

# ON THE CONSTRUCTIVE APPROXIMATION OF NON-LINEAR OPERATORS IN THE MODELLING OF DYNAMICAL SYSTEMS

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## Abstract

In this paper we propose a systematic theoretical procedure for the constructive approximation of non-linear operators and show how this procedure can be applied to the modelling of dynamical systems. We extend previous work to show that the model is stable to small disturbances in the input signal and we pay special attention to the role of real number parameters in the modelling process. The implications of computability are also discussed. A number of specific examples are presented for the particular purpose of illustrating the theoretical procedure.

## 1. Introduction

In this paper we propose a systematic theoretical procedure for the constructive approximation of non-linear operators and show how this procedure can be applied to the modelling of dynamical systems. There are several properties which we have sought to preserve in the modelling process. In many cases the only given information about such a system is information pertaining to an abstract operator  $F$ . We wish to construct an approximating operator  $S$  which can be realised in physical terms, will approximate  $F$  with a given accuracy and must be stable to *small* disturbances. The operator  $S$  defines our model of the real system and will be constructed from an algebra of elementary continuous functions by a process of finite arithmetic. For this reason we regard  $S$  as computer-processable.

A number of specific examples are presented for the particular purpose of illustrating the theoretical results. Although the examples have been simplified for computational convenience and are somewhat artificial they are none-the-less representative of real situations. In these examples we have used an underlying polynomial algebra but we note that this is simply a matter of theoretical convenience. A suitable wavelet

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algebra could be used instead. Another currently popular alternative is an algebra generated by superpositions of a sigmoidal function. Such algebras are discussed in detail by Cybenko [3] and Barron [1]. In general we require only that the underlying algebra satisfy the conditions of Stone's Algebra [11]. For the purposes of this paper we have assumed that the elementary continuous functions which generate the algebra can be evaluated by a process of finite arithmetic. While this assumption may not be strictly correct the errors involved are limited only by machine accuracy and in principle do not disrupt our analysis.

This research has been motivated by a desire to understand the nature of the modelling process for simulation of a real dynamical system. A dynamical system is defined by a mapping that transforms a set of input signals to a corresponding set of output signals. A signal is normally defined by a set of real number parameters. In practice these parameter sets could be uncountably infinite. For a computer simulation of the system each signal must be represented by a finite set of real number parameters and the mapping must be represented by a finite arithmetical process. We must nevertheless show that the simulated system is a good approximation to the real system.

To justify the approximation process we impose a basic topological structure and use the consequent notions of continuity to establish theorems of Weierstrass type. In the case of a general continuous map  $F : X \rightarrow Y$  where  $X, Y$  are locally convex topological vector spaces we will show that the approximation procedure can be used on any given compact subset  $K \subseteq X$ . Indeed if we assume that  $F$  is known only on  $K$  then for some suitable neighbourhood  $\epsilon$  of zero in  $X$  the construction of an extended operator  $S : K + \epsilon \subseteq X \rightarrow Y$  is an important ingredient in our approximation procedure. The extension of the domain allows us to consider the effect of a small disturbance in the input signal. Such disturbances are unavoidable in the modelling process.

We emphasise that our paper describes a generic approach and is concerned with applicable conditions that will allow the simulated system to represent the real system to within an arbitrarily prescribed accuracy. The problem of relating the various error bounds to the dimensions of the model is not considered and may be more effectively resolved in a specific context. One could certainly consider these questions at the level of our particular examples.

There are other aspects of the approximation process which we have not considered. A real system is normally causal and may also be stationary or have finite memory. It would be useful to know if these additional properties could be preserved in the modelling process.

## 2. Relation to previous studies

The extension of the classical Stone-Weierstrass theorem to the approximation of continuous mappings on topological vector spaces by polynomial mappings has been known for some time and was developed by Prenter [7], Istratescu [6], Prolla and Machado [8] and Bruno [2]. In these papers the approximation procedure relies directly on the classical theorem via an underlying algebra of real valued polynomials.

Our procedure is essentially an elaboration of the procedure used by Bruno but is more explicitly constructive and we believe more directly related to the representation of real dynamical systems. In particular we show that the model is stable to small disturbances in the input signal. We have also considered the role of parameters in the representation process and have adapted our methods accordingly. Our procedure is not limited to polynomial approximation. On the other hand our analysis is restricted to locally convex topological vector spaces. The present work is developed from an approach used by Torokhti [13–16].

## 3. A remark on the compactness condition

The assumption of compactness for the set  $K$  on which the operator  $F$  is to be approximated is an important part of the modelling process and cannot be totally removed. For a continuous real valued function on the real line it is well known that uniform approximation by a polynomial can be guaranteed only on a compact subset.

We believe that the compactness assumption is reasonable in practice. Suppose the dynamical system is defined by an operator  $F : X \rightarrow Y$  where  $X$  and  $Y$  are topological vector spaces. Some knowledge of the operator is necessary if we wish to simulate the given system. It may happen that  $F$  is known only on the basis of a finite subset

$$\{(x_n, y_n) \mid x_n \in X \text{ and } y_n = F(x_n) \in Y \text{ for } n = 1, 2, \dots, N\} \subseteq X \times Y$$

or alternatively on a set

$$\{(x_\gamma, y_\gamma) \mid x_\gamma \in X \text{ and } y_\gamma = F(x_\gamma) \in Y \text{ for } \gamma \in \Gamma \subseteq \mathbb{R}^n\} \subseteq X \times Y,$$

where  $\Gamma$  is compact. Such knowledge may be empirical or based on a restricted analysis of the system concerned. Of course there may be some situations where the compactness assumption is not reasonable. If the set on which the approximation is required is not compact then a stronger continuity condition is needed. In a subsequent paper we will use stronger topological assumptions to consider this more difficult problem.

### 4. Preliminaries

We begin with some preliminary results.

DEFINITION 1. Let  $X, Y$  be real Hausdorff topological vector spaces and let  $A$  be a subset of  $X$ . The map  $F : A \rightarrow Y$  is uniformly continuous on  $A$  if for each open neighbourhood of zero<sup>2</sup>  $\tau \subseteq Y$  we can find a neighbourhood of zero  $\sigma \subseteq X$  such that

$$F [(x + \sigma) \cap A] \subseteq F(x) + \tau$$

for all  $x \in A$ .

LEMMA 1. Let  $X, Y$  be real Hausdorff topological vector spaces and let  $K$  be a compact subset of  $X$ . If  $F : K \rightarrow Y$  is continuous on  $K$  then it is uniformly continuous on  $K$ .

PROOF. Let  $\tau$  be a neighbourhood of zero in  $Y$ . Choose a neighbourhood of zero  $\nu \subseteq Y$  with  $\nu - \nu \subseteq \tau$ . For each  $x \in K$  we choose a neighbourhood of zero  $\mu(x) \subseteq X$  such that

$$F [(x + \mu(x)) \cap K] \subseteq F(x) + \nu.$$

Now choose a neighbourhood of zero  $\sigma(x) \subseteq X$  such that  $\sigma(x) + \sigma(x) \subseteq \mu(x)$ . We write  $\Omega(x) = x + \sigma(x)$ . Since  $K \subseteq \bigcup_{x \in K} \Omega(x)$  and since  $K$  is compact we can find a finite subcollection  $\Omega_1, \Omega_2, \dots, \Omega_r$  (where we write  $\Omega_i = x_i + \sigma_i, \sigma_i = \sigma(x_i)$  and  $\mu_i = \mu(x_i)$ ) such that  $K \subseteq \bigcup_{i=1}^r \Omega_i$ . Define  $\sigma = \bigcap_{i=1}^r \sigma_i$ . It is clear that  $\sigma$  is an open neighbourhood of zero in  $X$ . If we choose any  $x \in K$  then we can find  $k$  such that  $x \in \Omega_k$ . Thus  $F(x) \in F(x_k) + \nu$ . Since

$$x + \sigma \subseteq \Omega_k + \sigma \subseteq (x_k + \sigma_k) + \sigma_k \subseteq x_k + \mu_k$$

it follows that

$$F [(x + \sigma) \cap K] \subseteq F(x_k) + \nu$$

and hence

$$\begin{aligned} F [(x + \sigma) \cap K] - F(x) &= \{F [(x + \sigma) \cap K] - F(x_k)\} - \{F(x) - F(x_k)\} \\ &\subseteq \nu - \nu \\ &\subseteq \tau. \end{aligned}$$

Therefore

$$F [(x + \sigma) \cap K] \subseteq F(x) + \tau.$$

<sup>2</sup>In a topological vector space a set  $\tau$  with  $0 \in \text{int}(\tau)$  will be called a *neighbourhood of zero*.

LEMMA 2. Let  $X$  be a normal<sup>3</sup> topological vector space and let  $Y$  be a locally convex topological vector space. Let  $K$  be a compact subset of  $X$  and  $F : K \rightarrow Y$  a continuous map. For each convex neighbourhood of zero  $\tau \subseteq Y$  there exists a neighbourhood of zero  $\sigma \subseteq X$  and a continuous map  $\mathcal{F}_\sigma : K + \sigma \rightarrow Y$  in the form

$$\mathcal{F}_\sigma(u) = \sum_{i=1}^r \kappa_i(u) F(x_i),$$

where  $x_i \in K$  for each  $i = 1, 2, \dots, r$  and where  $\kappa_i : K + \sigma \rightarrow \mathbb{R}$  is continuous with

(1)  $\kappa_i(u) \in [0, 1]$ , and

(2)  $\sum_{i=1}^r \kappa_i(u) = 1$ ,

such that  $F(x) - \mathcal{F}_\sigma(u) \in \tau$  whenever  $x \in K$  and  $x - u \in \sigma$ .

PROOF. Choose a neighbourhood of zero  $\mu \subseteq X$  so that for all  $x \in K$

$$F[(x + \mu) \cap K] \subseteq F(x) + \tau \quad (1)$$

and choose a neighbourhood of zero  $\sigma \subseteq X$  with  $\sigma + \sigma \subseteq \mu$ . Let  $\Omega_x = x + \sigma$ . Since

$$K \subseteq \bigcup_{x \in K} \Omega_x \quad (2)$$

we can find a finite subcollection  $\Omega_1, \Omega_2, \dots, \Omega_r$  such that

$$K \subseteq \bigcup_{i=1}^r \Omega_i. \quad (3)$$

Since  $X$  is a normal topological vector space we can construct a collection of continuous functions  $\kappa_i : K + \sigma \rightarrow \mathbb{R}$  for each  $i = 1, 2, \dots, r$  with the properties that

(1)  $\kappa_i(u) \in [0, 1]$ ,

(2)  $\sum_{i=1}^r \kappa_i(u) = 1$ , and

(3)  $\kappa_i(u) = 0$  for  $u \notin \Omega_i$ .

We define a map  $\mathcal{F}_\sigma : K + \sigma \rightarrow Y$  by the formula

$$\mathcal{F}_\sigma(u) = \sum_{i=1}^r \kappa_i(u) F(x_i).$$

<sup>3</sup>A topological vector space is said to be *normal* if for each pair of disjoint closed sets  $A, B \subseteq X$  there exists a pair of disjoint open sets  $U, V \subseteq X$  with  $A \subseteq U$  and  $B \subseteq V$ .

Now  $\kappa_i(u) \neq 0$  implies  $u \in \Omega_i$  and if  $x - u \in \sigma$  then we have  $x \in x_i + \mu$ . Hence if  $x \in K$  then  $F(x) - F(x_i) \in \tau$  and so

$$\begin{aligned} F(x) - \mathcal{F}_\sigma(u) &= \sum_{i=1}^r \kappa_i(u)[F(x) - F(x_i)] \\ &= \sum_{\{i|\kappa_i(u) \neq 0\}} \kappa_i(u)[F(x) - F(x_i)] \\ &\in \tau \end{aligned}$$

since the right hand side is a convex combination and  $\tau$  is a convex set.

**COROLLARY 1.** *If in addition to the conditions of Lemma 2 we have  $F(0) = 0$  then we can choose  $\mathcal{F}_\sigma^* : K + \sigma \rightarrow Y$  such that  $\mathcal{F}_\sigma^*$  satisfies the conditions of Lemma 2 and also satisfies  $\mathcal{F}_\sigma^*(0) = 0$ .*

**PROOF.** Choose a neighbourhood of zero  $\sigma_0 \subseteq X$  such that  $F(\sigma_0) \subseteq \tau$ . Choose another neighbourhood of zero  $\sigma \subseteq X$  such that  $\sigma + \sigma \subseteq \sigma_0$  and such that

$$F(x) - \mathcal{F}_\sigma(u) \in \tau$$

whenever  $x \in K$  and  $x - u \in \sigma$ . In accordance with Urysohn's Lemma [4] there is a continuous function  $f : X \rightarrow [0, 1]$  such that  $f(0) = 0$  and such that  $f(u) = 1$  when  $u \notin \sigma$ . Let

$$\mathcal{F}_\sigma^*(u) = f(u)\mathcal{F}_\sigma(u).$$

When  $u \notin \sigma$  we have

$$F(x) - \mathcal{F}_\sigma^*(u) = F(x) - \mathcal{F}_\sigma(u) \in \tau$$

and when  $u \in \sigma$  we have  $x \in \sigma_0$  and hence

$$F(x) - \mathcal{F}_\sigma^*(u) = [1 - f(u)]F(x) + f(u)[F(x) - \mathcal{F}_\sigma(u)] \in \tau,$$

since  $F(x) \in \tau$  and  $F(x) - \mathcal{F}_\sigma(u) \in \tau$  and the right hand side is a convex combination.

**REMARK 1.** The condition  $F(0) = 0$  in Corollary 1 can be interpreted as follows. If the operator  $F$  is the mathematical model of some dynamical system then the output  $y$  is related to the input  $x$  by  $y = F(x)$ . Thus the condition  $F(0) = 0$  means that a zero input produces a zero output.

## 5. The main results

Recall that our aim has been the constructive definition of an operator  $S$  to approximate the given operator  $F$ . Furthermore there are certain properties that must be satisfied by  $S$  if we wish to construct a useful model of the real system.

**DEFINITION 2.** Let  $X$  be a topological vector space. We say that the space  $X$  possesses the Grothendieck property of approximation [12] if one can find a sequence  $\{G_m\}_{m=1,2,\dots}$  of continuous linear operators  $G_m \in \mathcal{L}(X, X_m)^4$  where  $X_m \subseteq X$  is a subspace of dimension  $m$  and where the operators  $G_m$  are equicontinuous on compacta<sup>5</sup> and are uniformly convergent to unit operators on the same compacta<sup>6</sup>.

**REMARK 2.** The conditions in Definition 2 are related. The condition that the sequence of operators is equicontinuous on a compact set implies that a uniformly convergent subsequence can be found. On the other hand if the sequence of operators converges uniformly to the unit operator on a compact set then the sequence is equicontinuous.

Let  $X, Y$  be topological vector spaces with the Grothendieck property of approximation and with approximating sequences  $\{G_m\}_{m=1,2,\dots}, \{H_n\}_{n=1,2,\dots}$  of continuous linear operators  $G_m \in \mathcal{L}(X, X_m), H_n \in \mathcal{L}(Y, Y_n)$  where  $X_m \subseteq X, Y_n \subseteq Y$  are subspaces of dimension  $m, n$  as described in Definition 2. Write

$$X_m = \left\{ x_m \in X \mid x_m = \sum_{j=1}^m a_j u_j \right\}$$

and

$$Y_n = \left\{ y_n \in Y \mid y_n = \sum_{k=1}^n b_k v_k \right\},$$

where  $a = (a_1, a_2, \dots, a_m) \in \mathbb{R}^m, b = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n$  and  $\{u_j\}_{j=1,2,\dots,m}, \{v_k\}_{k=1,2,\dots,n}$  are bases in  $X_m, Y_n$  respectively. Let  $\{g\} = \mathcal{G}$  be an algebra of continuous functions  $g : \mathbb{R}^m \rightarrow \mathbb{R}$  that satisfies the conditions of Stone's Algebra [11]. Define

<sup>4</sup> $\mathcal{L}(X, X_m)$  denotes the set of all continuous linear mappings from  $X$  into  $X_m$ .

<sup>5</sup>The sequence  $\{G_m\}$  is said to be *equicontinuous on compacta*, if for any given compact set  $K \subseteq X$  and any given neighbourhood of zero  $\mu \subseteq X$  we can find a neighbourhood of zero  $\sigma = \sigma(\mu) \subseteq X$  such that  $G_m(x_1) - G_m(x_2) \in \mu$  for all  $m = 1, 2, \dots$  whenever  $x_1, x_2 \in K$  and  $x_1 - x_2 \in \sigma$ .

<sup>6</sup>The sequence  $\{G_m\}$  is said to *converge uniformly on compact set*  $K \subseteq X$  to the unit operator on  $K$  if for any given neighbourhood of zero  $\mu \subseteq X$  we can find  $M > 0$  such that  $G_m(x) - x \in \mu$  whenever  $x \in K$  and  $m > M$ .

the operators  $Q \in \mathcal{L}(X_m, \mathbb{R}^m)$ <sup>7</sup>,  $Z : \mathbb{R}^m \rightarrow \mathbb{R}^n$  and  $W \in \mathcal{L}(\mathbb{R}^n, Y_n)$  by the formulae

$$Q(x_m) = a, \quad Z(a) = (g_1(a), g_2(a), \dots, g_n(a)), \quad \text{and} \quad W(z) = \sum_{k=1}^n z_k v_k,$$

where each  $g_k \in \mathcal{G}$  and  $z_k = g_k(a)$ . Let  $S : X \rightarrow Y_n$  be defined by the composition

$$S = WZQG_m.$$

A block diagram for the realisation of  $S$  is depicted in Figure 1.

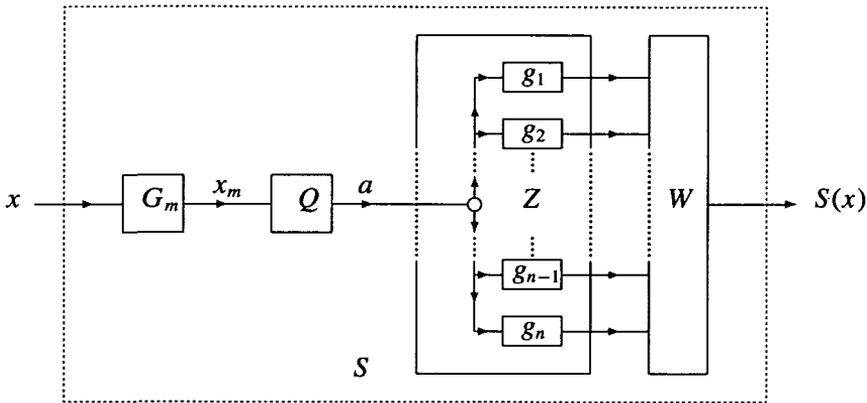


FIGURE 1. Block diagram for realisation of  $S$ .

Subject to an appropriate choice of the functions  $\{g_k\} \in \mathcal{G}$  we now show that  $S$  supplies the required approximation to  $F$ .

**THEOREM 1.** *Let  $X, Y$  be locally convex topological vector spaces with the Grothendieck property of approximation and let  $X$  be normal. Let  $K \subseteq X$  be a compact set and  $F : K \rightarrow Y$  a continuous map. For a given convex neighbourhood of zero  $\tau \subseteq Y$  there exists a neighbourhood of zero  $\sigma \subseteq X$  with an associated continuous operator  $S : X \rightarrow Y_n$  in the form  $S = WZQG_m$  and a neighbourhood of zero  $\epsilon \subseteq X$  such that for all  $x \in K$  and all  $x' \in X$  with  $x' - x \in \epsilon$  we have*

$$F(x) - S(x') \in \tau.$$

This theorem can be regarded as a generalisation of the famous Weierstrass approximation theorem. To prove the theorem we need to establish that certain sets are

<sup>7</sup>It is necessary to justify our assertion that  $Q \in \mathcal{L}(X_m, \mathbb{R}^m)$ . This will be done shortly.

compact. Since  $G_m \in \mathcal{L}(X, X_m)$  is continuous and since  $K$  is compact it follows that  $G_m(K)$  is compact. To show that the set  $QG_m(K)$  is compact we need to establish that  $Q \in \mathcal{L}(X_m, \mathbb{R}^m)$ . Since  $Q(\sum_{j=1}^m a_j u_j) = a$  we need to show that there exists a constant  $M_Q$  and a seminorm  $\rho : X \rightarrow \mathbb{R}$  with  $\|a\| \leq M_Q \rho(\sum_{j=1}^m a_j u_j)$ .

We have the following preliminary results.

LEMMA 3. *Let  $X$  be a locally convex topological vector space and let  $X_m$  be the subspace defined above. We can find a sequence  $\{\rho_s\}_{s=1,2,\dots,r}$  of seminorms  $\rho_s : X \rightarrow \mathbb{R}$  where  $r \leq m$  such that the function  $\rho : X \rightarrow \mathbb{R}$  defined by*

$$\rho(x) = \left[ \sum_{s=1}^r \{\rho_s(x)\}^2 \right]^{\frac{1}{2}}$$

is a norm on  $X_m$ .

PROOF. Let  $\{\rho_s\}_{s=1,2,\dots,r}$  be a sequence of seminorms and let  $\rho : X \rightarrow \mathbb{R}$  be the function defined above. Let  $N_s = \{x \mid x \in X_m \text{ and } \rho_s(x) = 0\}$  for each  $s = 1, 2, \dots, r$  and let  $N = \{x \mid x \in X_m \text{ and } \rho(x) = 0\}$ . It is easy shown that

- (1)  $\rho$  is a seminorm,
- (2)  $N_s$  is a subspace of  $X_m$  for each  $s = 1, 2, \dots, r$ ,
- (3)  $N = \bigcap_{s=1}^r N_s$  is also a subspace of  $X_m$ .

Since  $X$  is a locally convex linear topological space we can choose a sequence  $\{\rho_s\}_{s=1,2,\dots,r}$  of seminorms so that  $r \leq m$  and  $N = \{0\}$ . In this case the function  $\rho : X \rightarrow \mathbb{R}$  defined above is the required norm on  $X_m$ .

LEMMA 4. *Let  $X$  be a locally convex topological vector space and let  $X_m$  be the subspace defined above. If  $\rho : X \rightarrow \mathbb{R}$  is a norm on  $X_m$  then we can find  $\alpha > 0$  such that*

$$\rho \left( \sum_{j=1}^m a_j u_j \right) \geq \alpha \|a\|$$

for each  $a \in \mathbb{R}^m$ .

PROOF. It is sufficient to prove that there exists some  $\alpha > 0$  with

$$\rho \left( \sum_{j=1}^m a_j u_j \right) \geq \alpha$$

whenever  $\|a\| = 1$ . If not we can find  $\{a^{(p)}\}_{p=1,2,\dots}$  such that

$$\rho \left( \sum_{j=1}^m a_j^{(p)} u_j \right) < \frac{1}{p}$$

and  $\|a^{(p)}\| = 1$ . Thus we can find a convergent subsequence (which for convenience we also denote by  $\{a^{(p)}\}_{p=1,2,\dots}$ ) with  $a^{(p)} \rightarrow a$  as  $p \rightarrow \infty$  for some  $a \in \mathbb{R}^m$ . It now follows that

$$\rho \left( \sum_{j=1}^m a_j u_j \right) = 0$$

and also that  $\|a\| = 1$ . But

$$\rho \left( \sum_{j=1}^m a_j u_j \right) = 0 \Rightarrow \sum_{j=1}^m a_j u_j = 0 \Rightarrow a = 0.$$

Since  $\|a\| = 1$  this is a contradiction.

We are now able to prove Theorem 1.

PROOF. By the approximation property of the space  $X$ , for any neighbourhood of zero  $\xi \subseteq X$  and for all  $x \in K$ , we can find  $M = M(\xi) > 0$  such that  $G_m(x) - x \in \xi$  for  $m > M$ . By Lemma 2 we can choose  $\sigma$  and a continuous map  $\mathcal{F}_\sigma : K + \sigma \rightarrow Y$  given by

$$\mathcal{F}_\sigma(u) = \sum_{i=1}^r \kappa_i(u) F(x_i)$$

with the property that

$$F(x) - \mathcal{F}_\sigma(u) \in \frac{\tau}{4}$$

when  $x - u \in \sigma$  and hence if we choose  $\xi \subseteq \sigma$  and  $m > M$  then

$$F(x) - \mathcal{F}_\sigma G_m(x) \in \frac{\tau}{4}$$

for each  $x \in K$ . If we write

$$G_m(x) = \sum_{j=1}^m a_j u_j$$

then

$$\mathcal{F}_\sigma G_m(x) = \mathcal{F}_\sigma \left( \sum_{j=1}^m a_j u_j \right) = \sum_{i=1}^r \kappa_i \left( \sum_{j=1}^m a_j u_j \right) F(x_i)$$

and hence

$$H_n \mathcal{F}_\sigma G_m(x) = \sum_{k=1}^n b_k \left[ \sum_{i=1}^r \kappa_i \left( \sum_{j=1}^m a_j u_j \right) F(x_i) \right] v_k = \sum_{k=1}^n f_k(a) v_k.$$

We note that  $\mathcal{F}_\sigma G_m(K) \subseteq Y$  is a compact subset. By the approximation property of the space  $Y$ , for any given neighbourhood of zero  $\nu \subseteq Y$ , we can choose  $N_m > 0$  so that

$$H_n \mathcal{F}_\sigma G_m(x) - \mathcal{F}_\sigma G_m(x) \in \nu$$

for all  $x \in K$  when  $n > N_m$ . We also note that

$$\begin{aligned} H_n \mathcal{F}_\sigma G_m(x) - S(x) &= H_n \mathcal{F}_\sigma G_m(x) - WZQG_m(x) \\ &= \sum_{k=1}^n [f_k(a) - g_k(a)] v_k. \end{aligned}$$

If we suppose that the algebra  $\mathcal{G}$  satisfies the conditions of Stone's Algebra then since  $a \in QG_m(K)$  and since  $QG_m(K)$  is compact it follows that we can choose  $\{g_k\}_{k=1,2,\dots,n} \in \mathcal{G}$  so that

$$H_n \mathcal{F}_\sigma G_m(x) - S(x) \in \nu.$$

Thus, if we choose  $\nu \subseteq \frac{\tau}{8}$ , then

$$\mathcal{F}_\sigma G_m(x) - S(x) \in \frac{\tau}{8} + \frac{\tau}{8} \subseteq \frac{\tau}{4}$$

and hence

$$F(x) - S(x) \in \frac{\tau}{4} + \frac{\tau}{4} \subseteq \frac{\tau}{2}.$$

Finally we note that

$$S(x) - S(x + \Delta x) = \sum_{k=1}^n [g_k(a) - g_k(a + \Delta a)] v_k \in \frac{\tau}{2},$$

where  $\Delta a \in \mathbb{R}^m$  is defined by

$$G_m(x + \Delta x) = \sum_{j=1}^m (a_j + \Delta a_j) u_j,$$

provided we choose  $\Delta x \in \epsilon$  where  $\epsilon$  is a sufficiently small neighbourhood of zero in  $X$ . Now it follows that

$$F(x) - S(x') \in \frac{\tau}{2} + \frac{\tau}{2} \subseteq \tau,$$

where  $x' = x + \Delta x$ .

REMARK 3. Theorem 1 can have the following interpretation. The operator  $S$  represents a mathematical model of the real system [9, 14]. In this context  $x$  is the input signal,  $F(x)$  is the output signal from the real system,  $\Delta x$  is the noise that is added to the input signal in practice, and  $S(x + \Delta x)$  is the output signal from the constructed system. Thus  $S$  is a practical realisation of the given abstract operator  $F$ . Note that the noise term in the input signal could result from truncation of the parametric description.

Next we show how the operator  $S = WZQG_m$  could be constructed to give, in some definite sense, the best possible approximation to the operator  $F$ . Let  $X, Y$  be Banach spaces having the Grothendieck property of approximation and consider the following procedure. We now suppose that  $Z = Z_c$  with

$$Z_c(a) = (g_1(c_1; a), g_2(c_2; a), \dots, g_n(c_n; a))$$

and

$$g_k(c_k; a) = \sum_{s=0}^p c_{k,s} r_s(a),$$

where  $p = (p_1, p_2, \dots, p_m) \in \mathcal{Z}_+^m$  is given and  $\mathcal{Z}_+ = \{0, 1, 2, \dots\}$  denotes the set of non-negative integers and where  $c = (c_1, c_2, \dots, c_n)$ ,  $c_k = \{c_{k,s}\}_{s \in \mathcal{Z}_+^m}$  and  $c_{k,s} \in \mathbb{R}$  for each  $k = 1, 2, \dots, n$  and each permissible  $s \in \mathcal{Z}_+^m$ . We assume that each  $r_s : \mathbb{R}^m \rightarrow \mathbb{R}$  is continuous and that the collection  $\{r_s\}_{s \in \mathcal{Z}_+^m}$  generates an algebra that satisfies the conditions of Stone's Algebra. We could for example take  $r_s(a) = a^s = a_1^{s_1} a_2^{s_2} \dots a_m^{s_m}$ . We assume that the functions  $\{r_s\}_{s \in \mathcal{Z}_+^m}$  are linearly independent. Introduce the class  $\mathcal{S}$  of operators given by

$$\mathcal{S} = \{S \mid S : X \rightarrow Y_n \text{ and } S = S_c = WZ_cQG_m\}$$

with fixed operators  $G_m, Q, H_n$  and  $W$  and with fixed functions  $\{r_s\}_{s \in \mathcal{Z}_+^m}$ . Thus the operator  $S_c$  is completely defined by the coefficients  $\{c_{k,s}\}$ . Let  $\{g_k(c_k^*; a)\}$  denote the functions which best approximate the given functions  $\{f_k(a)\}$  on the set  $QG_m(K) \subseteq \mathbb{R}^m$ .

We can now state our second theorem.

THEOREM 2. Let  $X, Y$  be Banach spaces with the Grothendieck property of approximation, let  $K \subseteq X$  be a compact set and  $F : K \rightarrow Y$  a continuous map. Let the operator  $Z^* : \mathbb{R}^m \rightarrow \mathbb{R}^n$  be defined by

$$Z^* = Z_{c^*}.$$

Then for some fixed  $\epsilon > 0$  and for all  $x, x' \in X$  with  $\|x' - x\| < \epsilon$  the operator  $S^* = S_{c^*} : X \rightarrow Y_n$  in the form

$$S^* = WZ^*QG_m$$

satisfies the equality

$$\sup_{x \in K} \|F(x) - S^*(x')\| = \inf_{S \in \mathcal{S}} \left\{ \sup_{x \in K} \|F(x) - S(x')\| \right\}.$$

We illustrate these theorems with an example although we do not seek a best approximation in the strict sense of Theorem 2.

EXAMPLE 1. Let  $X = Y = \mathcal{C}[-1, 1]$  be the Banach space of continuous functions on  $[-1, 1]$  with the uniform norm  $\|f\| = \sup_{t \in [-1, 1]} |f(t)|$ . For each  $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^3$  define  $x_\gamma(t) = \gamma_1 \cos(\gamma_2 t + \gamma_3)$  and  $y_\gamma(t) = (\gamma_1 \cos \gamma_3)^4 \cosh(\gamma_2 t)$  and let  $K \subseteq X$  be the compact set given by

$$K = \{x \mid x = x_\gamma \text{ for some } \gamma \in \Gamma = [0, 1] \times [\frac{1}{2}, 1] \times [0, 2\pi] \subseteq \mathbb{R}^3\}.$$

Let the non-linear operator  $F : K \rightarrow L = F(K) \subseteq Y$  be defined by the formula  $\tilde{F}(x_\gamma) = y_\gamma$  and consider the dynamical system described by the mapping  $F : K \rightarrow L$ .

We wish to construct a practical model of the given system. We suppose that the input signal is disturbed by an additional noise term that is essentially unrelated in structure to the true input signal. In this example we choose to approximate the input signal by a polynomial and hence it is convenient for the noise term to be modelled by a polynomial of the same degree. Thus we assume that the actual input signal  $x'$  is given by

$$x' = x + \Delta x = x_\gamma + \Delta x$$

for some  $\gamma \in \Gamma$  where  $\Delta x = h$  is an appropriate polynomial. We will also approximate the output signal by a polynomial.

For some given tolerance  $\alpha > 0$  and a corresponding restriction  $h \in \epsilon$  on the magnitude of the noise term (in this context  $\epsilon$  is some suitable neighbourhood of zero) we wish to find an operator  $S : K + \epsilon \rightarrow Y$  such that  $\|F(x) - S(x')\| < \alpha$  for all  $x \in K$  and all  $x' - x \in \epsilon$ .

To construct the operator  $S$  it is necessary to extend the given set  $K$  of input signals to include the additional noise terms. Some initial discussion of calculation procedures is therefore desirable. To this end let  $\mathcal{P}_s$  denote the space of polynomials with real coefficients and of degree at most  $s - 1$ . We define a Chebyshev projection operator  $\Pi_s : \mathcal{C}[-1, 1] \rightarrow \mathcal{P}_s$  by the formula

$$\Pi_s(u) = \sum_{j=1}^s c_j(u) T_{j-1},$$

where  $T_{j-1}(\cos \theta) = \cos(j - 1)\theta$  is the Chebyshev polynomial of the first kind and where the coefficients  $c_j = c_j(u)$  are defined by the formulae

$$c_1 = \frac{1}{\pi} \int_{-1}^1 \frac{u(t)dt}{\sqrt{(1-t^2)}} \quad \text{and} \quad c_j = \frac{2}{\pi} \int_{-1}^1 \frac{u(t)T_{j-1}(t)dt}{\sqrt{(1-t^2)}}$$

for each  $j = 2, 3, \dots, s$ . In this example we will not use the integral formulae but will calculate approximate Chebyshev coefficients where necessary by using a standard economisation procedure [5].

Let  $X_m = \mathcal{P}_m$  and  $Y_n = \mathcal{P}_n$ . We define linear operators  $G_m : X \rightarrow X_m$ ,  $H_n : Y \rightarrow Y_n$  by setting  $G_m = \Pi_m$ ,  $H_n = \Pi_n$ . For convenience we will use the following approximate calculation procedure to determine  $X_m(x_\gamma) = x[\gamma, m]$  for  $x_\gamma \in K$  and  $H_n(y_\gamma) = y[\gamma, n]$  for  $y_\gamma \in L$ . For any given values of  $\mu, \nu > 0$  we can choose  $m = m(\mu)$ ,  $n = n(\nu)$  and polynomials  $p_m, q_m \in \mathcal{P}_m$  and  $r_n \in \mathcal{P}_n$  with  $p_m(\tau) = \sum_{j=1}^{\lfloor \frac{m+1}{2} \rfloor} p_{2j-1} \tau^{2j-2}$ ,  $q_m(\tau) = \sum_{j=1}^{\lfloor \frac{m+1}{2} \rfloor} q_{2j} \tau^{2j-1}$  and  $r_n(\tau) = \sum_{j=1}^{\lfloor \frac{n+1}{2} \rfloor} r_{2j-1} \tau^{2j-2}$  such that

$$|p_m(\tau) - \cos \tau| + |q_m(\tau) - \sin \tau| < \mu \quad \text{and} \quad |r_n(\tau) - \cosh \tau| < \nu$$

for all  $\tau \in [-1, 1]$ . Now define  $x[\gamma, m] \in \mathcal{P}_m$  and  $y[\gamma, n] \in \mathcal{P}_n$  by

$$x[\gamma, m](t) = \gamma_1 [(\cos \gamma_3) p_m(\gamma_2 t) - (\sin \gamma_3) q_m(\gamma_2 t)] = \sum_{j=1}^m a_j[\gamma, m] t^{j-1}$$

and

$$y[\gamma, n](t) = (\gamma_1 \cos \gamma_3)^4 r_n(\gamma_2 t) = \sum_{j=1}^n b_j[\gamma, n] t^{j-1}$$

and note that

$$|x_\gamma(t) - x[\gamma, m](t)| < \mu \quad \text{and} \quad |y_\gamma(t) - y[\gamma, n](t)| < \nu$$

for all  $t \in [-1, 1]$ . By observing that  $G_m(p) = p$ ,  $H_n(q) = q$  when  $p \in \mathcal{P}_m$ ,  $q \in \mathcal{P}_n$  and by using the linearity of the operators  $G_m$ ,  $H_n$  we can extend the above calculation procedure to polynomial neighbourhoods of  $K, L$ .

Since the hypothetical input signal  $x_\gamma \in K$  is approximated by a polynomial  $x[\gamma, m] \in \mathcal{P}_m$  we suppose that the noise term is also modelled by a polynomial  $h \in \mathcal{P}_m$ . Thus we assume that

$$h(t) = \sum_{j=1}^m w_j t^{j-1}.$$

where  $w = (w_1, w_2, \dots, w_m) \in \mathbb{R}^m$  is an unknown constant.

At this stage we need to point out that we will not follow the specific construction procedure described in our theoretical development. In this example the compact set  $K$  is described by a parameter  $\gamma \in \Gamma$  and the preceding theory suggests that we should choose an appropriate neighbourhood of zero  $\sigma \subseteq X$  and construct an operator  $\mathcal{F}_\sigma : K + \sigma \rightarrow Y$  by choosing a finite collection  $\{\gamma^{(i)}\}_{i=1,2,\dots,r}$  of points in  $\Gamma$  and an appropriate partition of unity. In practice it is often easier to choose a neighbourhood of zero  $\zeta \subseteq X_m \subseteq X$  and use direct methods to construct an operator  $\mathcal{R}_\zeta : G_m(K + \zeta) = G_m(K) + \zeta \rightarrow Y_n$  which effectively replaces the operator  $\mathcal{F}_\sigma$  used in the theoretical development by providing an approximate representation of the formal composition  $H_n \mathcal{F}_\sigma G_m^{-1}$ . We will show that the operator  $R : G_m(K) \rightarrow H_n(L)$  given by

$$R(x[\gamma, m]) = y[\gamma, n]$$

can be extended to provide the desired approximation. To define  $R$  we note that

$$a_k[\gamma, m] = \begin{cases} \gamma_1 \gamma_2^{k-1} (\cos \gamma_3) p_k & \text{if } k \text{ is odd} \\ -\gamma_1 \gamma_2^{k-1} (\sin \gamma_3) q_k & \text{if } k \text{ is even} \end{cases}$$

and

$$b_k[\gamma, m] = \begin{cases} \gamma_1^4 \gamma_2^{k-1} (\cos \gamma_3)^4 r_k & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even.} \end{cases}$$

In particular we note that

$$b_k[\gamma, m] = \left( \frac{a_1[\gamma, m]}{p_1} \right)^3 \left( \frac{a_k[\gamma, m]}{p_k} \right) r_k$$

for each  $k = 1, 2, \dots, n$ . Therefore if we define

$$g_k(a) = \left( \frac{a_1}{p_1} \right)^3 \left( \frac{a_k}{p_k} \right) r_k$$

for each  $k = 1, 2, \dots, n$  and set

- (1)  $G_m(x_\gamma) = x[\gamma, m]$ ,
- (2)  $Q(x[\gamma, m]) = (a_1[\gamma, m], a_2[\gamma, m], \dots, a_m[\gamma, m]) = a[\gamma, m]$ ,
- (3)  $Z(a) = (g_1(a), g_2(a), \dots, g_n(a)) = g(a)$ , and
- (4)  $W(b[\gamma, n]) = y[\gamma, n]$

then the desired operator  $R : G_m(K) \rightarrow H_n(L)$  is given by  $R = WZQ$ . For any fixed neighbourhood of zero  $\zeta \in \mathcal{P}_m$  the extended operator  $\mathcal{R}_\zeta : G_m(K + \zeta) \rightarrow Y_n$  is simply defined by noting that

(1)  $G_m(x_\gamma + h) = x[\gamma, m] + h$ , and

(2)  $Q(x[\gamma, m] + h) = (a_1[\gamma, m] + w_1, a_2[\gamma, m] + w_2, \dots, a_m[\gamma, m] + w_m)$ .

The operator  $S : K + \zeta \rightarrow Y_n$  is now given by the composition  $S = WZQG_m$ . We note that  $S(x[\gamma, m]) = y[\gamma, n]$  and that

$$S(x[\gamma, m] + h)(t) = \sum_{k=1}^n \left( \frac{(a_1[\gamma, m] + w_1)}{p_1} \right)^3 \left( \frac{a_k[\gamma, m] + w_k}{p_k} \right) r_k t^{k-1}.$$

Suppose that the actual level of approximation required is given by

$$\|F(x) - S(x')\| < .01.$$

Of course it is necessary to understand that the achievable level of approximation will be limited by the magnitude of  $h$ . By the same token we can only quantify this limitation when we have decided on the precise structure of  $S$ . To begin the process we let  $m = n = 3$  and construct the necessary polynomial approximations by applying a standard Chebyshev economization procedure [5] to the appropriate Maclaurin series. We have

$$\begin{aligned} \cos \tau &\approx 1 - \frac{\tau^2}{2} + \frac{\tau^4}{24} - \frac{\tau^6}{720} \\ &= \frac{1763}{2304} T_0(\tau) - \frac{353}{1536} T_2(\tau) + \frac{19}{3840} T_4(\tau) - \frac{1}{23040} T_6(\tau) \\ &\approx \frac{1763}{2304} T_0(\tau) - \frac{353}{1536} T_2(\tau) \\ &= \frac{4585}{4608} - \frac{353}{768} \tau^2 \\ &= p_3(\tau), \\ \sin \tau &\approx \tau - \frac{\tau^3}{6} + \frac{\tau^5}{120} \\ &= \frac{169}{192} T_1(\tau) - \frac{5}{128} T_3(\tau) + \frac{1}{1920} T_5(\tau) \\ &\approx \frac{169}{192} T_1(\tau) \\ &= \frac{169}{192} \tau \\ &= q_3(\tau), \end{aligned}$$

and

$$\begin{aligned}
 \cosh \tau &\approx 1 + \frac{\tau^2}{2} + \frac{\tau^4}{24} + \frac{\tau^6}{720} \\
 &= \frac{2917}{2304} T_0(\tau) + \frac{139}{512} T_2(\tau) + \frac{7}{1280} T_4(\tau) + \frac{1}{23040} T_6(\tau) \\
 &\approx \frac{2917}{2304} T_0(\tau) + \frac{139}{512} T_2(\tau) \\
 &= \frac{4583}{4608} - \frac{139}{256} \tau^2 \\
 &= r_3(\tau).
 \end{aligned}$$

With these approximations it can be seen that

$$|p_3(\tau) - \cos \tau| + |q_3(\tau) - \sin \tau| < .05 \quad \text{and} \quad |r_3(\tau) - \cosh \tau| < .006$$

for all  $\tau \in [-1, 1]$ . It follows that

$$\|x_\gamma - x[\gamma, 3]\| < .05 \quad \text{and} \quad \|y_\gamma - y[\gamma, 3]\| < .006.$$

Now we have

- (1)  $x[\gamma, 3](t) = \frac{4585}{4608} \gamma_1 \cos \gamma_3 - \frac{169}{192} \gamma_1 \gamma_2 (\sin \gamma_3) t - \frac{353}{768} \gamma_1 \gamma_2^2 (\cos \gamma_3) t^2$
- (2)  $a[\gamma, 3] = (\frac{4585}{4608} \gamma_1 \cos \gamma_3, -\frac{169}{192} \gamma_1 \gamma_2 \sin \gamma_3, -\frac{353}{768} \gamma_1 \gamma_2^2 \cos \gamma_3),$
- (3)  $g(a) = (\frac{4583 \times (4608)^3}{4585} a_1^4, 0, \frac{139 \times 768 \times (4608)^3}{256 \times 353 \times (4585)^3} a_1^3 a_3),$  and
- (4)  $y[\gamma, 3](t) = \frac{4583}{4608} (\gamma_1 \cos \gamma_3)^4 - \frac{139}{256} (\gamma_1 \cos \gamma_3)^4 \gamma_2^2 t^2.$

In this particular example we suppose that the noise term has the form  $h(t) = wt^2$  where  $|w| < .002$ . Therefore

$$S(x')(t) = y[\gamma, 3](t) + \frac{139 \times 768 \times 4608}{256 \times 353 \times 4585} (\gamma_1 \cos \gamma_3)^3 wt^2$$

and hence

$$\|S(x') - y[\gamma, 3]\| < .003.$$

Therefore

$$\begin{aligned}
 \|F(x) - S(x')\| &\leq \|y_\gamma - y[\gamma, 3]\| + \|y[\gamma, 3] - S(x')\| \\
 &< .006 + .003 \\
 &< .01.
 \end{aligned}$$

In the above example we note that the input signal depends on a finite number of real parameters. It is natural to investigate what happens when the error in the input signal is caused by an inherent uncertainty in our knowledge of the parameter values. We will motivate further discussion by considering a second example.

**EXAMPLE 2.** We consider the system described in Example 1 and suppose that the error in the input signal is due entirely to an inherent uncertainty  $\Delta\gamma$  in our knowledge of the value of  $\gamma$ . For convenience we write  $\gamma' = \gamma + \Delta\gamma$  and  $x' = x_{\gamma'}$ . For a sufficiently small neighbourhood of zero  $\theta \subseteq \mathbb{R}^3$  and with the same definitions as we used in Example 1 we can define an operator  $S : K_{\Gamma+\theta} \rightarrow Y_n$  such that

$$S(x_{\gamma'}) = y[\gamma', n]$$

whenever  $\gamma' - \gamma \in \theta$ . Since

$$F(x_{\gamma}) - S(x') = y_{\gamma} - y[\gamma', n]$$

it follows that

$$\|F(x_{\gamma}) - S(x')\| \leq \|y_{\gamma} - y_{\gamma'}\| + \|y_{\gamma'} - y[\gamma', n]\|.$$

It is now easy to see that the achievable level of approximation is limited by the uncertainty in  $\gamma$ . In particular we note that

$$\begin{aligned} y_{\gamma'}(t) - y_{\gamma}(t) &\approx \frac{\partial y}{\partial \gamma_1}(t)\Delta\gamma_1 + \frac{\partial y}{\partial \gamma_2}(t)\Delta\gamma_2 + \frac{\partial y}{\partial \gamma_3}(t)\Delta\gamma_3 \\ &= 4(\gamma_1 \cos \gamma_3)^3 (\cos \gamma_3) (\cosh \gamma_2 t) \Delta\gamma_1 + (\gamma_1 \cos \gamma_3)^4 (\sinh \gamma_2 t) t \Delta\gamma_2 \\ &\quad - 4(\gamma_1 \cos \gamma_3)^3 \gamma_1 (\sin \gamma_3) (\cosh \gamma_2 t) \Delta\gamma_3 \end{aligned}$$

and hence calculate that  $\|y_{\gamma'} - y_{\gamma}\| < \sqrt{(32 \cosh^2 1 + \sinh^2 1)} \|\Delta\gamma\|$ . If we suppose that  $\|\Delta\gamma\| < .0005$  we have  $\|y_{\gamma'} - y_{\gamma}\| < .042$ . Suppose the actual level of approximation required is given by  $\|F(x) - S(x')\| < .05$ . If we let  $m = n = 3$  as we did in the previous example then we again obtain  $\|y_{\gamma} - y[\gamma, 3]\| < .006$  and hence

$$\|F(x) - S(x')\| < .042 + .006 < .05.$$

**REMARK 4.** The problem of parameter estimation in signal analysis is well known to electrical engineers and has been studied extensively. The determination of a suitable estimate  $\hat{\gamma}$  for the parameter  $\gamma \in \mathbb{R}^3$  in the above examples is a classic single-tone estimation problem and is discussed in detail by Rife and Boorstyn [10]. The estimation procedure is based on the following observation. If we define

$$E(c) = \frac{1}{2} \int_{-1}^1 |\gamma_1 \exp[i(\gamma_2 t + \gamma_3)] - c_1 \exp[i(c_2 t + c_3)]|^2 dt$$

for each  $c \in \mathbb{R}^3$  then

$$E(c) = \gamma_1^2 - \gamma_1 c_1 \left[ \frac{\sin[(c_2 - \gamma_2) + (c_3 - \gamma_3)] - \sin[(c_2 - \gamma_2) - (c_3 - \gamma_3)]}{c_2 - \gamma_2} \right] + c_1^2$$

and it is now easy to establish that

$$\begin{aligned} \min_{c_3} E(c) &= E(c_1, c_2, \gamma_3) = \gamma_1^2 - 2\gamma_1 c_1 \left[ \frac{\sin(c_2 - \gamma_2)}{c_2 - \gamma_2} \right] + c_1^2, \\ \min_{c_2} E(c_1, c_2, \gamma_3) &= E(c_1, \gamma_2, \gamma_3) = \gamma_1^2 - 2\gamma_1 c_1 + c_1^2 \end{aligned}$$

and

$$\min_{c_1} E(c_1, \gamma_2, \gamma_3) = E(\gamma) = 0.$$

The estimate  $\hat{\gamma}$  for  $\gamma$  is found by an elementary search over a suitably chosen finite set  $\{c\} \subseteq \mathbb{R}^3$ . On the basis of the above analysis the search procedure can be seen to consist of three consecutive one dimensional searches. When the full signal  $\gamma_1 \exp[i(\gamma_2 t + \gamma_3)]$  is not known we define

$$E_1(c) = \frac{1}{2} \int_{-1}^1 [\gamma_1 \cos(\gamma_2 t + \gamma_3) - c_1 \cos(c_2 t + c_3)]^2 dt$$

and search over  $\{c\} \subseteq \mathbb{R}^3$  to find the minimum value  $E_1(\gamma) = 0$ . The search is a true three dimensional search because the problem is no longer separable. On the other hand if the signal  $\gamma_1 \cos(\gamma_2 t + \gamma_3)$  is observed for all  $t \in (-\infty, \infty)$  then we have

$$\gamma_1 \sin(\gamma_2 s + \gamma_3) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\gamma_1 \cos(\gamma_2 t + \gamma_3)}{s - t} dt$$

which reconstructs the full signal and allows us to use the original method. Although our explanation does not consider the influence of noise on the estimation process the above procedure is valid in the presence of additive Gaussian noise.

## 6. Simplification of the canonical structure in the approximating operator

Consider application of the above approach in the approximation of real non-linear dynamical systems where the system is completely described by a finite number of real parameters.

Let  $X, Y$  be locally convex topological vector spaces and let  $F : K \subseteq X \rightarrow Y$  be a given continuous map. As above we will consider  $F$  as an abstract model of some

dynamical system where the sets  $K$  and  $L = F(K) \subseteq Y$  are understood to be the sets of input and output signals respectively. It may be that both sets depend continuously on a finite number of real parameters. In this regard we will therefore assume the existence of closed and bounded intervals  $\Gamma \subseteq \mathbb{R}^m$  and  $\Delta \subseteq \mathbb{R}^n$  and continuous maps  $\varphi : \Gamma \rightarrow K_\Gamma = K$  and  $\psi : \Delta \rightarrow L_\Delta = L$  with  $\varphi(\gamma) = x_\gamma$  and  $\psi(\delta) = y_\delta$ , where  $F(x_\gamma) = y_\delta$ . If we also assume that  $\psi$  is a homeomorphism (thus we assume that  $\psi$  is a one to one map and that  $\psi^{-1}$  is continuous) then we effectively assume the existence of a continuous map  $R : \Gamma \rightarrow \Delta$  defined by the composition  $R = \psi^{-1}F\varphi$  which describes the continuous dependence  $\delta = R(\gamma)$  of the output parameters on the input parameters. The non-linear system described by the continuous map  $F : K_\Gamma \rightarrow L_\Delta$ , where

$$K_\Gamma = \{x \mid x = x_\gamma \text{ where } \gamma \in \Gamma\}$$

and

$$L_\Delta = \{y \mid y = y_\delta \text{ where } \delta \in \Delta\}$$

can now be represented in alternative form on the compact set  $\Gamma \subseteq \mathbb{R}^m$  by the continuous map  $R : \Gamma \rightarrow \Delta$ .

For each neighbourhood of zero  $\eta \subseteq \mathbb{R}^n$  we can use Lemma 2 to find a neighbourhood of zero  $\zeta \subseteq \mathbb{R}^m$  and a continuous map  $\mathcal{R}_\zeta : \Gamma + \zeta \rightarrow \mathbb{R}^n$  such that  $R(\gamma) - \mathcal{R}_\zeta(\hat{\gamma}) \in \eta$  whenever  $\gamma \in \Gamma$  and  $\gamma - \hat{\gamma} \in \zeta$ . We choose  $\zeta$  to be closed and bounded and assume that the map  $\mathcal{R}_\zeta$  can be represented approximately on the compact set  $\Gamma + \zeta$  by a continuous map  $Z : \Gamma + \zeta \rightarrow \mathbb{R}^n$  with the property that  $Z(\hat{\gamma}) - \mathcal{R}_\zeta(\hat{\gamma}) \in \eta$  whenever  $\hat{\gamma} \in \Gamma + \zeta$ . This representation is normally constructed from a given algebra  $\{g\} = \mathcal{G}$  of continuous functions  $g : \mathbb{R}^m \rightarrow \mathbb{R}^n$  that satisfies the conditions of Stone's Algebra.

In practice our knowledge of the parameter values is subject to systematic and pseudo-random instrumental errors. Thus the assumed parameter value is given by  $\gamma' = \gamma + \Delta\gamma$  where  $\gamma$  is the true value and where the error  $\Delta\gamma$  is bounded by  $\Delta\gamma \in \theta$  for some known neighbourhood of zero  $\theta \subseteq \mathbb{R}^m$ . We will assume that  $\theta + \theta \subseteq \zeta$ .

In addition to the problem of instrumental errors it may be necessary to use some prescribed method of approximate calculation to determine the parameter values from measurements of the input signal. To this end we assume that for each  $\theta$  we can choose a neighbourhood of zero  $\xi \subseteq \theta$  and a continuous operator  $\hat{V}_\xi : K_{\Gamma+\theta} \rightarrow \Gamma + \zeta$  which is used to calculate the parameter value  $\gamma$  and for which the calculated value  $\hat{\gamma}' = \hat{V}_\xi(x_{\gamma'})$  satisfies the constraint  $\hat{\gamma}' - \gamma' \in \xi$  for all  $\gamma' \in \Gamma + \theta$ . Thus  $\hat{\gamma}' \in \Gamma + \zeta$ .

To describe the system we define an operator  $\hat{S} : K_{\Gamma+\theta} \rightarrow L_\Delta + \tau$  in the form of a composition

$$\hat{S} = WZ\hat{V}_\xi.$$

We can now state the following theorem.

**THEOREM 3.** *Let  $X, Y$  be locally convex linear topological spaces and let  $F : K_\Gamma \subseteq X \rightarrow L_\Delta \subseteq Y$  be a continuous map as described above. Then, for each given neighbourhood of zero  $\tau \subseteq Y$ , we can find neighbourhoods of zero  $\xi \subseteq \theta \subseteq \mathbb{R}^m$  and an operator  $\hat{S} : K_{\Gamma+\theta} \rightarrow L_\Delta + \tau$  in the form of a composition  $\hat{S} = WZ\hat{V}_\xi$  such that*

$$F(x_\gamma) - \hat{S}(x_{\gamma'}) \in \tau$$

whenever  $\gamma \in \Gamma$  and  $\gamma' - \gamma \in \theta$ .

**PROOF.** In terms of the notation introduced above we define  $W = \psi$  and let  $\eta = \psi^{-1}(\nu)$  where the neighbourhood of zero  $\nu \subseteq Y$  is chosen so that  $\nu + \nu \subseteq \tau$ . We also suppose that the neighbourhoods of zero  $\xi \subseteq \theta \subseteq X$  are chosen in the manner suggested in the above preamble. Now we can write

$$\begin{aligned} F(x_\gamma) - W\mathcal{R}_\xi \hat{V}_\xi(x_{\gamma'}) &= \psi[\psi^{-1}F\varphi](\gamma) - \psi\mathcal{R}_\xi \hat{V}_\xi(x_{\gamma'}) \\ &= \psi[R(\gamma) - \mathcal{R}_\xi(\hat{\gamma}')]. \end{aligned}$$

We choose  $\zeta$  so that  $R(\gamma) - \mathcal{R}_\xi(\hat{\gamma}') \in \eta$  whenever  $\gamma \in \Gamma$  and  $\hat{\gamma}' - \gamma \in \zeta$ . Hence

$$\psi[R(\gamma) - \mathcal{R}_\xi(\hat{\gamma}')] \in \nu.$$

Since  $\hat{\gamma}' \in \Gamma + \zeta$  it follows that  $\mathcal{R}_\xi(\hat{\gamma}') - Z(\hat{\gamma}') \in \eta$  and hence

$$\begin{aligned} F(x_\gamma) - \hat{S}(x_{\gamma'}) &= [F(x_\gamma) - W\mathcal{R}_\xi \hat{V}_\xi(x_{\gamma'})] + [W\mathcal{R}_\xi \hat{V}_\xi(x_{\gamma'}) - \hat{S}(x_{\gamma'})] \\ &= \psi[R(\gamma) - \mathcal{R}_\xi(\hat{\gamma}')] + \psi[\mathcal{R}_\xi(\hat{\gamma}') - Z(\hat{\gamma}')] \\ &\in \nu + \nu \\ &\in \tau. \end{aligned}$$

This completes the proof.

**REMARK 5.** Practical considerations allow us, as a rule, to determine  $R$  on a set  $\Gamma + \zeta$  for some neighbourhood of zero  $\zeta \subseteq \mathbb{R}^m$  and hence we can set  $\mathcal{R}_\zeta = R$ .

Now consider the idea of best approximation to the operator  $F : K_\Gamma \rightarrow Y$  by the operator  $\hat{S} = WZ\hat{V}_\xi$ . Let  $X, Y$  be Banach spaces. As before we suppose that  $Z = Z_c$  with

$$Z_c(\gamma) = (g_1(c_1; \gamma), g_2(c_2; \gamma), \dots, g_n(c_n; \gamma))$$

and

$$g_k(c_k; \gamma) = \sum_{s=0}^p c_{k,s} r_s(\gamma)$$

where  $p = (p_1, p_2, \dots, p_m) \in \mathcal{L}_+^m$  is fixed and where

$$r_s(\gamma) = \gamma^s = \gamma_1^{s_1} \gamma_2^{s_2} \dots \gamma_m^{s_m}.$$

The neighbourhoods of zero  $\xi, \theta, \zeta \subseteq \mathbb{R}^m$  can be chosen to be closed and bounded. Fix  $\xi, \theta, \zeta$  and the method of calculation of the parameters and introduce the class  $\hat{\mathcal{S}}$  of operators given by

$$\hat{\mathcal{S}} = \{ \hat{S} \mid \hat{S} : K_{\Gamma+\theta} \rightarrow Y \text{ and } \hat{S} = \hat{S}_c = WZ_cQ\hat{V}_\xi \}.$$

Thus the operator  $\hat{S}_c$  is completely defined by the coefficients  $\{c_{k,s}\}$ . We suppose the map  $\mathcal{B}_\zeta : \Gamma + \zeta \rightarrow \mathbb{R}^n$  is written in the form  $\mathcal{B}_\zeta(\gamma) = (f_1(\gamma), f_2(\gamma), \dots, f_n(\gamma))$  and let  $\{g_k(c_k^*; \gamma)\}$  denote the functions which best approximate the given functions  $\{f_k(\gamma)\}$  on the closed and bounded interval  $\Gamma + \zeta \subseteq \mathbb{R}^m$ .

We have the following theorem.

**THEOREM 4.** *Let  $X, Y$  be Banach spaces, let  $K \subseteq X$  be a compact set and  $F : K \rightarrow Y$  a continuous map. Let the operator  $Z^* : \mathbb{R}^m \rightarrow \mathbb{R}^n$  be defined by*

$$Z^* = Z_{c^*}.$$

*Then for some fixed  $\alpha > 0$  and for all  $x, x' \in X$  with  $\|x' - x\| < \alpha$  the operator  $\hat{S}^* = \hat{S}_{c^*} : X \rightarrow Y$  in the form*

$$\hat{S}^* = WZ^*\hat{V}_\xi$$

*satisfies the equality*

$$\sup_{x \in K} \|F(x) - \hat{S}^*(x')\| = \inf_{\hat{S} \in \hat{\mathcal{S}}} \left\{ \sup_{x \in K} \|F(x) - \hat{S}(x')\| \right\}.$$

The scheme of numerical realisation of the operator  $\hat{S}$  consists of the following steps. Firstly it is necessary to implement a method for approximate determination of the parameter  $\gamma$ . Secondly it is necessary to construct the functions  $g_1, g_2, \dots, g_n$  and consequently the operator  $Z$  and thirdly it is necessary to construct an appropriate operator  $W$ . We will illustrate these procedures with an example involving parameter estimation.

EXAMPLE 3. Consider the following situation. Let  $X$  be a space of measurable functions. We will suppose that a set of incoming signals has the form  $K = \{x_\gamma\}_{\gamma \in \Gamma}$  where each signal  $x_\gamma \in X$  is completely specified by the value of a parameter  $\gamma \in \Gamma \subseteq \mathbb{R}^m$ . By observing an individual signal from this set we obtain a measurement  $\delta = R(\gamma) \in \Delta \subseteq \mathbb{R}^n$  from which we wish to estimate the value of the unknown parameter  $\gamma$ .

Therefore the natural estimation procedure can be regarded as a dynamical system represented by a mapping  $R : \Gamma \rightarrow \Delta$  with input  $\gamma \in \Gamma \subseteq \mathbb{R}^m$  and output  $\delta \in \Delta \subseteq \mathbb{R}^n$ . We wish to construct a best possible approximation to this system in the sense of Theorem 4. Thus we must show that the mapping  $R : \Gamma \rightarrow \Delta$  can be approximated by an operator  $\hat{S} : \mathbb{R}^m \rightarrow \mathbb{R}^n$ . Since the output from the system is the parameter  $\delta$  itself we have  $y_\delta = \delta$  and the general output structure is simplified.

In our example we let  $X = L \times L$  where  $L$  is the space of all measurable functions  $x : [0, \infty) \rightarrow \mathbb{R}$  such that

$$\|x\| = \int_0^\infty |x(t)| dt < \infty.$$

We assume that the observed signal has the form  $x_\gamma = (x_{\gamma,1}, x_{\gamma,2}) \in X$  where

$$x_{\gamma,1}(t) = \exp(-t) \left( \cos \gamma_1 t + \frac{\sin \gamma_2 t}{t} \right) \quad \text{and} \quad x_{\gamma,2}(t) = t x_{\gamma,1}(t)$$

and where  $\gamma = (\gamma_1, \gamma_2) \in [-1, 1] \times [-1, 1] = \Gamma \subseteq \mathbb{R}^2$  is the unknown parameter. To estimate  $\gamma$  we take a Fourier cosine transform for  $x_\gamma$ . In particular the transform is used to determine the DC-component of each signal. Define  $\mathcal{X}_\gamma(\omega) = (\mathcal{X}_{\gamma,1}(\omega), \mathcal{X}_{\gamma,2}(\omega))$ . It is easily shown that

$$\begin{aligned} \mathcal{X}_{\gamma,1}(\omega) &= \int_0^\infty x_{\gamma,1}(t) \cos \omega t dt \\ &= \frac{1}{2} \left[ \frac{1}{1 + (\omega + \gamma_1)^2} + \frac{1}{1 + (\omega - \gamma_1)^2} \right] \\ &\quad + \frac{1}{2} \left[ \arctan(\omega + \gamma_2) - \arctan(\omega - \gamma_2) \right] \end{aligned}$$

and

$$\begin{aligned} \mathcal{X}_{\gamma,2}(\omega) &= \int_0^\infty x_{\gamma,2}(t) \cos \omega t dt \\ &= \frac{1}{2} \left[ \frac{1 - (\omega + \gamma_1)^2}{[1 + (\omega + \gamma_1)^2]^2} + \frac{1 - (\omega - \gamma_1)^2}{[1 + (\omega - \gamma_1)^2]^2} \right] \\ &\quad + \frac{1}{2} \left[ \frac{\omega + \gamma_2}{1 + (\omega + \gamma_2)^2} - \frac{\omega - \gamma_2}{1 + (\omega - \gamma_2)^2} \right]. \end{aligned}$$

Thus we calculate

$$\delta_1 = \mathcal{X}_{\gamma,1}(0) = \frac{1}{1 + \gamma_1^2} + \arctan \gamma_2$$

and

$$\delta_2 = \mathcal{X}_{\gamma,2}(0) = \frac{1 - \gamma_1^2}{[1 + \gamma_1^2]^2} + \frac{\gamma_2}{1 + \gamma_2^2}.$$

In effect we have defined a non-linear system which is described by a map  $R : \Gamma \rightarrow \mathbb{R}^2$  given by  $\delta = R(\gamma)$  where  $\delta = (\delta_1, \delta_2) \in \mathbb{R}^2$ . The non-linear system has input  $\gamma \in \Gamma$  and output  $\delta \in R(\Gamma) = \Delta$ . It is clear that the above formulae for  $\delta = \mathcal{X}_\gamma(0)$  can be applied to all  $\gamma \in \mathbb{R}^2$  to define an extended map  $\mathcal{R} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ .

We seek the best possible approximation to the extended operator  $\mathcal{R}$  in the following sense. Let  $H$  be the Hilbert space of measurable functions  $f : [-1, 1] \rightarrow \mathbb{R}$  such that

$$\int_{-1}^1 \frac{[f(s)]^2 ds}{\sqrt{1 - s^2}} < \infty$$

with inner product

$$\langle f, g \rangle = \int_{-1}^1 \frac{f(s)g(s)ds}{\sqrt{1 - s^2}}.$$

Let  $\mathcal{P}_m \subseteq H$  be the subspace of polynomials of degree at most  $m - 1$ . For each  $f \in H$  there exists a unique polynomial  $p_m = p_m(f) \in \mathcal{P}_m$  which minimises the integral

$$E(f, p) = \|f - p\|^2 = \langle f - p, f - p \rangle$$

over all  $p \in \mathcal{P}_m$ . It is well known that  $p_m(f) = \Pi_m(f)$  where  $\Pi_m : H \rightarrow \mathcal{P}_m$  is the Chebyshev projection operator defined in Example 1. Therefore

$$p_m = \sum_{j=1}^m c_j T_{j-1},$$

where  $T_{j-1}$  is the Chebyshev polynomial of the first kind of degree  $j - 1$  and where the coefficients  $c_j = c_j(f)$  are calculated using the integral formulae given in Example 1. We write  $f \sim p_m(f)$ .

In this example we will take  $m = 6$ . Define functions  $\{f_{ij}\}_{i,j \in \{1,2\}} \in H$  by the formulae

$$f_{11}(s) = \frac{1}{1 + s^2}, \quad f_{12}(s) = \arctan s, \quad f_{21}(s) = \frac{1 - s^2}{(1 + s^2)^2}, \quad \text{and} \quad f_{22}(s) = \frac{s}{1 + s^2}.$$

The corresponding projections  $\{p_{ij}\}_{i,j \in \{1,2\}} \in \mathcal{P}_6$  are given by

$$\begin{aligned}
 p_{11} &= \frac{\sqrt{2}}{2}T_0 - (3\sqrt{2} - 4)T_2 + (17\sqrt{2} - 24)T_4 \\
 &\approx (.7071)T_0 - (.2426)T_2 + (.0416)T_4, \\
 p_{12} &= (2\sqrt{2} - 2)T_1 - \frac{(10\sqrt{2} - 14)}{3}T_3 + \frac{(58\sqrt{2} - 82)}{5}T_5 \\
 &\approx (.8284)T_1 - (.0474)T_3 + (.0049)T_5, \\
 p_{21} &= \frac{\sqrt{2}}{4}T_0 - \frac{(8 - 5\sqrt{2})}{2}T_2 + \frac{(112 - 79\sqrt{2})}{2}T_4 \\
 &\approx (.3536)T_0 - (.4645)T_2 + (.1386)T_4, \quad \text{and} \\
 p_{22} &= (2 - \sqrt{2})T_1 - (10 - 7\sqrt{2})T_3 + (58 - 41\sqrt{2})T_5 \\
 &\approx (.5858)T_1 - (.1005)T_3 + (.0172)T_5.
 \end{aligned}$$

The theoretical system is therefore replaced by a more practical system described by a map  $Z^* : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by  $\delta = Z^*(\gamma)$  where  $\delta_1 = p_{11}(\gamma_1) + p_{12}(\gamma_2)$  and  $\delta_2 = p_{21}(\gamma_1) + p_{22}(\gamma_2)$ . In actual fact the calculations will be based on one further approximation. In practice we choose a large value of  $T$  and calculate  $\delta_T = \mathcal{R}_T(\gamma)$  using

$$\delta_{T,1} = \int_0^T x_{\gamma,1}(t)dt \quad \text{and} \quad \delta_{T,2} = \int_0^T x_{\gamma,2}(t)dt.$$

Associated with each  $\delta_T$  there is a uniquely defined (virtual) measurement  $\hat{\gamma}$  defined by

$$\hat{\gamma} = \mathcal{R}^{-1}(\delta_T). \tag{4}$$

Therefore we have a (virtual) measurement scheme defined by an operator  $\hat{V}_T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by  $\hat{V}_T = \mathcal{R}^{-1}\mathcal{R}_T$  and written in the form  $\hat{\gamma} = \hat{V}_T(\gamma)$ . The practical measurement system is now described by an operator  $\hat{S} = Z^*\hat{V}_T = Z^*\mathcal{R}^{-1}\mathcal{R}_T$  with output given by

$$\delta = \hat{S}(\gamma) = Z^*\hat{V}_T(\gamma) = Z^*(\hat{\gamma}).$$

The operator  $\hat{S}$  is the best possible approximation in the sense of Theorem 4.

To estimate the parameter  $\gamma$  we take the (observed) value  $\delta_T = \mathcal{R}_T(\gamma)$  of  $\delta$  and compute  $\gamma_{\text{est}} = Z^{*-1}(\delta_T) = Z^{*-1}\mathcal{R}_T(\gamma)$ . For example this could be done by using a Newton iteration to solve the equation  $Z^*(\gamma_{\text{est}}) = \delta_T$ .

## 7. Summary

We have shown that realistic models can be constructed for non-linear dynamical systems in such a way that the model provides an accurate representation of the *input-output* behaviour of the given system and is stable to small disturbances. We have used several examples to illustrate the proposed construction procedure. By assuming a stronger topological structure for the spaces involved it should be possible to strengthen the conclusions. In particular it may be useful to extend the approximation procedure to systems defined on non-compact sets. For systems that incorporate more algebraic structure alternative approximation procedures may be possible.

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