $<$ on X, there is the *Šmirnov compactification* Z of X for which the associated completely regular proximity is precisely $<$. The *Q-closure* of X in Z is defined to be the set of points $z \in Z$ such that for each $f \in C(Z)$ with $f(z) = 0$, there is $x \in X$ with $f(x) = 0$; it is known to be a realcompact embedding of X.

Suppose that $<$ is a completely regular proximity for X. The *realcompact (compact) structure* of $<$ is the collection of all covers α of X such that for some countable (finite) cover β there is for each $B \in \beta$ an $A \in \alpha$ such that $B < A$. It is readily shown using the axioms of completely regular proximities that the realcompact (compact) structure of $<$ is indeed a realcompact (compact) structure. The compact structure of $<$ is in fact the totally bounded uniform structure defined by $<$, as is shown by Šmirnov in [**7**].

A filter γ on X is *round* if for each $A \in \gamma$ there is $B \in \gamma$ such that $B < A$. The following result is readily obtained:

7.1. (a) [**7**] *A filter is minimal Cauchy in the compact structure of* $<$ *if and only if it is maximal round.*

(b) *A filter is minimal Cauchy in the realcompact structure of* $<$ *if and only if it is maximal round and has the countable intersection property.*

The following result is a characterization of the Q-closure of a space in a compactification, expressed in terms of the induced proximity.

THEOREM F. *Let* $<$ *be a completely regular proximity that induces the topology of* X.

(a) [**7**] *The completion of* X *in the compact structure of* $<$ *is the Šmirnov compactification.*

(b) *The completion of* X *in the realcompact structure of* $<$ *is the* Q*-closure of* X *in the Šmirnov compactification.*

Proof of (b). Let Z be the Šmirnov compactification and let Y be the Q-closure of X in Z. For each $z \in Z$ let N^z be the trace on X of the neighbourhood filter of z. Then, by (a), the maximal round filters on X are precisely the filters N^z for $z \in Z$. It follows from 7.1(b) that the completion W of X in the realcompact structure of $<$ is a subspace of Z.

Suppose that $z \in Z - Y$; then there is $f \in C(Z)$ with $f(z) = 0$ and $f > 0$ on X. Setting $A_n = \{x \in X: f(x) < 1/n\}$, it follows that $A_n \in N^z$ and $A_{n+1} < A_n$ for each n, while $\bigcap\{A_n\} = \emptyset$. Thus N^z does not have the countable intersection property, and so, by 7.1(b), $z \notin W$.

Conversely, suppose that $z \in Z - W$. Then N^z does not have the countable intersection property; thus there are members A_n of N^z with $A_{n+1} < A_n$ for each n. It follows that there is, for each n, a function $f_n \in C(Z)$, that is 0 on A_{n+1} and 1 on $X - A_n$. On defining f as in the proof of Theorem B, it follows that $f \in C(Z)$, $f(z) = 0$, and $f > 0$ on X. Thus $z \notin Y$.

8. Other internal characterizations. Shirota [**6**] has characterized realcompact spaces as the spaces that are complete in the uniform structure whose basis consists of all countable normal coverings. In this connection it is clear that any uniform structure with a base of countable covers is a realcompact structure; however a realcompact structure need not be a uniform structure, as follows from an example given (for another purpose) by Šmirnov

[**7**, p. 32]. Shirota's famous theorem regarding spaces that are complete in a uniform structure is also a characterization, if spaces with measurable cardinal are excluded.

Internal characterizations in terms of closed sets have recently been given by Johnson and Mandelker [**4**] and by McArthur [**5**].

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