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RIESZ'S FUNCTIONS IN WEIGHTED HARDY AND BERGMAN SPACES

Dedicated to Professor Fumi-Yuki Maeda on his sixtieth birthday

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ABSTRACT. Let μ be a finite positive Borel measure on the closed unit disc $\bar{D}$. For each a in $\bar{D}$, put

$$S(a) = \inf \int_{\bar{D}} |f|^p \, d\mu$$

where f ranges over all analytic polynomials with $f(a) = 1$. This upper semicontinuous function $S(a)$ is called a *Riesz's function* and studied in detail. Moreover several applications are given to weighted Bergman and Hardy spaces.

1. **Introduction.** Let D be the open unit disc in the complex plane $\mathbf{C}$. P denotes a set of all analytic polynomials and H denotes a set of all analytic functions on D. Suppose $0 < p < \infty$. When μ is a finite positive Borel measure on $\bar{D}$ and $a \in \bar{D}$, put

$$S(\mu, a) = S(\mu, p, a) = \inf\left\{ \int_{\bar{D}} |f|^p \, d\mu \,; f \in P \text{ and } f(a) = 1 \right\}$$

and

$$R(\mu, a) = R(\mu, p, a) = \sup\left\{ |f(a)|^p \,; f \in P \text{ and } \int_{\bar{D}} |f|^p \, d\mu \leq 1 \right\}.$$

When μ is a finite positive Borel measure on D and $a \in D$, put

$$s(\mu, a) = s(\mu, p, a) = \inf\left\{ \int_{D} |f|^p \, d\mu \,; f \in H \text{ and } f(a) = 1 \right\}$$

and

$$r(\mu, a) = r(\mu, p, a) = \sup\{ |f(a)|^p \,; f \in H \text{ and } \int_{D} |f|^p \, d\mu \leq 1 \}.$$

The four functions S, R, s and r are called *Riesz's functions*. In this paper we study these four Riesz's functions. M. Riesz used such functions to solve the moment problem on the real line (*cf.* [6, Chapter 5]). T. Kriete and T. Trent [7] also investigated the relationship between μ and $R(\mu, 2, a)$. In the investigations of Riesz's functions, the most fundamental and important result is the following theorem by G. Szegő (*cf.* [5, Chapter 3]). He proved it only when $p = 2$ but it can be proved for arbitrary p. In the statement of the theorem, we note that the integral kernel $(1 - |a|^2)/|1 - \bar{a}e^{i\theta}|^2$ is called the *Poisson kernel.*

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SZEGŐ'S THEOREM. *Suppose* $0 < p < \infty$, μ *is a finite positive Borel measure on* $\bar{D}$ *with* $\operatorname{supp}\mu \subseteq \partial D$ *and* $d\mu/(d\theta/2\pi) = w(e^{i\theta})$. *Then,*

$$S(\mu,p,a) = (1 - |a|^2)\exp(\log w)^{\wedge}a) \quad (a \in D)$$

where $(\log w)^{\wedge}(a) = \int_0^{2\pi} \log w(e^{i\theta}) \frac{1-|a|^2}{|1-\bar{a}e^{i\theta}|^2}\, d\theta/2\pi$.

It is most desirable to describe $S(\mu,p,a)$ using μ as in Szegő's Theorem, when μ is an arbitrary finite Borel measure on $\bar{D}$. However such a problem is very difficult except for some special measures μ. In Section 2, we study the behaviour of $S(\mu,p,a)$ as $|a| \longrightarrow 1$ for an arbitrary measure on $\bar{D}$. Moreover we note that

$$S(\mu,p,a)R(\mu,p,a) = 1 \quad (a \in \bar{D}).$$

Thus, we need to know only S or R. In this paper, the results and the proofs about s and r are very similar to those about S and R. Hence we concentrate on only S or R in Sections 2, 3 and 4. Let m be the normalized area measure on D, that is, $dm = r\,dr\,d\theta/\pi$. In Section 3, we give several lower estimates of S using $d\mu/dm$. It is more difficult to give the upper estimates of S. We do it only in very special cases. In Section 4, we show that $R(\mu,p,a)$ is not in $L^1(\mu)$ if $\operatorname{supp}\mu$ is not a finite set.

Suppose $0 < p < \infty$. $H^p(\mu)$ denotes the closure of P in $L^p(\mu)$ when μ is a finite positive Borel measure on $\bar{D}$. $H^p(\mu)$ is called a *weighted Hardy space*. If $d\mu = d\theta/2\pi, H^p(\mu) = H^p$ is the classical Hardy space. When μ is a finite positive Borel measure on D, then one defines $L^p_a(\mu) = H \cap L^p(\mu)$. $L^p_a(\mu)$ is called a *weighted Bergman space*. If $\mu = m, L^p_a(\mu) = L^p_a$ is the usual Bergman space. H^p can be embedded in H. $L^p_a = H^p(m)$, and hence L^p_a is closed. We are interested in the following questions:

(1) When can $H^p(\mu)$ be embedded in H?

(2) When is $L^p_a(\mu)$ closed?

(3) When can $H^p(\mu)$ be embedded in $L^p_a(\mu)$?

Of course it is very interesting to know when $L^p_a(\mu) = H^p(\mu)$, where μ is a measure on D. This problem is classical and important (*cf.* [2]). However, in this paper we are not going to consider this problem. Question (2) was studied by M. Yamada [13]. If μ is a measure on D, question (1) is equivalent to (3). Note that the measure μ for (2) satisfies (3). In Section 5, we study the three questions given above. For example, for some compact set K in D, if $\int_{\bar{D}\setminus K} \log W\, dm > -\infty$ then $H^p(\mu)$ can be embedded in H where $W = d\mu/dm$. This result follows from the lower estimate of $S(\mu,p,a)$ in Section 3.

In this paper, we will use the following notation. For each $a \in D$, let ϕ_a be the Möbius function on D, that is,

$$\phi_a(z) = \frac{a - z}{1 - \bar{a}z} \quad (z \in D),$$

and put

$$\beta(a,z) = \frac{1}{2}\log\frac{1 + |\phi_a(z)|}{1 - |\phi_a(z)|} \quad (a,z \in D).$$

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For $0 < r \leq \infty$ and $a \in D$, let

$$D_r(a) = \{z \in D\,; \beta(a,z) < r\}$$

be the Bergman disc with 'center' a and 'radius' r. For $u \in L^1(m)$,

$$\tilde{u}(a) = \int_D u \circ \phi_a(z)\, dm(z) \quad (a \in D).$$

Then $\tilde{u}$ may be bounded on D even if u is not bounded on D.

2. **Riesz's function.** If $\mu = m$, then for $0 < p < \infty$ $S(m,p,a) = (1-|a|^2)^2$. Hence $\mu = m$ or $\operatorname{supp}\mu \subseteq \partial D$, by Szegő's Theorem $\lim_{r\to 1-} S(\mu,p,re^{i\theta}) = 0$ a. e. θ. In this section, we show that this is true in general. In particular, R is not bounded on D. In fact, for arbitrary μ, we show that $\lim_{r\to 1-} S(\mu,p,re^{i\theta}) = 0$ except for a countable set of θ.

PROPOSITION 1. *Suppose $0 < p < \infty$ and μ is a finite positive Borel measure. Then the following are valid for $R(a) = R(\mu,p,a)$ and $S(a) = S(\mu,p,a)$.*

(1) $R(\mu,p,a)\, S(\mu,p,a) = 1$ for $a \in \bar{D}$, assuming $\infty \times 0 = 1$.

(2) $R(\mu)$ is lower semicontinuous on $(0,\infty) \times D$, and $S(\mu)$ is upper semicontinuous on the same set. Moreover $R(\mu,p,a) \geq 1/\mu(\bar{D})$ and $S(\mu,p,a) \leq \mu(\bar{D})$.

(3) If $\log R$ or R is in $L^1(m)$, then for $a \in D$

$$R(a) \leq \exp(\log R)^{\sim}(a) \leq \tilde{R}(a).$$

(4) If $r < \infty$, then for $a \in D$

$$\log R(a) \leq \left(\frac{1+s|a|}{1-s|a|}\right)^2 \frac{1}{m\big(D_r(a)\big)} \int_{Dr(a)} \log R\, dm$$

where $s = \tanh r$. Hence for $a \in D$

$$\log R(a) \leq \left(\frac{1+|a|}{1-|a|}\right) \int_D \log R\, dm.$$

These inequalities are also valid for R instead of $\log R$.

(5) For $a \in D$,

$$S(\mu,p,a) \geq S\big(S(\mu)\, dm, p, a\big).$$

(6) R is not bounded on D and $\bar{D}$.

PROOF. (1) It is easy to see that $1 \leq R(a)S(a)$ for $1 \in \bar{D}$. If $1 < R(a)S(a)$, then there exists a positive constant γ such that $1 \leq \gamma S(a)$ and $\gamma < R(a)$. Hence $1 \leq \gamma \int |g|^p d\mu$ for any $g \in P$ with $g(a) = 1$ and so

$$|f(a)|^p \leq \gamma \int_{\bar{D}} |f|^p\, d\mu \text{ for any } f \in P.$$

This implies $\gamma \geq R(a)$. This contradiction shows that $1 = R(a)\, S(a)$. (2) is clear by (1).
(3) If $f \in P$, then $\log|f|$ is subharmonic on D and hence for any $a \in D$,

$$\log|f(a)|^p \leq \int_D \log|f(z)|^p \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4}\, dm(z).$$

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Assuming $\int |f|^p \, d\mu \leq 1$, by definition of R

$$\log R(a) \leq \int_D \log R(z) \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} \, dm(z).$$

This implies $R(a) \leq \exp(\log R)^{\sim}(a) \leq \tilde{R}(a)$. (4) If $0 < r < \infty$, for any $a \in D_r(0)$ and any $f \in P$,

$$\log |f(a)|^p \leq \frac{1}{m\big(D_r(0)\big)} \int_{D_r(a)} \log |f(z)|^p \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} \, dm(z)$$

and hence

$$\log |f(a)|^p \leq \frac{1}{m\big(D_r(a)\big)} \left(\frac{1+s|a|}{1-s|a|} \right)^2 \int_{D_r(a)} \log |f|^p \, dm$$

where $s = \tanh r$. This proof is the same as that of [14, Proposition 4.3.8.]. Assuming $\int |f|^p \, d\mu \leq 1$, we get (4) as in (3). (5) By (1),

$$\int |f|^p \, d\mu \geq S(\mu, z)|f(z)|^p \quad (z \in D),$$

and hence $\int |f|^p \, d\mu \geq \int |f|^p S(\mu) \, dm$. Assuming $f(a) = 1$ and $a \in D$, we get $S(\mu, a) \geq S\big(S(\mu) \, dm, a\big)$. (6) If $R(\mu, p, a)$ is bounded on $\bar{D}$, then $H^p(\mu) \subset L^\infty(\mu)$. By [11, Theorem 5.2], $H^p(\mu)$ is finitely dimensional. It is easy to see that $\operatorname{supp} \mu$ is a finite set. Then trivially $R(\mu, p, a) = \infty$ except for $a \in \operatorname{supp} \mu$. The proof of the statement for D is same to that for $\bar{D}$, assuming $\mu = \mu | D$.

Even if v is not bounded, $\tilde{v}$ may be bounded. However (3) and (6) of Proposition 1 show that $\tilde{R}$ is also not bounded. The following theorem gives a stronger result.

THEOREM 2. *Suppose* $0 < p < \infty$ *and* μ *is a finite positive Borel measure on* $\bar{D}$. *If* $a \in \partial D$, *then the following are valid.*

(1) $\mu(\{a\}) = 0$ *if and only if* $S(\mu, p, a) = 0$.
(2) $\lim_{r \to 1-} S(\mu, p, ra) = 0$ *except for a countable set of* a *in* ∂D.
(3) If $\mu(\{a\}) = 0$ *and* $\{a_n\}$ *is a sequence in* D *with* $\lim a_n = a$, *then* $\lim_{n \to \infty} S(\mu, p, a_n) = 0$.
(4) If $\mu(\{a\}) > 0$, *then for each* n, *the set* $\{z \in D; |z - a| < 1/n\} \cap \{z \in D; S(\mu, p, z) < 1/n\}$ *is a nonempty open set.*
(5) If $b < c$ *and* $E = \{z \in D; z = re^{i\theta}, 0 \leq r < 1$ *and* $b \leq \theta \leq c\}$, *then* R *is not bounded on* E.

PROOF. We may assume $a = 1$. (1) If $\mu(\{1\}) > 0$, then $|f(1)|^p \leq \int |f|^p \, d\mu / \mu(\{1\})$ and so $R(\mu, p, 1) \leq 1/\mu(\{1\})$. (1) of Proposition 1 implies $S(\mu, p, 1) > 0$. Conversely suppose $\mu(\{1\}) = 0$. If $z \in \bar{D}$ and $z \neq 1$, then $\lim_{t \to 1+} |(1-t)/(z-t)| = 0$ and

$$\left| \frac{z-1}{z-t} - 1 \right| = \left| \frac{1-t}{z-t} \right| < 1 \quad (t > 1).$$

For any $t > 1$,

$$S(\mu, p, 1) \leq \int_{\bar{D}} \left| 1 - \frac{z-1}{z-t} \right|^p d\mu(z) = \int_{\bar{D} \setminus \{1\}} \left| \frac{1-t}{z-t} \right|^p d\mu(z).$$

As $t \longrightarrow 1$, by the Lebesgue's dominated convergence theorem, $S(\mu,p,1) = 0$. (2) Suppose $\mu(\{1\}) = 0$. If there exist a sequence $\{r_n\}$ and a positive constant ε such that $0 < r_n < 1$ with $r_n \longrightarrow 1$ and $S(\mu,p,r_n) \geq \varepsilon > 0$, then

$$|f(r_n)|^p \leq \frac{1}{\varepsilon}\int_{\bar D}|f|^p\,d\mu \text{ and so } |f(1)|^p \leq \frac{1}{\varepsilon}\int_{\bar D}|f|^p\,d\mu.$$

This implies $S(\mu,p,1) > 0$ and contradicts (1). Hence if $\mu(\{1\}) = 0$, then $\lim_{r\to 1-} S(\mu,p,r) = 0$. This implies (2) because $\{a \in \partial D; \mu(\{a\}) > 0\}$ is a countable set. (3) is clear by the proof of (2). (4) Suppose $\mu(\{1\}) > 0$ and for each n, put

$$G_n = \left\{z \in \bar D; |z-1| < \frac{1}{n}\right\} \cap \left\{z \in \bar D; S(\mu,p,z) < \frac{1}{n}\right\}.$$

Since $\{z \in \partial D; \mu(\{z\}) > 0\}$ is a countable set, for each n there exists $b_n \in \{z \in \partial D; |z-1| < \frac{1}{n}\}$ with $\mu(\{b_n\}) = 0$. Then $S(\mu,p,b_n) = 0$ by (1) and hence G_n is not empty. G_n is a relatively open set in $\bar D$ by (2) of Proposition 1 and so $G_n \cap D$ is a nonempty open set. (5) follows from (2).

If $R(\mu,2,a) < \infty$, then the point $a \in D$ is a bounded point evaluation for $H^2(\mu)$. Therefore, there exists k_a in $H^2(\mu)$ such that $f(a) = \int f(z)\overline{k_a(z)}\,d\mu(z)$ for any f in $H^2(\mu)$ and hence $R(\mu,2,a) = \int |k_a(z)|^2\,d\mu(z)$. Thus the results in this section give some information about the reproducing kernel k_a.

3. **Estimate of Riesz's function.** In this section we give upper and lower estimates of S. The lower ones will be used later. The following proposition is a generalization of Szegő's theorem in the Introduction. In fact, if $\mu|D$ is a zero measure, then it gives Szegő's Theorem.

PROPOSITION 3. *Suppose $0 < p < \infty$ and μ is a finite positive Borel measure such that $(d\mu|\partial D)/(d\theta/2\pi) = w(e^{i\theta})$, $\mu|D = \sum a_j\delta_{z_j}$ and $\sum(1-|z_j|) < \infty$. Let b be a Blaschke product of $\{z_\ell\}$ and b_j a Blaschke product of $\{z_\ell\}_{\ell\neq j}$. Then for all $a \in D$, $(1-|a|^2)\exp(\log w)^\wedge(a) \leq S(\mu,p,a)$. If $a \in D\setminus\{z_\ell\}$, then*

$$S(\mu,p,a) \leq |b(a)|^{-p}(1-|a|^2)\exp(\log w)^\wedge(a).$$

If $a = z_j$, then

$$S(\mu,p,a) \leq |b_j(a)|^{-p}(1-|a|^2)\exp(\log w)^\wedge(a) + a_j.$$

In particular, $S(\mu,p,a) > 0$ if and only if $\log w \in L^1(d\theta)$.

PROOF. Since $S(\mu,p,a) \geq S(w\,d\theta/2\pi,p,a)$ for all $a \in D$, by Szegő's Theorem $(1-|a|^2)\exp(\log w)^\wedge(a) \leq S(\mu,p,a)$ for all $a \in D$. Let B_n be a finite Blaschke product of $\{z_1,z_2,\ldots,z_n\}$. If $a \in D\setminus\{z_\ell\}$, then

$$\begin{aligned} S(\mu,p,a) &\leq \inf\left\{\int\left|\frac{B_n}{B_n(a)}g\right|^p d\mu|\partial D + \sum_{j=1}^{\infty} a_j\left|\frac{B_n(z_j)}{B_n(a)}g(z_j)\right|^p; g \in P \text{ and } g(a) = 1\right\} \\ &= \frac{1}{|B_n(a)|^p}\inf\left\{\int |B_n g|^p\,d\mu|\partial D + \sum_{j=n+1}^{\infty} a_j|B_n(z_j)|^p|g(z_j)|^p;\right. \\ &\qquad \left. g \in P \text{ and } g(a) = 1\right\}. \end{aligned}$$

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As $n \longrightarrow \infty$,

$$S(\mu,p,a) \leq \frac{1}{|b(a)|^p} \inf\left\{ \int |g|^p \, d\mu|\partial D \,; g \in P \text{ and } g(a) = 1 \right\}.$$

Now by Szegő's Theorem, for each $a \in D$, $S(\mu,p,a) \leq |b(a)|^{-p}\,(1-|a|^2)\exp(\log w)^\wedge(a)$. Let $B_{j,n}$ be a finite Blaschke product of $\{z_1,z_2,\ldots,z_n\} \setminus \{z_j\}$. If $a = z_j$ and $n > j$, then

$$\begin{aligned} S(\mu,p,a) &\leq \inf\left\{ \int \left|\frac{B_{j,n}}{B_{j,n}(a)} g\right|^p d\mu \,; g \in P \text{ and } g(a) = 1 \right\} \\ &= \frac{1}{|B_{j,n}(a)|^p} \inf\left\{ \int |B_{j,n} g|^p \, d\mu|\partial D + a_j |B_{j,n}(a)|^p \right. \\ &\qquad \left. + \sum_{\ell \geq n+1} a_\ell |B_{j,n}(z_\ell)|^p |g(z_\ell)|^p \,; g \in P \text{ and } g(a) = 1 \right\}. \end{aligned}$$

As $n \longrightarrow \infty$, by Szegő's Theorem, for $a = z_j$,

$$S(\mu,p,a) \leq |b_j(a)|^{-p}(1-|a|^2)\exp(\log w)^\wedge(a) + a_j.$$

The following proposition is related to Theorem 2 in this paper and the Theorem in [7]. In fact, if $\tilde{W}$ is bounded on D, then $(1-|a|^2)^{-2} S(W\,dm,p,a)$ is bounded on D. Moreover if W is continuous on $\bar{D}$, then for all $e^{i\theta}$,

$$\lim_{a \longrightarrow e^{i\theta}} (1-|a|^2)^2 R(W\,dm,p,a) = 1 / W(e^{i\theta}),$$

since for a function u continuous on $\bar{D}$ we have $lim_{a \to e^{i\theta}} \tilde{u}(a) = u(e^{i\theta})$.

PROPOSITION 4. *Suppose* $0 < p < \infty$ *and* μ *is a finite positive Borel measure on* $\bar{D}$.

(1) $\tilde{\mu}(a) \geq \big(S(\mu)\big)^\sim(a) \quad (a \in D)$.

(2) If $d\mu = W\,dm$ *and* $a \in D$, *then*

$$(1-|a|^2)^2 \exp(\log W)^\sim(a) \leq S(\mu,p,a) \leq (1-|a|^2)^2 \tilde{W}(a).$$

(3) $S(W\,dm,a) = (1-|a|^2)^2 S(W \circ \phi_a\,dm, 0)$ *for* $a \in D$.

PROOF. (1) For all $z \in D$

$$\int |f|^p \, d\mu \geq |f(z)|^p S(z) \text{ and so } \int |f|^p \, d\mu \geq \int |f|^p S\,dm.$$

Assuming $f(z) = \{(1-|a|^2)/(1-\bar{a}z)^2\}^{2/p}$ for $a \in D$, $\tilde{\mu}(a) \geq \tilde{S}(a)$. (2) If $\log W \in L^1(m)$, then

$$\begin{aligned} &S(W\,dm,p,a) \\ &= \inf\left\{ \int |f|^p W\,dm \,; f \in P \text{ and } f(a) = 1 \right\} \\ &= \inf\left\{ \int |g|^p W \circ \phi_a \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} \, dm \,; g \in H^p(W \circ \phi_a\,dm) \text{ and } g(0) = 1 \right\} \\ &= (1-|a|^2)^2 \inf\left\{ \int |k|^p W \circ \phi_a \, dm \,; k \in H^p(W \circ \phi_a\,dm) \text{ and } k(0) = 1 \right\} \\ &\geq (1-|a|^2)^2 \exp \int (\log W) \circ \phi_a \, dm = (1-|a|^2)^2 \exp(\log W)^\sim(a). \end{aligned}$$

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The inequality above is proved by the fact that $\log|k(0)| \leq \int_0^{2\pi} \log|k(re^{i\theta})|d\theta/2\pi$ for $0 < r < 1$ if $k \in H$, and by two Jensen's inequalities. The other inequality in (2) follows by setting $k \equiv 1$ in the infimum above. (3) is clear by the proof of (2).

In (2) of Proposition 4, we can get estimates of $S(\mu, p, a)$ as in Proposition 3 when $d\mu = W\,dm + \sum_{j=1}^{\infty} a_j\delta_{z_j}, \{z_j\} \subset D$ and $\sum(1 - |z_j|) < \infty$. The following theorem is important in this paper and the following lemma is used to prove it.

LEMMA 1. *Let $\Delta_s(a)$ be the set $\{z \in D; |(a-z)/(1-\bar{a}z)| < s\}$ where $a \in D$ and $s \in (0,1)$. If $t \in (0,1)$ and $1 - s^2 = (1-|a|^2)(1-t^2)/5$, then $\overline{\Delta_t(0)} \subset \Delta_s(a)$.*

PROOF. Without loss of generality $a \neq 0$. the Euclidean center and radius of $\Delta_s(a)$ are

$$C = \frac{1-s^2}{1-s^2|a|^2}a, \quad R = \frac{1-|a|^2}{1-s^2|a|^2}s$$

respectively. Hence to prove $\overline{\Delta_t(0)} \subset \Delta_s(a)$, it is sufficient to show that

$$t + \frac{1-s^2}{1-s^2|a|^2}|a| < \frac{1-|a|^2}{1-s^2|a|^2}s.$$

If $1 - s^2 = (1-|a|^2)(1-t^2)/5$, then

$$1 - s^2 \leq \frac{(1-|a|^2)(1-t^2)}{5-|a|^2}$$

and hence $s^2 \geq \{4 + (1-|a|^2)t^2\}/(5-|a|^2)$. The last inequality is equivalent to

$$1 - s^2 \leq \frac{(1-|a|^2)(s^2-t^2)}{4}.$$

Then

$$1 - s^2 \leq \frac{(1-|a|^2)(s-t)}{2}\frac{s+t}{2} < \frac{(1-|a|^2)(s-t)}{|a|(t|a|+1)}$$

because $s + t < 2$ and $|a|(t|a|+1) < 2$. This is equivalent to the inequality

$$t + \frac{1-s^2}{1-s^2|a|^2}|a| < \frac{1-|a|^2}{1-s^2|a|^2}s.$$

THEOREM 5. *Suppose $0 < p < \infty$ and μ is a finite positive Borel measure on $\bar{D}$. Set $d\mu/dm = W\,dm$, suppose K is an arbitrary compact set in D and let $t = \max\{|z|; z \in K\}$. Then, for $a \in D$*

$$S(\mu, p, a) \geq \frac{(1-|a|^2)^3(1-t^2)}{5}\exp\left[\frac{2^4\cdot 5}{(1-|a|^2)^3(1-t^2)}\int_{K^c}\log(W \wedge 1)\,dm\right].$$

If $1 \leq p < \infty$ and $a \in D$, then

$$S(\mu,p,a) \geq \frac{(1-|a|^2)^{3(2-\frac{1}{p})}(1-t^2)^{2-\frac{1}{p}}}{2^{4(1-\frac{1}{p})}\cdot 5^{2-\frac{1}{p}}}\left(\int_{K^c} W^{-\frac{1}{p-1}}\,dm\right)^{\frac{1}{p}-1}.$$

PROOF. By two Jensen's inequalities, for $a \in D$

$$\begin{aligned}
S(\mu,p,a) &\geq S(W\,dm,p,a)\\
&= \inf\left\{\int |g|^p W\circ\phi_a \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4}\,dm\,;g(0)=1\right\}\\
&= (1-|a|^2)^2\inf\left\{\int |k|^p W\circ\phi_a\,dm\,;k(0)=1\right\}\\
&\geq (1-|a|^2)^2\int_0^1 2r\,dr\exp\left[\int_0^{2\pi}\log W\circ\phi_a\,d\theta/2\pi\right]\\
&\geq (1-|a|^2)^2(1-s^2)\int_s^1 \frac{2r}{1-s^2}\,dr\exp\left[\int_0^{2\pi}\log W\circ\phi_a\,d\theta/2\pi\right]\\
&\geq (1-|a|^2)^2(1-s^2)\exp\left[\frac{1}{1-s^2}\int_s^1 2r\,dr\int_0^{2\pi}\log W\circ\phi_a\,d\theta/2\pi\right]\\
&= (1-|a|^2)^2(1-s^2)\exp\left[\frac{1}{1-s^2}\int_{D\setminus\Delta_s(0)}\log W\circ\phi_a\,dm\right]\\
&= (1-|a|^2)^2(1-s^2)\exp\left[\frac{1}{1-s^2}\int_{D\setminus\Delta_s(a)}\log W\frac{(1-|a|^2)^2}{|1-\bar{a}z|^4}\,dm\right]\\
&\geq (1-|a|^2)^2(1-s^2)\exp\left[\frac{(1-|a|^2)^2}{(1-|a|)^4}\frac{1}{1-s^2}\int_{D\setminus\Delta_s(a)}\log(W\wedge 1)\,dm\right]
\end{aligned}$$

where $s \in (0,1)$ and $\Delta_s(a) = \{z \in D\,;|(a-z)/(1-\bar{a}z)| < s\}$. For each compact set $K \subset D$, if $t = \max\{|z|\,;z \in K\}$ and $1-s^2 = (1-|a|^2)(1-t^2)/5$, then by Lemma 1 $\overline{\Delta_t(0)} \subset \Delta_s(a)$. Hence $K \subset \Delta_s(a)$ and so $K^c \supset D\setminus\Delta_s(a)$. Thus, if $1-s^2 = (1-|a|^2)(1-t^2)/5$, then

$$\frac{(1-|a|^2)^2}{(1-|a|)^4}\frac{1}{1-s^2} = \frac{(1+|a|)^4}{(1-|a|^2)^2(1-s^2)} \leq \frac{2^4\cdot 5}{(1-|a|^2)^3(1-t^2)}$$

and hence for all $a \in D$

$$S(\mu,p,a) \geq \frac{(1-|a|^2)^3(1-t^2)}{5}\exp\left[\frac{2^4\cdot 5}{(1-|a|^2)^3(1-t^2)}\int_{K^c}\log(W\wedge 1)\,dm\right].$$

Now we will prove the second inequality. Instead of Jensen's two inequalities, we will use the Kolmogoroff's inequality (*cf.* [12, Theorem 4.3.1]). For $a \in D$, if $1 \leq p < \infty$ and $1/p+1/q = 1$,

$$
\begin{aligned}
S(\mu,p,a) &\geq (1-|a|^2)^2 \int_0^1 2r\,dr\Big(\int_0^{2\pi}(W\circ\phi_a)^{-\frac{1}{p-1}}\,d\theta/2\pi\Big)^{-\frac{1}{q}} \\
&\geq (1-|a|^2)^2(1-s^2)\int_s^1 \frac{2r}{1-s^2}\,dr\Big(\int_0^{2\pi}(W\circ\phi_a)^{-\frac{1}{p-1}}\,d\theta/2\pi\Big)^{-\frac{1}{q}} \\
&\geq (1-|a|^2)^2(1-s^2)\Big(\frac{1}{1-s^2}\int_s^1 2r\,dr\int_0^{2\pi}(W\circ\phi_a)^{-\frac{1}{p-1}}\,d\theta/2\pi\Big)^{-\frac{1}{q}} \\
&= (1-|a|^2)^2(1-s^2)^{1+\frac{1}{q}}\Big(\int_{D\setminus\Delta_s(0)}(W\circ\phi_a)^{-\frac{1}{p-1}}\,dm\Big)^{-\frac{1}{q}} \\
&= (1-|a|^2)^2(1-s^2)^{1+\frac{1}{q}}\Big(\int_{D\setminus\Delta_s(a)}W^{-\frac{1}{p-1}}\frac{(1-|a|^2)^2}{|1-\bar{a}z|^4}\,dm\Big)^{-\frac{1}{q}} \\
&\geq (1-|a|^2)^2(1-s^2)^{1+\frac{1}{q}}\Big\{\frac{(1-|a|^2)^2}{(1-|a|)^4}\int_{D\setminus\Delta_s(a)}W^{-\frac{1}{p-1}}\,dm\Big\}^{-\frac{1}{q}} \\
&\geq \frac{(1-|a|^2)^{2(1+\frac{1}{q})}(1-s^2)^{1+\frac{1}{q}}}{2^{\frac{4}{q}}}\Big(\int_{D\setminus\Delta_s(a)}W^{-\frac{1}{p-1}}\,dm\Big)^{-\frac{1}{q}}
\end{aligned}
$$

where $s\in(0,1)$. As in the proof of the first inequality, for each compact set $K\subset D$, if $t=\max\{|z|\,;z\in K\}$ and $1-s^2=(1-|a|^2)\,(1-t^2)/5$, then $K^c\supset D\setminus\Delta_s(a)$. Thus, if $1-s^2=(1-|a|^2)\,(1-t^2)/5$, then for all $a\in D$

$$
S(\mu,p,a)\geq\frac{(1-|a|^2)^{3(1+\frac{1}{q})}(1-t^2)^{1+\frac{1}{q}}}{2^{\frac{4}{q}}\cdot 5^{1+\frac{1}{q}}}\Big(\int_{K^c}W^{-\frac{1}{p-1}}\,dm\Big)^{-\frac{1}{q}}.
$$

The second inequality of Theorem 5 implies

$$
S(\mu,1,a)\geq\ (1-|a|^2)^3\times(1-t^2)(1/5)\operatorname{ess\,inf}\{W(x)\,;x\in K^c\}.
$$

Let σ be a finite positive Borel measure on [0,1]. Then, $\mu(re^{i\theta})=\sigma(r)\times W(re^{i\theta})\,d\theta/2\pi$ is more general than $W\,dm=2r\,dr\times W(re^{i\theta})\,d\theta/2\pi$. If $\sigma(r)$ is singular to the Lebesgue measure on [0,1], then μ is singular to m. However we can give an interesting lower estimate. It is different from that of Theorem 5 in case of $\mu=W\,dm$.

THEOREM 6. *Suppose* $0<p<\infty$ *and* $d\mu=\sigma(r)\times W(re^{i\theta})\,d\theta/2\pi$ *where* $\sigma(r)$ *is a finite positive Borel measure on* $[0,1]$. *If* $\mathbf{W}(e^{i\theta})=\sup_r W(re^{i\theta})$ *and* $W_r(e^{i\theta})=W(re^{i\theta})$, *then for* $a\in D$

$$
\begin{aligned}
&(1-|a|^2)\int_{|a|}^1 \exp(\log W_r)^\wedge(a)\,d\sigma(r) \\
&\quad\leq S(\mu,p,a) \\
&\quad\leq \sigma([0,1])\inf\Big\{\sup_r\int_0^{2\pi}|f(re^{i\theta})|^p\,W(re^{i\theta})\,d\theta/2\pi\,;f(a)=1\Big\} \\
&\quad\leq \sigma([0,1])\inf\Big\{\sup_r\int_0^{2\pi}|f(re^{i\theta})|^p\,\mathbf{W}(e^{i\theta})\,d\theta/2\pi\,;f(a)=1\Big\}.
\end{aligned}
$$

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PROOF. For $a \in D$,

$$\begin{aligned} S(\mu,p,a) &= \inf\left\{\int_0^1 d\sigma(r)\int_0^{2\pi} |f(re^{i\theta})|^p W(re^{i\theta})\,d\theta/2\pi\,; f(a)=1\right\} \\ &\geq \int_0^1 d\sigma(r)\inf\left\{\int_0^{2\pi} |f(re^{i\theta})|^p W(re^{i\theta})\,d\theta/2\pi\,; f(a)=1\right\} \\ &= \int_{|a|}^1 d\sigma(r)\inf\left\{\int_0^{2\pi} |f(re^{i\theta})|^p W(re^{i\theta})\,d\theta/2\pi\,; f(a)=1\right\} \\ &= \int_{|a|}^1 (1-|a|^2)\exp(\log W_r)^\wedge(a)\,d\sigma(r). \end{aligned}$$

We used Szegő's Theorem in the last equality. The upper estimates are trivial.

COROLLARY 1. *Let* $d\mu = \sigma(r)\times W(re^{i\theta})\,d\theta/2\pi$ *as in Theorem 6 and* $0<p<\infty$.
(1) If $W(re^{i\theta}) \equiv 1$, *then for* $a \in D$

$$(1-|a|^2)\sigma([|a|,1]) \leq S(\mu,p,a) \leq (1-|a|^2\sigma([0,1]).$$

In particular, $S(\mu,p,0) = \sigma([0,1])$.
(2) If $W(re^{i\theta}) = |h(re^{i\theta})|$ *for some outer function* h *in* $H^1(d\theta)$, *then for* $a \in D$

$$(1-|a|^2)\int_{|a|}^1 W(ra)\,d\sigma(r) \leq S(\mu,p,a) \leq (1-|a|^2)W(a)\sigma([0,1]).$$

(3) If $1<p<\infty$ *and* $\mathbf{W}(e^{i\theta}) = \sup W(re^{i\theta})$ *satisfies the* A_p *condition, then there exists a positive constant* γ *such that for* $a \in D$

$$S(\mu,p,a) \leq \gamma(1-|a|^2)\exp(\log \mathbf{W})^\wedge(a)\sigma([0,1]).$$

PROOF. (1) is a special case of (2). (2) Since h is an outer function in H^1, for $a \in D$

$$\exp(\log W_r)^\wedge(a) = \exp(\log|h_r|)^\wedge(a) = |h(ra)| = W(ra)$$

and

$$\begin{aligned} &\inf_f\left\{\sup_r \int_0^{2\pi} |f(re^{i\theta})|^p W(re^{i\theta})\,d\theta/2\pi\right\} \\ &\quad = \inf_f \int_0^{2\pi} |f(e^{i\theta})|^p\,|h(e^{i\theta})|\,d\theta/2\pi = (1-|a|^2)|h(a)| = (1-|a|^2)W(a). \end{aligned}$$

Now Theorem 6 implies (2). (3) By a theorem of M. Rosenblum (*cf.* [10] and [9, Theorem 2.2]), there exists a positive constant γ such that for any $f \in P$

$$\sup_r \int_0^{2\pi} |f(re^{i\theta})|^p \mathbf{W}(e^{i\theta})\,d\theta/2\pi \leq \gamma\int_0^{2\pi} |f(e^{i\theta})|^p \mathbf{W}(e^{i\theta})\,d\theta/2\pi$$

because $\mathbf{W} \in A_p$. By Theorem 6 and Szegő's Theorem, for $a \in D$

$$\inf_f \left\{ \sup_r \int_0^{2\pi} |f(re^{i\theta})|^p \mathbf{W}(e^{i\theta})\, d\theta/2\pi \right\} \leq \gamma \inf_f \int_0^{2\pi} |f(e^{i\theta})|^p \mathbf{W}(e^{i\theta})\, d\theta/2\pi$$
$$= \gamma(1-|a|^2)\exp(\log \mathbf{W})^\wedge(a)$$

This implies (3).

In (2) of Corollary 1, the referee pointed out that the identity $S(\mu,p,a) = W(a)S(\nu,p,a)$ is valid where $d\nu = \sigma(r) \times d\theta/2\pi$. Applying Theorem 6 for ν, we have the estimates $(1-|a|^2)\sigma([|a|,1])W(a) \leq S(\mu,p,a) \leq (1-|a|^2)\sigma([0,1])W(a)$.

4. **The Carleson inequality and Riesz's function.** Let ν and μ be finite positive Borel measures on $\bar{D}$ and $1 \leq p < \infty$. We say that ν and μ satisfy the (ν,μ,p)-Carleson inequality, if there exists a constant $\gamma > 0$ such that

$$\int_{\bar{D}} |f|^p\, d\nu \leq \gamma \int_{\bar{D}} |f|^p\, d\mu$$

for all $f \in P$ (see [8]). ν and μ satisfy the (ν,μ,p)-Carleson inequality if and only if $H^p(\mu) \subset H^p(\nu)$ and the inclusion mapping $i_p: H^p(\mu) \longrightarrow H^p(\nu)$ is bounded. We say that for $p > 1$, ν and μ satisfy the (ν,μ,p)-vanishing Carleson inequality if $H^p(\mu) \subset H^p(\nu)$ and $i_p: H^p(\mu) \longrightarrow H^p(\nu)$ is compact. We say that for $p = 1$, ν and μ satisfy the (ν,μ,p)-vanishing Carleson inequality if i_p is star-compact. We could not prove Theorem 7 for $p = 1$ because we do not know anything about the predual of $H^1(\mu)$. Using Riesz's functions, we will show vanishing Carleson inequalities. As a result, we show that $R(\mu,p) \notin L^1(\mu)$ if supp μ is not a finite set. Moreover, from a given measure μ, we will show how to construct a measure ν such that the (ν,μ,p)-vanishing Carleson inequality is valid.

THEOREM 7. *Suppose $1 < p < \infty$, and ν and μ are finite positive Borel measures on $\bar{D}$.*

(1) If $\int R(\mu,p)\, d\nu < \infty$, then ν and μ satisfy the (ν,μ,p)-vanishing Carleson inequality and

$$R(\mu,p,a) \leq \left(\int R(\mu,p)\, d\nu \right) R(\nu,p,a) \quad (a \in \bar{D}).$$

(2) If V is a Borel function such that $0 \leq V \leq S$ on $\bar{D}$, then $V|g|^p$ is bounded on $\bar{D}$ for each g in $H^p(\mu)$, and $V\,dm$ and μ satisfy the $(V\,dm,\mu,p)$-vanishing Carleson inequality.

PROOF. (1) By definition of $R(\mu,p,a)$, for $a \in \bar{D}$,

$$|f(a)|^p \leq R(\mu,p,a) \int |f|^p\, d\mu \quad (f \in P).$$

Hence if $\gamma = \int R(\mu,p)\, d\nu < \infty$, then $\int |f|^p\, d\nu \leq \gamma \int |f|^p\, d\mu$ ($f \in P$) and so $i_p: H^p(\mu) \longrightarrow H^p(\nu)$ is bounded. We will show that i_p is compact. If $f_n \longrightarrow f$ weakly in $H^p(\mu)$, then there exists a finite positive constant γ' such that

$$\int |f_n - f|^p\, d\mu \leq \gamma' \text{ for all } n.$$

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By the hypothesis, $R(\mu,p,a) < \infty$ ν-a.e. on $\bar{D}$ and so $f_n \longrightarrow f$ ν-a.e. on $\bar{D}$ because $f_n \longrightarrow f$ weakly. Moreover by definition of $R(\mu,p,a)$, $|f_n(a) - f(a)|^p \leq \gamma' R(\mu,p,a)$ and by the hypothesis, $R(\mu,p,a) \in L^1(\nu)$. Thus

$$\int |f_n - f|^p \, d\nu \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

by Lebesgue's dominated convergence theorem. This implies i_p is compact. Since $\int |f|^p \, d\nu \leq \gamma \int |f|^p \, d\mu$ and $\gamma = \int R(\mu,p) \, d\nu$, assuming $f(a) = 1$, we get $S(\nu,p,a) \leq \gamma S(\mu,p,a)$. Now by (1) of Proposition 1, we get the inequality of (1). (2) If $0 \leq V \leq S$, then $VR \leq 1$ and hence $V(a)|f(a)|^p$ is bounded on $\bar{D}$ by $\int |f|^p \, d\mu$, for each $f \in H^p(\mu)$. Moreover if $\nu = V\,dm$ and $0 \leq V \leq S$, then $\int R(\mu,p)\, d\nu \leq \int dm = 1$ and hence by (1) ν and μ satisfy the (ν,μ,p)-vanishing Carleson inequality.

COROLLARY 2. *If $0 < p < \infty$ and $\operatorname{supp}\mu$ is not a finite set, then $R(\mu,p) \notin L^1(\mu)$.*

PROOF. Suppose $1 < p < \infty$. If $R(\mu,p) \in L^1(\mu)$, then the inclusion map $i_p \colon H^p(\mu) \longrightarrow H^p(\mu)$ is compact. It is easy to see that i_p is an identity operator. Hence the unit ball of $H^p(\mu)$ is compact with respect to the norm. Therefore $H^p(\mu)$ is finitely dimensional. This contradicts that $\operatorname{supp}\mu$ is not a finite set. This implies that $R(\mu,p) \notin L^1(\mu)$. For $0 < p \leq 1$, the proof is due to the referee. Choose n sufficiently large that $np > 1$. If $g(a) = 1$ then $g^n(a) = 1$ as well, and g^n is a polynomial if g is a polynomial. Thus,

$$\begin{aligned} S(\mu,p,a) &= \inf\left\{ \int_{\bar{D}} |f|^p \, d\mu \,; f \in P, f(a) = 1 \right\} \\ &\leq \inf\left\{ \int_{\bar{D}} |g^n|^p \, d\mu \,; g \in P, g(a) = 1 \right\} = S(\mu,np,a). \end{aligned}$$

This implies that $R(\mu,p) \notin L^1(\mu)$ for $0 < p \leq 1$.

By Proposition 4 and Theorem 5 we obtain the following result.

COROLLARY 3. *Suppose $1 < p < \infty$ and $d\mu/dm = W$.*

(1) If $\log W \in L^1(m)$ and $d\nu = (1 - |z|^2)^2 \exp(\log W)^{\sim}\, dm$, then ν and μ satisfy the (ν,μ,p)-vanishing Carleson inequality.

(2) If $\chi_{K^c} \log(W \wedge 1) \in L^1(m)$ for some compact set K in D, then there exists a nonnegative constant b such that $d\nu = \exp\{-b(1-|z|^2)^{-3}\}\, dm$ and μ satisfy the (ν,μ,p)-vanishing Carleson inequality.

(3) Suppose $\chi_{K^c} W^{-\frac{1}{p-1}} \in L^1(m)$ for some compact set K in D. If $d\nu = c(1-|z|^2)^{3(2-\frac{1}{p})}\, dm$, then ν and μ satisfy the (ν,μ,p)-vanishing Carleson inequality.

Suppose $1 < p < \infty$ and $d\mu/dm = W$. If $\chi_{K^c} \log W \in L^1(m)$ for some compact set K in D, then there exists a positive constant a and a nonnegative constant b such that

$$a(1 - |z|^2)^3 \exp\{-b(1 - |z|^2)^{-3}\} |f(z)|^p$$

is bounded on D for each $f \in H^p(\mu)$. Here a and b do not depend on f, but only on W and the choice of K. This is a corollary of (2) in Theorem 7.

5. $H^p(\mu)$ **and** $L^p_a(\mu)$. The following is a result of Theorem 5. If $d\mu/dm = W$ and $\log W$ is integrable on the complement K^c of a compact set in D, then $H^p(\mu) \subseteq L^p_a(\mu)$. In this section, we show that if $\log W$ is locally integrable on K^c, then the same result is true. We give a necessary and sufficient condition for $H^p(\mu) \subset L^p_a(\mu)$ using Riesz's function, providing $(\operatorname{supp}\mu) \cap D$ is a uniqueness set for H. A subset E of D is a uniqueness set if E satisfies the following: If f in H is zero on E, then $f \equiv 0$ on D. Theorem 8 is a joint work with K. Takahashi.

LEMMA 2. *Suppose* $0 < p < \infty$ *and* μ *is a finite positive Borel measure on D. Then the following (1)–(3) are equivalent.*

(1) $\sup_{a\in K} R(\mu,p,a) < \infty$ *for all compact sets K in D.*

(2) $\int_K R(\mu,p)\,dm < \infty$ *for all compact sets K in D.*

(3) $\int_K \log R(\mu,p)\,dm < \infty$ *for all compact sets K in D.*

PROOF. Both (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are trivial. We will show (3) $\Rightarrow$ (1). We may assume that $\mu(D) = 1$. For any $f \in P$,

$$\log|f(0)|^p \leq \frac{1}{m\big(D_r(0)\big)} \int_{D_r(0)} \log|f|^p\,dm.$$

If $a \in D_r(0)$, then for all $f \in P$

$$\log|f(a)|^p \leq \frac{1}{m\big(D_r(0)\big)} \int_{D_r(a)} \log|f|^p \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4}\,dm.$$

Assuming $\int |f|^p\,d\mu \leq 1$, we get

$$\log R(\mu,p,a) \leq \frac{1}{m\big(D_r(0)\big)} \frac{(1+|a|)^2}{(1-|a|)^2} \int_{D_r(a)} \log R(\mu,p)\,dm.$$

Since $D_r(a) \subset D_{2r}(0)$ and $R(\mu,p,a) \geq 1$, there exists a finite positive constant γ_r such that for each $a \in D_r(0)$ we have

$$\log R(\mu,p,a) \leq \gamma_r \int_{D_{2r}(0)} \log R(\mu,p)\,dm.$$

This implies (1).

LEMMA 3. *Let X be a Banach space which consists of analytic functions on D and contains* 1. *Suppose there exists a dense subspace Y of X such that if f in Y, then* $\big(f - f(a)\big)/(z-a)$ *belongs to Y for some* $a \in D$. *If* $(z-a)X$ *is not dense in X, then the functional* $f \longmapsto f(a)$ *is bounded on Y.*

PROOF. By the hypothesis, if $f \in Y$, then $f = f(a) + (z-a)g$ for some $g \in Y$. Since $(z-a)X$ is not dense in X, there exists a nonzero $\phi \in X^*$ such that $\langle (z-a)h, \phi\rangle = 0$. Then, for $f \in Y$ we have $\langle f,\phi\rangle = f(a)\langle 1,\phi\rangle$. Since ϕ is not identically zero we have $\langle 1,\phi\rangle \neq 0$. Thus $|f(a)| \leq \gamma\|f\|$ for all $f \in Y$ where $\gamma = |\langle 1,\phi\rangle|^{-1}\|\phi\|_*$.

THEOREM 8. *Suppose* $1 \leq p < \infty$ *and* μ *is a finite positive Borel measure on* D *such that* $(\operatorname{supp}\mu) \cap D$ *is a uniqueness set for* H.

(1) $L^p_a(\mu)$ *is closed if and only if for all compact sets* K *in* D

$$\int_K \log r(\mu,p)\, dm < \infty \text{ or } \int_K \log s(\mu,p)\, dm > -\infty.$$

(2) $H^p(\mu) \subset L^p_a(\mu)$ *if and only if for all compact sets* K *in* D

$$\int_K \log R(\mu,p)\, dm < \infty \text{ or } \int_K \log S(\mu,p)\, dm > -\infty.$$

PROOF. (1) First suppose that $L^p_a(\mu)$ is closed. If $f \in L^p_a(\mu)$, then $\big(f - f(0)\big)/z$ belongs to H. Since $\big(f - f(0)\big)/z$ is bounded on $|z| \leq t < 1$ and $1/z$ is bounded on $|z| \geq t$, $\big(f - f(0)\big)/z$ belongs to $L^p_a(\mu)$. This implies that $\{f \in L^p_a(\mu)\,; f(0) = 0\} = zL^p_a(\mu)$ and hence $L^p_a(\mu) = \mathbf{C} \oplus zL^p_a(\mu)$. If $Af = zf$ for $f \in L^p_a(\mu)$, then A is a bounded operator on $L^p_a(\mu)$ and the range of A is algebraically complemented in $L^p_a(\mu)$ by what was just proved. By [4, Part III, Corollary 2.3], the range of A is closed and hence $zL^p_a(\mu)$ is not dense in $L^p_a(\mu)$. Applying Lemma 3 with $X = Y = L^p_a(\mu)$, it follows that $r(\mu,p,a) < \infty$ for $a = 0$. The same argument is true for all $a \in D \setminus \{0\}$ and hence $r(\mu,p,a) < \infty$ for all $a \in D$. By the boundedness of holomorphic functions on compact sets and the uniform boundedness principle, $\sup_{a \in K} r(\mu,p,a) < \infty$ for all compact sets K in D. As Lemma 2 also holds for $r(\mu,p,a)$,

$$\int_K \log r(\mu,p)\, dm < \infty \text{ or } \int_K \log s(\mu,p)\, dm > -\infty.$$

Conversely, suppose $\int_K \log r(\mu,p)\, dm < \infty$ for every compact sets K. Then by the above lemma, $\sup_K r(\mu,p) < \infty$ for every compact sets K. If f is in the $L^p(\mu)$-norm closure of $L^p_a(\mu)$, then there exists a sequence $\{f_n\}$ in $L^p_a(\mu)$ such that $\int |f - f_n|^p\, d\mu \longrightarrow 0$. Then for any fixed $r < \infty$ if we let $k_r = \sup_{a \in D_r(0)} r(\mu,p,a)$, then we will have $\sup\{|g(z)|\,; z \in D_r(0)\} \leq k_r \|g\|_{L^p\mu}$. Applying this with $g = f_n - f_m$ we see that the f_n are uniformly Cauchy on $D_r(0)$ and hence converge uniformly to an analytic function on $D_r(0)$. Since r was arbitrary, the f_n converge uniformly on compacta to an analytic function g on D, and we must have $g = f$, μ-a.e. on D.

(2) The 'if' part is same as (1) and hence we will show the 'only if' part. If we put $M = \{f \in L^p(\mu)\,; zf \in H^p(\mu)\}$, then M is a closed subspace of $L^p(\mu)$ such that

$$M \supseteq H^p(\mu) \supseteq zM \supseteq H^p(\mu)_0$$

where $H^p(\mu)_0 = \{f \in H^p(\mu)\,; f(0) = 0\}$. $H^p(\mu)_0$ is well defined because $H^p(\mu) \subset L^p_a(\mu)$. Suppose $H^p(\mu) \neq zM$. Then $H^p(\mu) = \mathbf{C} + H^p(\mu)_0 = \mathbf{C} + zM$ and $\mathbf{C} \cap zM = \{0\}$. As in the proof of (1), by [4, Part III, Corollary 2.3], zM is closed in $H^p(\mu)$ and hence $zH^p(\mu)$ is not dense in $H^p(\mu)$. Applying Lemma 3 with $X = H^p(\mu)$ and $Y = P$, it follows that $R(\mu,p,a) < \infty$ for $a = 0$. Suppose $H^p(\mu) = zM$. Then $z^{-1} \in L^p(\mu)$ and hence $\mu(\{0\}) = 0$. If $Af = zf$ for $f \in M$, then A is a one-one bounded operator from M onto $H^p(\mu)$. Therefore A is invertible and hence $A(zM) = zH^p(\mu)$ is closed. Since $H^p(\mu) \subset L^p_a(\mu), zH^p(\mu) \neq H^p(\mu)$ and hence by Lemma 3, $R(\mu,p,0) < \infty$ follows. The same argument implies that $R(\mu,p,a) < \infty$ for all $a \in D$. Now, as in the proof of (1), Lemma 2 implies the 'only if' part of (2).

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COROLLARY 4. *Suppose* $1 \leq p < \infty$ *and* $d\mu/dm = W$. *If* $\log W$ *is locally integrable on* K_0^c *for some compact set* K_0 *in* D, *then* $L_a^p(\mu)$ *is closed and* $H^p(\mu) \subseteq L_a^p(\mu)$.

PROOF. By (1) of Theorem 8, it is sufficient to prove that for any compact set K in D, $\inf_K \log s(\mu,p) > -\infty$. If $\log W$ is integrable on K_0^c, then by the proof of Theorem 5 $\inf_K \log s(\mu,p) > -\infty$. For a more general W in this corollary, we have to proceed as follows. Suppose $a \in D$ and $0 < \varepsilon < \delta < 1$. As in the proof of Theorem 5,

$$
\begin{aligned}
&s(\mu,p,a)\\
&\geq (1-|a|^2)^2 \int_\varepsilon^\delta \exp\Big(\int_0^{2\pi} \log W \circ \phi_a \, d\theta/2\pi\Big) 2r\, dr\\
&\geq (1-|a|^2)^2(\delta^2-\varepsilon^2)\exp\Big(\frac{1}{\delta^2-\varepsilon^2}\int_{\Delta_\delta(0)\setminus\Delta_\varepsilon(0)} \log W \circ \phi_a \, dm\Big)\\
&\geq (1-|a|^2)^2(\delta^2-\varepsilon^2)\exp\Big(\frac{2^4}{(1-|a|^2)^2(\delta^2-\varepsilon^2)}\int_{\Delta_\delta(a)\setminus\Delta_\varepsilon(a)} \log(W\wedge 1)\, dm\Big).
\end{aligned}
$$

Suppose K is an arbitrary compact set in D. Put $t = \max\{|z|\,; z \in K_0\}$ and $k = \max\{|z|\,; z \in K\}$. The Euclidean center and radius of $\Delta_\gamma(k)$ $(0 < \gamma < 1)$ are

$$C(\gamma) = \frac{1-\gamma^2}{1-\gamma^2 k^2}k, R(\gamma) = \frac{1-k^2}{1-\gamma^2k^2}\gamma$$

respectively. Put $\ell = R(\delta) + C(\delta)$ and $s = R(\varepsilon) - C(\varepsilon)$. There exist δ and ε such that $0 < \varepsilon < \delta < 1$ and

$$\overline{\Delta_\ell(0) \setminus \Delta_s(0)} \subset D \setminus \Delta_t(0).$$

Then for all $a \in K$

$$\Delta_\delta(a) \setminus \Delta_\varepsilon(a) \subset \Delta_\ell(0) \setminus \Delta_s(0).$$

Hence for all $a \in K$

$$\overline{\Delta_\delta(a) \setminus \Delta_\varepsilon(a)} \subset K_0^c$$

and so for all $a \in K$

$$s(\mu,p,a) \geq (1-|a|^2)^2(\delta^2-\varepsilon^2)\exp\Big(\frac{2^4}{(1-|a|^2)^2(\delta^2-\varepsilon^2)}\int_{\overline{\Delta_\delta(a)\setminus\Delta_\varepsilon(a)}} \log(W\wedge 1)\, dm\Big),$$

since $\overline{\Delta_\delta(a) \setminus \Delta_\varepsilon(a)}$ is a compact subset of $D\setminus K_0$ and $\log W$ is locally integrable on $D\setminus K_0$. This shows the corollary.

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