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KMS states on uniform Roe algebras

Bruno M. Braga and Ruy Exel

Abstract. We initiate the treatment of KMS states on uniform Roe algebras $C^*_u(X)$ for a class of naturally occurring flows on these algebras. We show that KMS states on $C^*_u(X)$ always factor through the diagonal operators $\ell_\infty(X)$. We show the study of those states splits into understanding their strongly continuous KMS states and the KMS states which vanish on the ideal of compact operators. We show strongly continuous states are always unique when they exist and we give explicit formulas for them. We link the study of KMS states which vanish on the compacts to the Higson corona of X and provide lower bounds for the cardinality of the set of extreme KMS states. Lastly, we apply our theory to the n-branching tree: in this example, $\beta = \log(n)$ is a phase transition admitting $2^{2^{\aleph_0}}$ KMS states, no KMS states for smaller inverse temperatures, and a unique one for larger ones (the Gibbs state). Moreover, we show that the behavior of the KMS states around $\beta = \log(n)$ is chaotic.

1 Introduction

In noncommutative geometry, given a metric space X, one defines certain C^*-algebras of operators on a Hilbert space with the goal of coding certain aspects of the geometry of X in C^*-algebraic terms. When interested in the large scale geometric properties of X, that is, in its coarse geometry, a well-known C^*-algebra is to be considered: the *uniform Roe algebra of X*. This C^*-algebra was introduced by Roe to study the index theory of elliptic operators on noncompact manifolds [Roe88, Roe93]. The interest in these algebras was then boosted due to their connection with the coarse Baum-Connes conjecture [Yu00]. More recently, these C^*-algebras entered the realm of mathematical physics and researchers interested in topological insulators have been using them as observable algebras in order to describe topological phases. We refer the reader to [Kub17, EM19, Jon21, LT21, Bou22] for the rapidly growing literature about uniform Roe algebras in mathematical physics.

The goal of this article is to look at uniform Roe algebras under yet another point of view motivated by mathematical physics: We study KMS states on uniform Roe algebras. Named after mathematical physicists Kubo, Martin, and Schwinger, KMS states are states defined on any C^*-algebra A admitting a flow, that is, a strongly continuous one-parameter group $\{\sigma_t\}_{t\in\mathbb{R}}$ of automorphisms, thought of as the time development of observables of an idealized infinite system of particles. Among the

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many equivalent definitions of such states, we adopt the one that requires our state φ to satisfy the relation

$$\varphi(ba) = \varphi\big(a\sigma_{i\beta}(b)\big),$$

for every a in A and every analytic element b in A. This condition has been noted by Kubo, Martin, and Schwinger in the late 1950's, as being satisfied by the *grand canonical ensembles* in the Gibbs equilibrium formalism for finite systems. Observing that this condition in fact characterizes the Gibbs states, Haag, Hugenholtz, and Winnink later proposed this as a criterion for equilibrium.

The parameter β appearing above is the same parameter weighing the *average energy* and the *entropy* in the expression for the *free energy* in the variational deduction of Gibbs states, and it is often thought of as the reciprocal of the *temperature*. While our abstract treatment of KMS states will not really involve the physical meaning of β, it is crucial to realize that the existence and uniqueness of KMS states depend in a very fundamental way on β, so much so that we shall refer to states satisfying the above condition as (σ, β)-KMS states, following the the modern literature standards.

Crucially, among the most interesting features of KMS states is the abrupt change in behavior as β crosses certain thresholds. In classical infinite particle systems, a sudden change with temperature is often referred to as a *phase transition*, which is what one observes when a gas liquefies when cooled down or when a magnet spontaneously loses its magnetization when heated beyond a critical temperature. Thus, if for example there is a unique (σ, β)-KMS state for every β greater than some fixed β_0, while there are many (σ, β_0)-KMS states, one says that a phase transition has happened at the critical value β_0.

It is well known that uniform Roe C*-algebras may be described as the reduced groupoid C*-algebra of a principal, ample, étale groupoid [STY02]. Moreover, in his 1980 thesis [Ren80, Proposition II.5.4], Renault described a method to study KMS states on groupoid C*-algebras in terms of *quasi-invariant measures* satisfying a certain *Radon-Nikodym condition*. Even though we do not directly employ Renault's result here, much of what we do here may be interpreted as studying such quasi-invariant measures.

Before giving a detailed description of this article and our main findings, we start with some basic definitions.

1.1 Coarse geometry and uniform Roe algebras

A map $h\colon (X, d) \to (Y, \partial)$ between metric spaces is called *coarse* if for all $r > 0$, there is $s > 0$ such that

$$d(x, y) < r \text{ implies } \partial(h(x), h(y)) < s.$$

With coarse maps being the morphisms of interest, local properties of the metric spaces are irrelevant in coarse geometry and one usually restricts themselves to discrete spaces. In fact, for our goals, we will assume the metric spaces to be *uniformly*

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locally finite (abbreviated as *u.l.f.*), that is, they have the property that for each $r > 0$ their balls of radius r are uniformly bounded in size by a finite quantity.[1]

Given a set X, $\ell_2(X)$ denotes the Hilbert space of square-summable maps $X \to \mathbb{C}$ and $(\delta_x)_{x\in X}$ denotes its canonical orthonormal basis. The space of bounded operators on $\ell_2(X)$ is denoted by $\mathcal{B}(\ell_2(X))$ and $\mathcal{K}(\ell_2(X))$ denotes its ideal of compact operators.

Definition 1.1 Let (X,d) be a u.l.f. metric space. The propagation of an operator $a \in \mathcal{B}(\ell_2(X))$ is defined by

$$\mathrm{prop}(a) = \sup\left\{d(x,y) \mid a_{x,y} \coloneqq \langle a\delta_y, \delta_x\rangle \neq 0\right\}.$$

The *-algebra of all operators with finite propagation, denoted by $\mathrm{C}^*_u[X]$, is the *algebraic uniform Roe algebra of* (X,d). The norm closure of $\mathrm{C}^*_u[X]$, denoted by $\mathrm{C}^*_u(X)$, is the *uniform Roe algebra of* (X,d).

Uniform Roe algebras code coarse geometric properties of X in terms of C^*-algebraic properties. For instance, it is known that X has Yu's property A if and only if $\mathrm{C}^*_u(X)$ is nuclear [BO08, Theorem 5.5.7]. Also, it has been recently shown that this construction is *rigid* in the sense that if the C^*-algebras $\mathrm{C}^*_u(X)$ and $\mathrm{C}^*_u(Y)$ are isomorphic, then X and Y must be coarsely equivalent [BBF$^+$22b, Theorem 1.2].

1.2 Flows and KMS states on uniform Roe algebras

Given a C^*-algebra A, an action $\sigma\colon \mathbb{R} \curvearrowright A$ is a *flow* if it is strongly continuous[2] and $\sigma_t\colon A \to A$ is an isomorphism for all $t \in \mathbb{R}$.

Quantum mechanical systems in thermal equilibrium can be described by their so called *KMS states*. The number β in the definition below should be interpreted as the inverse of the temperature of the system.

Definition 1.2 Let A be a C^*-algebra and σ be a flow on A. For $\beta \in \mathbb{R}$, we say that a state φ on A is a (σ,β)*-KMS state* if

$$\varphi(a\sigma_{i\beta}(b)) = \varphi(ba)$$

for all $a \in A$ and all analytic $b \in A$.[3]

In order to study KMS states on uniform Roe algebras, one must first identify natural flows in them. We now introduce such flows. Given a set X and a map $h\colon X \to \mathbb{R}$, we denote by $\bar{h}$ the X-by-X diagonal matrix of reals such that its (x,x)-entry is $h(x)$ for all $x \in X$ and all other entries are zero. Notice that $\bar{h}$ canonically induces a bounded operator on $\ell_2(X)$ if and only if h is bounded.

[1] A metric space with this property is also often called a *metric space with bounded geometry* in the literature. Other authors call a space with bounded geometry one that is coarsely equivalent to a u.l.f. space.

[2] The action σ is *strongly continuous* if $t \in \mathbb{R} \mapsto \sigma_t(a) \in A$ is continuous for all $a \in A$.

[3] An element $b \in A$ is *analytic for* σ if the map $t \in \mathbb{R} \mapsto \sigma_t(b) \in A$ extends to an entire analytic map $\mathbb{C} \to A$.

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Definition 1.3 Let X be a u.l.f. metric space and $h\colon X \to \mathbb{R}$ be a coarse map. We denote by σ_h the flow on $\mathrm{C}^*_u(X)$ given by,

$$\sigma_{h,t}(a) = e^{it\bar{h}} a e^{-it\bar{h}}$$

for all $t \in \mathbb{R}$ and all $a \in \mathrm{C}^*_u(X)$.

Notice that the hypothesis on $h\colon X \to \mathbb{R}$ being coarse is important so that σ_h is indeed a flow. Indeed, the action σ_h is strongly continuous if and only if h is coarse (see Proposition 2.1). All flows on uniform Roe algebras considered in this article will be of the form above for some appropriate $h\colon X \to \mathbb{R}$. In order to have any hope of understanding the KMS states for those flows, we must first understand the analytic elements of $\mathrm{C}^*_u(X)$ or, more precisely, a *-subalgebra of analytic operators of $\mathrm{C}^*_u(X)$ which is dense in it. We have:

Proposition 1.4 *Let X be a u.l.f. metric space and $h\colon X \to \mathbb{R}$ be a map.*

1. *If h is bounded, then every element of $\mathrm{C}^*_u(X)$ is analytic for σ_h*
2. *If h is coarse, then every element of $\mathrm{C}^*_u[X]$ is analytic for σ_h.*

The reader may wonder how strong is the restriction of only working with flows of the form above. As we show in Proposition 2.2, if $\sigma\colon \mathbb{R} \curvearrowright \mathrm{C}^*_u(X)$ is an arbitrary flow which leaves the Cartan masa $\ell_\infty(X)$ invariant, that is, $\sigma_t(\ell_\infty(X)) \subseteq \ell_\infty(X)$ for all $t \in \mathbb{R}$, then there is a coarse map $h\colon X \to \mathbb{R}$ such that $\sigma = \sigma_h$.[4] This corroborates to our claim that such flows form a very natural and general class of flows on those algebras.

1.3 Main results

It is often common in the study of KMS states on a given C^*-algebra A that there is some "natural" C^*-subalgebra $B \subseteq A$ and a conditional expectation $E\colon A \to B$ such that the KMS states $\varphi\colon A \to \mathbb{C}$ factor through E. We show that this is also the case in our setting with the "natural" C^*-subalgebra through which the KMS states factor being the C^*-algebra of all bounded maps $X \to \mathbb{C}$, denoted by $\ell_\infty(X)$. Precisely, throughout these notes, we identify $\ell_\infty(X)$ with the C^*-algebra of diagonal operators on $\ell_2(X)$ in the usual way: Given $a = (a_x)_{x\in X} \in \ell_\infty(X)$ and $\xi = (\xi_x)_{x\in X} \in \ell_2(X)$, we let

$$a\xi = (a_x \xi_x)_{x\in X} \in \ell_2(X).$$

Given $A \subseteq X$, $\chi_A \in \ell_\infty(X)$ denotes the canonical orthogonal projection $\ell_2(X) \to \ell_2(A)$.

We show the following:

Theorem 1.5 *Let X be a u.l.f. metric space, $h\colon X \to \mathbb{R}$ be a coarse map, and $\beta \in \mathbb{R}$. If φ is a (σ_h, β)-KMS state on $\mathrm{C}^*_u(X)$, then $\varphi = \varphi \circ E$, where $E\colon \mathrm{C}^*_u(X) \to \ell_\infty(X)$ is the canonical conditional expectation (see Figure 1).*

[4] We thank Stuart White for raising the possibility that this could be true.

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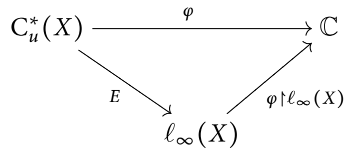

Figure 1: KMS states on $C^*_u(X)$ factor through $\ell_\infty(X)$, see Section 2.2 for the precise definition of E.

Theorem 1.5 is an extremely powerful tool in our study of KMS states on uniform Roe algebras and most of our results deeply depend on it. For instance, it allows us to understand the case of a flow given by a bounded map $h\colon X\to\mathbb{R}$ in terms of amenability: for h bounded, $C^*_u(X)$ has a (σ_h,β)-KMS states if and only if X is amenable (see Theorem 2.7). Moreover, Theorem 1.5 allows us to reduce the study of KMS states on uniform Roe algebras to two cases (see Proposition 4.1):

(I) strongly continuous KMS states, and
(II) KMS states which vanish on the the ideal of compact operators.

The strongly continuous case is the simplest one and the next result summarizes what happens:

Theorem 1.6 *Let X be a u.l.f. metric space, $h\colon X\to\mathbb{R}$ be a coarse map, and $\beta\geq 0$. There are strongly continuous (σ_h,β)-KMS states on $C^*_u(X)$ if and only if*

$$Z(\beta):=tr(e^{-\beta\bar{h}})=\sum_{x\in X}e^{-\beta h(x)}<\infty.$$

Moreover, a function $\varphi\colon C^*_u(X)\to\mathbb{C}$ is a strongly continuous (σ_h,β)-KMS state on $C^*_u(X)$ if and only if

$$\varphi(a)=\frac{tr(e^{-\beta\bar{h}}a)}{tr(e^{-\beta\bar{h}})}=\frac{1}{Z(\beta)}\sum_{x\in X}a_{x,x}e^{-\beta h(x)} \tag{1.1}$$

for all $a=[a_{x,y}]\in C^*_u(X)$. In particular, whenever they exist, strongly continuous (σ_h,β)-KMS states are unique.

In other words, the strongly continuous KMS states are exactly the Gibbs states provided $e^{-\beta\bar{h}}$ is trace class (see [BR97, Section 6.2.2]). This is of course no big surprise since the strongly continuous states on any operator algebra containing the compacts correspond precisely with the strongly continuous states defined on the whole $\mathcal{B}(\ell_2(X))$.

With the strongly continuous case being well understood, we then proceed to study the much more interesting case of KMS states which vanish on the compact operators. This property allows us to factor those states through the *uniform Roe corona of* X.

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Definition 1.7 [BFV21, Definition 1.2] Let X be a u.l.f. metric space. The *uniform Roe corona of* X is the C*-algebra given by

$$\mathrm{Q}^*_u(X) = \mathrm{C}^*_u(X)/\mathcal{K}(\ell_2(X)).$$

We denote by $\pi = \pi_X \colon \mathrm{C}^*_u(X) \to \mathrm{Q}^*_u(X)$ the canonical quotient map.

A state φ on $\mathrm{C}^*_u(X)$ which vanishes on $\mathcal{K}(\ell_2(X))$ gives rise to a well-defined state ψ on $\mathrm{Q}^*_u(X)$ determined by

$$\psi(\pi(a)) = \varphi(a), \text{ for all } a \in \mathrm{C}^*_u(X).$$

Moreover, given a coarse map $h \colon X \to \mathbb{R}$, the flow σ_h canonically induces a flow on the corona $\mathrm{Q}^*_u(X)$. Precisely, as σ_h leaves $\mathcal{K}(\ell_2(X))$ invariant, that is,

$$\sigma_{h,t}(\mathcal{K}(\ell_2(X))) \subseteq \mathcal{K}(\ell_2(X)) \text{ for all } t \in \mathbb{R},$$

we obtain a flow σ^∞_h on $\mathrm{Q}^*_u(X)$ by letting

$$\sigma^\infty_{h,t}(\pi(a)) = \pi(\sigma_{h,t}(a)) \text{ for all } a \in \mathrm{C}^*_u(X) \text{ and all } t \in \mathbb{R}.$$

In other words, σ^∞_h is a flow on $\mathrm{Q}^*_u(X)$ which makes the following diagram commute.

$$\begin{array}{ccc} \mathrm{C}^*_u(X) & \xrightarrow{\sigma_{h,t}} & \mathrm{C}^*_u(X) \\ \pi \downarrow & & \downarrow \pi \\ \mathrm{Q}^*_u(X) & \xrightarrow[\sigma^\infty_{h,t}]{} & \mathrm{Q}^*_u(X). \end{array}$$

We show that the study of (σ_h, β)-KMS states on $\mathrm{C}^*_u(X)$ which vanish on the ideal of compact operators completely reduces to the study of (σ^∞_h, β)-KMS states on $\mathrm{Q}^*_u(X)$ in a canonical way. Precisely:

Proposition 1.8 *Let X be a u.l.f. metric space, $h \colon X \to \mathbb{R}$ be a coarse map, and $\beta \in \mathbb{R}$. A state ψ on $\mathrm{Q}^*_u(X)$ is a (σ^∞_h, β)-KMS state if and only if $\varphi = \psi \circ \pi$ is a (σ_h, β)-KMS state on $\mathrm{C}^*_u(X)$. Moreover, the assignment*

$$\psi \mapsto \varphi = \psi \circ \pi$$

is an affine isomorphism from the set of all (σ^∞_h, β)-KMS states on $\mathrm{Q}^*_u(X)$ to the set of all (σ_h, β)-KMS states on $\mathrm{C}^*_u(X)$ which vanish on $\mathcal{K}(\ell_2(X))$.

Guided by Proposition 1.8, we then focus on KMS states on the corona algebra $\mathrm{Q}^*_u(X)$. For that, we show some general results about KMS states on arbitrary C*-algebras with respect to arbitrary flows (see Section 3 for details). In a nutshell, we show that the extreme KMS states on an arbitrary C*-algebra A are influenced by the center of A, denoted by $\mathcal{Z}(A)$, and its C*-subalgebras. Returning to our coarse setting, this brings up a seemingly unexpected link between KMS states on uniform Roe algebras and the *Higson corona* of metric spaces. More precisely, given a u.l.f. metric space X, we denote its *Higson compactification* by hX and its *Higson corona* by $\nu X = hX \backslash X$.[5] The space of continuous functions on the Higson compactification, $C(hX)$, is canonically seen as a C*-subalgebra of $\ell_\infty(X)$, which in turn allow us to canonically identify the continuous functions on its corona, $C(\nu X)$, with a C*-subalgebra of $\mathrm{Q}^*_u(X)$. Under this identifications, it has been recently shown that

$$\mathcal{Z}(\mathrm{Q}^*_u(X)) = C(\nu X)$$

(see [BBF+22a, Proposition 3.6]).

This link between KMS states and the Higson corona is essential in the analysis of KMS states which vanish on the compacts. Precisely, the next result summarizes our findings on the topic.

Theorem 1.9 *Let X be a u.l.f. metric space, $h\colon X \to \mathbb{R}$ be a coarse map, and $\beta \in \mathbb{R}$.*

1. *For any extreme (σ_h^∞, β)-KMS state ψ on $\mathrm{Q}^*_u(X)$, there is $x \in \nu X$ such that*
$$\psi(a) = a(x) \text{ for all } a \in C(\nu X).$$
2. *Suppose there is a (σ_h^∞, β)-KMS state on $\mathrm{Q}^*_u(X)$ whose restriction to $C(\nu X)$ is faithful. Then, for any $x \in \nu X$, there is an extreme (σ_h^∞, β)-KMS state ψ on $\mathrm{Q}^*_u(X)$ such that*
$$\psi(a) = a(x) \text{ for all } a \in C(\nu X).$$

Our methods give us a strong control on the support of KMS states on $\mathrm{C}^*_u(X)$. In order to state this control, a definition is in place.

Definition 1.10 Let X be a u.l.f. metric space, $x \in \nu X$, and φ be a state on $\mathrm{C}^*_u(X)$. We say that φ is *supported on x* if for all neighborhoods $U \subseteq hX$ of x, we have $\varphi(\chi_{U\cap X}) = 1$.

Theorem 1.11 *Let X be a u.l.f. metric space, $h\colon X \to \mathbb{R}$ be a coarse map, and $\beta \in \mathbb{R}$. The following holds:*

1. *Any extreme (σ_h, β)-KMS state on $\mathrm{C}^*_u(X)$ which vanishes on the compacts is supported at some element of νX.*
2. *If there is a (σ_h, β)-KMS state on $\mathrm{C}^*_u(X)$ which vanishes on the compacts and such that its induced state on $\mathrm{Q}^*_u(X)$ is faithful on $C(\nu X)$, then for every $x \in \nu X$ there is a (σ_h, β)-KMS state on $\mathrm{C}^*_u(X)$ supported on x.*

In fact, both Theorems 1.9 and 1.11 have versions that hold with $C(\nu X)$ being substituted by arbitrary unital C*-subalgebras of $C(\nu X)$ (see Theorems 4.4 and 4.6).

In Section 4.1, we show that the Higson corona of any infinite u.l.f. metric space contains $2^{2^{\aleph_0}}$ elements (see Theorem 4.14). This result has been first obtained in [Kee94, Theorem 3], but we chose to present an alternative and self-contained proof here for the readers convenience. As a consequence of this result, Theorem 1.9 and Proposition 1.8 above imply that if there is a (σ_h^∞, β)-KMS state on $\mathrm{Q}^*_u(X)$ whose

[5] For brevity, we refer the reader to Definition 4.2 for the precise definition of the Higson compactification/corona.

restriction to $C(\nu X)$ is faithful, then there are $2^{2^{\aleph_0}}$ extreme KMS states in both $\mathrm{Q}^*_u(X)$ and $\mathrm{C}^*_u(X)$ (see Corollary 4.16).

1.4 Applications

Our methods can be applied to specific metric spaces. Notice that Theorem 1.6 implies that if the balls of X have polynomial growth, then $\mathrm{C}^*_u(X)$ will have (σ_h, β)-KMS states for any $\beta > 0$ and any "reasonable" $h\colon X \to \mathbb{R}$. Indeed, suppose h is such that there is $L > 0$ and $x_0 \in X$ for which

$$h(x) \geq \frac{d(x, x_0)}{L} - L \text{ for all } x \in X.$$

Suppose now p is a polynomial controling the growth of the balls of X, that is, every ball in X centered at x_0 of radius r has at most $p(r)$ elements. Then, the series $\sum_{x \in X} e^{-\beta h(x)}$ converges to a finite number for any $\beta > 0$. Therefore, in order to find examples with interesting phase transition, it is advisable to look for metric spaces with large growth. This makes the n-branching trees natural spaces to apply our theory to.

We point out that, due to the technical aspects of Theorems 1.9 and 1.11, the result below is not a mere corollary of the results above and a deeper analysis of Higson coronas as well as of the weak*-limit of their strongly continuous KMS states is needed. The study of invariant means on semigroups developed by Chou in [Cho69] is also essential for the precise computation of the cardinality of extreme KMS states presented below.

Given $n \in \mathbb{N}$, let T_n denote the n-branching tree, that is, $T_n = \{\varnothing\} \cup \bigcup_{k=1}^{\infty} \{1, \ldots, n\}^k$ and we endow T_n with its canonical graph distance (see Section 5 for details). The *branches* of T_n are denoted by $[T_n]$, that is, $[T_n] = \{1, \ldots, n\}^{\mathbb{N}}$. Given $\bar{x} = (x_i)_{i=1}^{\infty} \in [T_n]$, we let $\bar{x}|k = (x_1, \ldots, x_k) \in T_n$ and $\bar{x}|k^\frown T_n$ denotes the words in T_n which start with $\bar{x}|k$.

Theorem 1.12 *Given $n \in \mathbb{N}$, let T_n denote the n-branching tree endowed with its graph distance d and let $\varnothing$ denote its root. Let $h\colon T_n \to \mathbb{R}$ be given by $h(x) = d(x, \varnothing)$ for all $x \in T_n$. Then, there is a (σ_h, β)-KMS state on $\mathrm{C}^*_u(T_n)$ if and only if $\beta \geq \log(n)$. Moreover,*

1. *For $\beta > \log(n)$, there is a unique (σ_h, β)-KMS state on $\mathrm{C}^*_u(T_n)$ and this state is strongly continuous.*
2. *For $\beta = \log(n)$, the (σ_h, β)-KMS states on $\mathrm{C}^*_u(T_n)$ all vanish on $\mathcal{K}(\ell_2(T_n))$. Moreover, for all $\bar{x} \in [T_n]$, there are $2^{2^{\aleph_0}}$ extreme (σ_h, β)-KMS states φ on $\mathrm{C}^*_u(T_n)$ such that*

$$\varphi(\chi_{\bar{x}|k^\frown T_n}) = 1 \text{ for all } k \in \mathbb{N}.$$

Conversely, any extreme (σ_h, β)-KMS state on $\mathrm{C}^*_u(T_n)$ satisfies the above for an appropriate $\bar{x} \in [T_n]$.

For inverse temperature $\beta > \log(n)$, we actually have a precise formula for its unique KMS state (see Theorem 5.4).

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Finally, in Section 5.4, we discuss a somewhat unusual phenomenon known as *chaotic convergence* of KMS states. In order to explain what this means, consider a flow σ on a C^*-algebra A admitting a unique KMS state at inverse temperature β, say φ_β, for every β in an interval of the form $(\beta_0, \beta_0 + \varepsilon)$, so that it makes sense to ask whether or not the limit

$$\lim_{\beta \to \beta_0^+} \varphi_\beta$$

exists (here the limit should be taken with respect to the weak* topology). The most commonly observed behavior (see [vER07, CH10, CRL15, BGT18]) is when this limit exists, even when β_0 is critical, that is, even when there are multiple (σ, β_0)-KMS states.

By *chaotic convergence* of KMS states it is meant a situation where the above fails in the sense that there are different sequences β_n converging to β_0 from above for which the corresponding limit states differ. This chaotic behavior has been observed for ground states [BGT18], that is, regarding the limit as $\beta \to \infty$, but we are not aware of too many situations where this phenomenon happens at finite temperatures.

As detailed in Theorem 5.15 below, we analyze this question for $C_u^*(T_n)$ as β approaches $\log(n)$ from above, showing that such chaotic behavior is indeed present.

2 Basics on KMS states on uniform Roe algebras

In this section, we start our study of KMS states of uniform Roe algebras and prove several general properties which will be essential throughout these notes. We also present some simple examples by studying the KMS states of the simplest coarse space: $\{n^2 \mid n \in \mathbb{N}\}$. We start this section introducing some notation which was left out from Section 1.

Given a set X and $x, y \in X$, we let $e_{x,y} \in \mathcal{B}(\ell_2(X))$ be the rank 1 partial isometry sending δ_y to δ_x. If $A \subseteq X$, we let

$$\chi_A = \text{SOT-}\sum_{x \in A} e_{x,x};$$

where the letters SOT above mean that the sum converges with respect to the strong operator topology. In other words, χ_A is the canonical orthogonal projection $\ell_2(X) \to \ell_2(A)$. Under the identification of $\ell_\infty(X)$ with the C^*-subalgebra of $C_u^*(X)$ consisting of the diagonal operators, we have that $\chi_A \in \ell_\infty(X)$ for all $A \subseteq X$. The C^*-algebra of functions $X \to \mathbb{C}$ which vanish at infinity is identified with the compact operators in $\ell_\infty(X)$, that is,

$$c_0(X) = \ell_\infty(X) \cap \mathcal{K}(\ell_2(X)).$$

The following description of operators in $C_u^*[X]$ will be very useful for our goals: Firstly, recall that a *partial bijection of* X is a bijection $f\colon A \to B$ between subsets A and B of X. If moreover

$$\sup_{x \in A} d(x, f(x)) < \infty,$$

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then we say that f is a *partial translation*. Given any partial translation $f\colon A \subseteq X \to B \subseteq X$, we define an operator v_f on $\ell_2(X)$ by letting

$$v_f\delta_x = \begin{cases} \delta_{f(x)}, & x \in A, \\ 0, & x \notin A. \end{cases} \tag{2.1}$$

So, each v_f is a partial isometry and the algebraic uniform Roe algebra is linearly spanned by products of elements in $\ell_\infty(X)$ by those partial isometries. Precisely, we have

$$\mathrm{C}^*_u[X] = \mathrm{span}\Big\{av_f \mid a \in \ell_\infty(X) \text{ and } f \text{ is a partial translation on } X\Big\}$$

(see [ŠW17, Lemma 2.4] for details).

2.1 Flows and analytic elements

Our very first result shows that the actions $\sigma_h\colon \mathbb{R} \curvearrowright \mathrm{C}^*_u(X)$ are indeed flows if and only if h is coarse.

Proposition 2.1 *Let X be a u.l.f. metric space and $h\colon X \to \mathbb{R}$ be a map. Then h is coarse if and only if the action σ_h given by Definition 1.3 is strongly continuous, that is,*

$$t \in \mathbb{R} \mapsto \sigma_{h,t}(a) \in \mathrm{C}^*_u(X) \tag{2.2}$$

is continuous for all $a \in \mathrm{C}^*_u(X)$.

Proof Suppose first that h is coarse. Since $\mathrm{C}^*_u[X]$ is dense in $\mathrm{C}^*_u(X)$, it is enough to show that the map in (2.2) is continuous for each $a \in \mathrm{C}^*_u[X]$. Moreover, since $\mathrm{C}^*_u[X]$ is spanned by the subset of all av_f, for $a \in \ell_\infty(X)$ and $f\colon A \subseteq X \to B \subseteq X$ a partial translation, it is enough to notice that (2.2) holds for all such elements av_f. Fix such a and f. Then, as f is a partial bijection, we have that

$$\begin{aligned} \|\sigma_{h,t}(av_f) - \sigma_{h,s}(av_f)\| &= \|e^{it\bar{h}}av_fe^{-it\bar{h}} - e^{is\bar{h}}av_fe^{-is\bar{h}}\| \\ &= \sup_{x\in A}\Big|e^{it(h(f(x))-h(x))} - e^{is(h(f(x))-h(x))}\Big|a_{f(x),f(x)}. \end{aligned} \tag{2.3}$$

Since f is a partial translation and h is coarse, we have

$$\sup_{x\in A}|h(f(x)) - h(x)| < \infty.$$

Therefore, it follows from (2.3) and the intermediate value theorem that

$$t \in \mathbb{R} \mapsto \sigma_t(av_f) \in \mathrm{C}^*_u(X)$$

is continuous.

Suppose now that the action σ_h is strongly continuous. Suppose towards a contradiction that h is not coarse. Then there is $r > 0$, and sequences $(x_i)_i$ and $(y_i)_i$ in X such that $\lim_i |h(x_i) - h(y_i)| = \infty$ and $d(x_i, y_i) \le r$ for all $i \in \mathbb{N}$. As X is u.l.f., those sequences cannot be bounded, so, by going to a subsequence if necessary, we

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assume that $(x_i)_i$ and $(y_i)_i$ are sequences of distinct points of X. We can then define a map

$$f\colon \{x_i \mid i \in \mathbb{N}\} \to \{y_i \mid i \in \mathbb{N}\}$$
$$x_i \mapsto y_i$$

and this map is a partial translation. So, $v_f \in \mathrm{C}^*_u[X]$ and, since σ_h is strongly continuous, we have that

$$\lim_{t\to 0} \|\sigma_t(v_f) - v_f\| = 0.$$

Fix $\delta > 0$ such that

$$|t| < \delta \ \text{ implies } \ \|\sigma_t(v_f) - v_f\| < 2.$$

Notice now that

$$\begin{aligned} \|\sigma_t(v_f) - v_f\| &= \|e^{it\bar{h}} v_f e^{-it\bar{h}} - v_f\| \\ &= \sup_{x\in X} \left| e^{it(h(f(x))-h(x))} - 1 \right| \\ &\geq \sup_{i\in\mathbb{N}} \left| e^{it(h(y_i)-h(x_i))} - 1 \right|. \end{aligned}$$

Hence, picking $i \in \mathbb{N}$ large enough so that

$$t = \frac{\pi}{|h(y_i) - h(x_i)|} < \delta,$$

we obtain that $\|\sigma_t(v_f) - v_f\| \geq 2$; contradiction. ∎

We now show that our choice of only dealing with flows of the form σ_h for some coarse map $h\colon X \to \mathbb{R}$ does not represent a big restriction in a sense.

Proposition 2.2 *Let X be a u.l.f. metric space and let $\sigma\colon \mathbb{R} \curvearrowright \mathrm{C}^*_u(X)$ be a flow leaving $\ell_\infty(X)$ invariant, that is, $\sigma(\ell_\infty(X)) \subseteq \ell_\infty(X)$ for all $t \in \mathbb{R}$. Then, there is a coarse map $h\colon X \to \mathbb{R}$ such that $\sigma = \sigma_h$.*

Proof We first notice that the condition of $\sigma\colon \mathbb{R} \curvearrowright \mathrm{C}^*_u(X)$ leaving $\ell_\infty(X)$ invariant implies that σ_t is the identity on $\ell_\infty(X)$ for all $t \in \mathbb{R}$. Indeed, as σ_0 is by hypothesis the identity on $\mathrm{C}^*_u(X)$, we have that $\sigma_0(e_{x,x}) = e_{x,x}$ for all $x \in X$. As σ_t is an isomorphism for all $t \in \mathbb{R}$, $\sigma_t(e_{x,x})$ must be a projection for all $t \in \mathbb{R}$ and all $x \in X$. Therefore, since $t \in \mathbb{R} \mapsto \sigma_t(e_{x,x}) \in \ell_\infty(X)$ is continuous, this shows that $\sigma_t(e_{x,x}) = e_{x,x}$ for all $t \in \mathbb{R}$ and all $x \in X$. Hence, σ_t must be the identity on $c_0(X)$ for all $t \in \mathbb{R}$. As isomorphisms of uniform Roe algebras are strongly continuous [ŠW13, Lemma 3.1], this shows that each σ_t is the identity on $\ell_\infty(X)$ are desired.

Fix $x \in X$. For each $\xi \in \ell_2(X)$, let r_ξ be the rank one operator given by

$$r_\xi \zeta = \langle \zeta, \delta_x \rangle \xi \ \text{ for all } \ \zeta \in \ell_2(X).$$

For each $t \in \mathbb{R}$, define an operator u_t on $\ell_2(X)$ by letting

$$u_t \xi = \sigma_t(r_\xi)\delta_x \ \text{ for all } \ \xi \in \ell_2(X).$$

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Claim 2.3 *We have*

$$\sigma_t(a) = u_t a u_{-t} \ \textit{for all}\ a \in \mathrm{C}^*_u(X) \ \textit{and all}\ \ t \in \mathbb{R}.$$

In particular, $u_t \in \ell_\infty(X)$ *for all* $t \in \mathbb{R}$.

Proof First notice that

$$ae_{x,x} = r_{a\delta_x} \ \text{ for all } a \in \mathrm{C}^*_u(X). \tag{2.4}$$

Hence, by the arbitrariness of a above, this implies that

$$u_t a u_{-t}\xi = u_t a\sigma_{-t}(r_\xi)\delta_x = \sigma_t(a\sigma_{-t}(r_\xi))\delta_x = \sigma_t(a)r_\xi\delta_x = \sigma_t(a)\xi$$

for all $\xi \in \ell_2(X)$, all $t \in \mathbb{R}$, and all $a \in \mathrm{C}^*_u(X)$.

For the last claim, notice that, as each σ_t is the identity on $\ell_\infty(X)$, the previous paragraph implies that each u_t commutes with the elements of $\ell_\infty(X)$. As $\ell_\infty(X)$ is a maximal abelian subalgebra of $\mathrm{C}^*_u(X)$, this gives that $u_t \in \ell_\infty(X)$ for all $t \in \mathbb{R}$. ∎

Claim 2.4 The family $(u_t)_t$ is a one-parameter unitary group, that is, $t \in \mathbb{R} \mapsto u_t\xi \in \ell_2(X)$ is continuous for all $\xi \in \ell_2(X)$, $u_{t+s} = u_t u_s$ for all $t, s \in \mathbb{R}$, and each u_t is a unitary,

Proof First notice that, as $t \in \mathbb{R} \mapsto \sigma_t(r_\xi) \in \mathrm{C}^*_u(X)$ is continuous, $t \in \mathbb{R} \mapsto u_t\xi \in \ell_2(X)$ is also continuous for all $\xi \in \ell_2(X)$. Also, using (2.4), we have

$$u_t(u_s\xi) = u_t(\sigma_s(r_\xi)\delta_x) = \sigma_t(\sigma_s(r_\xi))\delta_x = \sigma_{t+s}(r_\xi)\delta_x = u_{t+s}\xi$$

for all $\xi \in \ell_2(X)$ and all $t, s \in \mathbb{R}$. Finally, as each u_t is an element of $\ell_\infty(X)$ with norm at most one satisfying $u_t u_{-t} = 1$, this also shows that u_t is a unitary. ∎

By Claims 2.3 and 2.4, there is map $h\colon X \to \mathbb{R}$ such that

$$u_t = e^{it\bar{h}} \ \text{ for all } \ t \in \mathbb{R}.$$

Therefore, by Claim 2.3, we have that $\sigma = \sigma_h$. By Proposition 2.1, it follows that h must be coarse. ∎

In order to study the KMS states on uniform Roe algebras which are given by the flows defined above, it is essential to understand the analytic elements of this flow. This is precisely the content of Proposition 1.4.

Proof of Proposition 1.4 (1) If h is bounded, $\bar{h}$ is a bounded operator on $\ell_2(X)$. Therefore, the analyticity of e^z gives that

$$z \in \mathbb{C} \to e^{-iz\bar{h}}ae^{iz\bar{h}} \in \mathrm{C}^*_u(X)$$

is analytic for all $a \in \mathrm{C}^*_u(X)$.

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(2) Since $C^*_u[X]$ is spanned by the subset of all av_f, for $a \in \ell_\infty(X)$ and $f\colon A \subseteq X \to B \subseteq X$ a partial translation, it is enough to show that each such av_f is analytic. Fix such a and $f\colon A \subseteq X \to B \subseteq X$, and let $g\colon X \to \mathbb{R}$ be given by

$$g(x) = \begin{cases} h(f(x)) - h(x), & x \in A, \\ 0, & x \notin A. \end{cases}$$

A simple computation gives that

$$\sigma_{h,t}(av_f) = e^{it\bar{g}} av_f$$

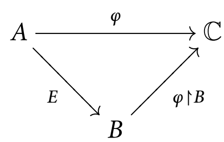
for all $t \in \mathbb{R}$. As $d(f(x), x) \leq r$ for all $x \in A$, g is bounded. Then, the analyticity of e^z implies the that

$$z \in \mathbb{C} \to e^{iz\bar{g}} av_f \in C^*_u(X)$$

is analytic; so, av_f is analytic. ∎

2.2 Factoring KMS-states through $\ell_\infty(X)$

It is often common in the study of KMS states on a given C*-algebra A that there is some "natural" C*-subalgebra $B \subseteq A$ and a conditional expectation $E\colon A \to B$ such that the KMS states $\varphi\colon A \to \mathbb{C}$ factor through E; precisely, $\varphi = \varphi \circ E$, so the diagram below commutes.



We now show that this also happens with KMS state on uniform Roe algebras.

Recall, $\ell_\infty(X)$ is a Cartan masa of $C^*_u(X)$ and the conditional expectation $E\colon C^*_u(X) \to \ell_\infty(X)$ is simply deleting the matrix entries of the operators on $C^*_u(X)$ which are not in the main diagonal. Precisely, the canonical conditional expectation $E\colon C^*_u(X) \to \ell_\infty(X)$ is defined as follows:

$$\langle E(a)\delta_x, \delta_y \rangle = \begin{cases} a_{x,x}, & x = y, \\ 0, & x \neq y, \end{cases}$$

for all $a = [a_{x,y}] \in C^*_u(X)$ and all $x, y \in X$.

Proof of Theorem 1.5 As $C^*_u[X]$ is dense in $C^*_u(X)$, it is enough to show that $\varphi(a) = \varphi(E(a))$ for all $a \in C^*_u[X]$. Moreover, as $C^*_u[X]$ is the span of all av_f, where $a \in \ell_\infty(X)$ and f is a partial translation on X, it is enough to show that $\varphi(av_f) = 0$ for all $a \in \ell_\infty(X)$ and all partial translations $f\colon A \subseteq X \to B \subseteq X$ such that $f(x) \neq x$ for all $x \in A$; fix a and f as such.

Let $r = \sup_{x \in A} d(x, f(x))$; as f is a partial translation, r is finite. As X is u.l.f., there is a partition

$$A = A_1 \sqcup \ldots \sqcup A_n,$$

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such that each A_i is $2r$-separated, that is, $d(x,y) > 2r$ for all $i \in \{1,\dots,n\}$ and all distinct $x, y \in A_i$. Therefore,

$$d(x, f(y)) \geq d(x,y) - d(y, f(y)) > r$$

for all $i \in \{1,\dots,n\}$ and all distinct $x, y \in A_i$; in particular, $x \neq f(y)$. Moreover, as $f(x) \neq x$ for all $x \in A$, this shows that

$$A_i \cap f(A_i) = \varnothing \tag{2.5}$$

for all $i \in \{1,\dots,n\}$.

For each $i \in \{1,\dots,n\}$, let $f_i = f \restriction A_i$. So, (2.5) implies that $\chi_{A_i} v_{f_i} = 0$ for all $i \in \{1,\dots,n\}$. Therefore, since

$$\chi_{A_i}\sigma_{h,i\beta}(av_{f_i}) = \chi_{A_i} e^{-\beta\bar{h}} a v_{f_i} e^{\beta\bar{h}} = e^{-\beta\bar{h}} a \chi_{A_i} v_{f_i} e^{\beta\bar{h}} = 0,$$

we conclude that

$$\varphi(av_{f_i}) = \varphi(av_{f_i}\chi_{A_i}) = \varphi(\chi_{A_i}\sigma_{h,i\beta}(av_{f_i})) = 0.$$

Since $v_f = v_{f_1} + \dots + v_{f_n}$, this finishes the proof. ■

As KMS states on uniform Roe algebras factor through the canonical conditional expectation $E\colon \mathrm{C}^*_u(X) \to \ell_\infty(X)$, it will be very useful to have a condition on when a state φ on $\mathrm{C}^*_u(X)$ satisfies the KMS condition which depends only on operators on $\ell_\infty(X)$. We first introduce some notation which will be used in the next proof. Given $a = (a_y)_y \in \ell_\infty(X)$ and a partial bijection $f\colon A \subseteq X \to B \subseteq X$, we let $a_{\circ f} \in \ell_\infty(X)$ be the operator given by

$$a_{\circ f}\delta_x = \begin{cases} a_{f(x)}\delta_x, & x \in A, \\ 0, & x \notin A, \end{cases}$$

for all $x \in X$.[6] The relevance of this notation is due to the fact that

$$v_f^* a v_f = a_{\circ f}.$$

Theorem 2.5 *Let X be a u.l.f. metric space, $h\colon X \to \mathbb{R}$ be coarse, and $\beta \in \mathbb{R}$. Suppose φ is a state on $\ell_\infty(X)$. Then φ satisfies*

$$\varphi(\chi_{f(A)}) = \varphi\Big(\chi_A e^{\beta(\overline{h - h\circ f})}\Big) \tag{2.6}$$

for all partial translations $f\colon A \to f(A)$ on X if and only if $\varphi \circ E$ is a (σ_h,β)-KMS state on $\mathrm{C}^*_u(X)$; where $E\colon \mathrm{C}^*_u(X) \to \ell_\infty(X)$ is the canonical conditional expectation.

Proof Suppose first that φ is a (σ_h,β)-KMS on $\mathrm{C}^*_u(X)$. Let $f\colon A \to f(A)$ be a partial translation on X. Then, $\chi_{f(A)} = \chi_{f(A)} v_f v_f^*$. As

$$v_f^*\sigma_{h,i\beta}(\chi_{f(A)}v_f) = v_f^* e^{-\beta\bar{h}} \chi_{f(A)} v_f e^{\beta\bar{h}} = \chi_A e^{\beta(\overline{h - h\circ f})},$$

[6] Here is a justification for this cumbersome notation: if $a \in \ell_\infty(X)$, then one can see a as a bounded sequence, say $a = (a_x)_{x\in X}$. Then $a_{\circ f}$ is the extension of $(a_{f(x)})_{x\in A}$ to the whole X by letting the coordinates not in A be zero.

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the KMS condition gives that

$$\varphi(\chi_{f(A)}) = \varphi(v_f^* \sigma_{h,i\beta}(\chi_{f(A)} v_f)) = \varphi\Big(\chi_A e^{\beta(\overline{h - h\circ f})}\Big).$$

Suppose now that φ satisfies (2.6). First, notice that as $\ell_\infty(X)$ is linearly generated by the characteristic functions on X, this implies that

$$\varphi(c) = \varphi\Big(c_{\circ f} e^{\beta(\overline{h - h\circ f})}\Big) \tag{2.7}$$

for all partial translations f on X and all $c \in \ell_\infty(\mathrm{Im}(f))$. By abuse of notation, we extend φ to the whole $\mathrm{C}_u^*(X)$ and still denote it by φ, that is, $\varphi = \varphi \circ E$. In order to show that $\varphi \circ E$ is a (σ_h, β)-KMS state on $\mathrm{C}_u^*(X)$, it is enough to show the KMS condition for elements of the form av_f, where $a \in \ell_\infty(X)$ and f is a partial translation of X.

Fix $a, b \in \ell_\infty(X)$ and partial translations f and g on X. Let

$$A = \Big\{x \in \mathrm{Dom}(f) \mid f(x) \in \mathrm{Dom}(g) \text{ and } g(f(x)) = x\Big\}.$$

and notice that $g \restriction f(A) = (f \restriction A)^{-1}$. We can then write

$$\begin{aligned} v_g a v_f &= v_{g\restriction f(A)} a v_{f\restriction A} + v_{g\restriction \mathrm{Dom}(g)\setminus f(A)} a v_{f\restriction A} \\ &\quad + v_{g\restriction f(A)} a v_{f\restriction \mathrm{Dom}(f)\setminus A} + v_{g\restriction \mathrm{Dom}(g)\setminus f(A)} a v_{f\restriction \mathrm{Dom}(f)\setminus A} \\ &= v_{(f\restriction A)^{-1}} a v_{f\restriction A} + v_{g\restriction f(A)} a v_{f\restriction \mathrm{Dom}(f)\setminus A} + v_{g\restriction \mathrm{Dom}(g)\setminus f(A)} a v_{f\restriction \mathrm{Dom}(f)\setminus A}. \end{aligned}$$

Notice that the last two terms in the right handside of the equality above are in the kernel of the conditional expectation E. Therefore,

$$E(bv_g a v_f) = E(bv_{(f\restriction A)^{-1}} a v_{f\restriction A}).$$

For this reason, it is enough to check the KMS condition for partial translations of X which are inverse of each other. For now on, assume that $g = f^{-1}$.

Let us now show the KMS condition holds. Firstly, notice that

$$bv_f a v_f^* = ba_{\circ f^{-1}} \text{ and } av_f^* e^{-\beta\bar{h}} b v_f e^{\beta\bar{h}} = ab_{\circ f} e^{\beta(\overline{h - h\circ f})}. \tag{2.8}$$

Then, letting $c = ba_{\circ f^{-1}}$, we have that $c \in \ell_\infty(\mathrm{Im}(f))$ and

$$c_{\circ f} = v_f^* c v_f = v_f^* b v_f a v_f^* v_f = v_f^* b v_f a = ab_{\circ f}.$$

Therefore, (2.7) gives that

$$\begin{aligned} \varphi(bv_f a v_f^*) &= \varphi(ba_{\circ f^{-1}}) \\ &= \varphi(c) \\ &= \varphi(c_{\circ f} e^{\beta(\overline{h - h\circ f})}) \\ &= \varphi(ab_{\circ f} e^{\beta(\overline{h - h\circ f})}) \\ &= \varphi\Big(av_f^* e^{-\beta\bar{h}} b v_f e^{\beta\bar{h}}\Big) \\ &= \varphi(av_f^* \sigma_{h,i\beta}(bv_f)). \end{aligned}$$

This shows that φ is a (σ_h, β)-KMS state on $\mathrm{C}_u^*(X)$. ∎

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2.3 Amenable spaces

A priori, our flows of interest σ_h are given by any coarse map $h\colon X \to \mathbb{R}$ (see Proposition 1.4). Therefore, being automatically coarse, bounded maps form a natural class of maps to produce flows in uniform Roe algebras. However, as we show in this subsection, the existence of KMS states for such flows reduces to the amenability of the metric space, equivalently, to the uniform Roe algebra having a positive unital trace (see [Roe03, Theorem 4.6]). Recall:

Definition 2.6 A u.l.f. metric space X is *amenable* if there is a nonzero finitely additive measure $\mu\colon \mathcal{P}(X) \to [0, \infty)$ which is *invariant*, that is, $\mu(A) = \mu(B)$ for all $A, B \subseteq X$ such that there is a partial translation $f\colon A \to B$. We call such measure an *invariant mean*.

Theorem 2.7 *Let X be a u.l.f. metric space, $h\colon X \to \mathbb{R}$ be a bounded map, and $\beta \in \mathbb{R}$. Then, $\mathrm{C}^*_u(X)$ has a (σ_h, β)-KMS state if and only if X is amenable.*

Before proving Theorem 2.7, we isolate a straightforward lemma which highlights the relation between the trace and the KMS condition when the KMS state is given by elements in the C^*-algebra.

Lemma 2.8 *Let A be a C^*-algebra and $u \in A$ be invertible. Consider the following assignments:*

1. *For each functional τ on A, let $\varphi_{\tau,u}$ be the functional given by $\varphi_{\tau,u}(a) = \tau(au)$ for all $a \in A$.*
2. *For each functional φ on A, let $\tau_{\varphi,u}$ be the functional given by $\tau_{\varphi,u}(a) = \varphi(au^{-1})$ for all $a \in A$.*

The assignment $\tau \mapsto \varphi_\tau$ defines a bijection between the functionals τ on A such that $\tau(ab) = \tau(ba)$ and the functionals φ on A such that $\varphi(ab) = \varphi(buau^{-1})$ for all $a, b \in A$; the inverse of this assignment is $\varphi \mapsto \tau_{\varphi,u}$ with the appropriate domain/codomain.

Proof of Theorem 2.7 We start recalling a well-known fact about uniform Roe algebras: a u.l.f. metric space has a positive unital trace if and only if it is amenable [Roe03, Theorem 4.6]. In fact, if μ is a nontrivial invariant mean on X, say $\mu(X) = 1$, and $E\colon \mathrm{C}^*_u(X) \to \ell_\infty(X)$ is the canonical conditional expectation, then

$$\tau(a) = \int_X E(a)\,d\mu, \quad \text{for all } a \in \mathrm{C}^*_u(X),$$

defines a positive unital trace on $\mathrm{C}^*_u(X)$. On the other hand, if τ is a positive unital trace on $\mathrm{C}^*_u(X)$, then

$$\mu(A) = \tau(\chi_A) \quad \text{for all } A \subseteq X$$

defines an invariant mean on X.

Suppose then that X is amenable and that τ is the trace on $\mathrm{C}^*_u(X)$ given by a nontrivial invariant mean μ on X as above. By Lemma 2.8, $\varphi_{\tau, e^{\beta \bar{h}}}$ satisfies the (σ_h, β)-KMS condition. Moreover, using the formula of τ, we have that

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$$\varphi_{\tau,e^{\beta\bar{h}}}(a) = \int_X E(a)e^{\beta\bar{h}}d\mu \text{ for all } a \in \mathrm{C}_u^*(X).$$

Therefore, φ is positive and, as $t = \sup_{x\in X}|h(x)| < \infty$, we have that

$$\varphi_{\tau,e^{\beta\bar{h}}}(\chi_X) = \int_X e^{\beta\bar{h}}d\mu \geq e^{-|\beta|t}\mu(X) > 0.$$

Therefore, normalizing φ, we obtain a (σ_h,β)-KMS state on $\mathrm{C}_u^*(X)$.

Suppose now that φ is a (σ_h,β)-KMS state on $\mathrm{C}_u^*(X)$. By Lemma 2.8, $\tau_{\varphi,e^{\beta\bar{h}}}$ satisfies the trace condition, that is, $\tau_{\varphi,e^{\beta\bar{h}}}(ab) = \tau_{\varphi,e^{\beta\bar{h}}}(ba)$ for all $a,b \in \mathrm{C}_u^*(X)$. As φ is positive and factors through the canonical conditional expectation $\mathrm{C}_u^*(X) \to \ell_\infty(X)$ (Theorem 1.5), $\tau_{\varphi,e^{\beta\bar{h}}}$ is also positive. Finally, it follows form our definition of t that

$$\tau_{\varphi,e^{\beta\bar{h}}}(\chi_X) = \varphi(e^{-\beta\bar{h}}) \geq \varphi(e^{-|\beta|t}\chi_X) > 0.$$

So, normalizing τ, we obtain a positive unital trace on X. ■

As Theorem 2.7 completely takes care of bounded maps, we can now restrict our analyses to unbounded coarse maps $h\colon X \to \mathbb{R}$.

2.4 Strongly continuous KMS states

This section deals with strongly continuous KMS states. As we shall see below, those states are the easiest to get and, whenever they exist, they are unique (Theorem 1.6). We also show that the set of β's for which a strongly continuous KMS state exists must be either of the form (t,∞) or $[t,\infty)$, for some $t \geq 0$ (Corollary 2.10 for a precise statement).

Proposition 2.9 *Let X be a u.l.f. metric space, $h\colon X \to [0,\infty)$ be an unbounded coarse map, and $\beta < 0$. If φ is a (σ_h,β)-KMS state, then $\varphi(e_{x,x}) = 0$ for all $x \in X$. In particular, there are no strongly continuous (σ_h,β)-KMS states on $\mathrm{C}_u^*(X)$.*

Proof Fix $x \in X$. As h is unbounded, there is a sequence $(x_n)_n$ in X such that $\lim_n h(x_n) = \infty$. Then, if φ is a (σ_h,β)-KMS state on $\mathrm{C}_u^*(X)$, we have

$$\varphi(e_{x,x}) = \varphi(e_{x,x_n}e_{x_n,x}) = \varphi(e_{x_n,x}\sigma_{h,i\beta}(e_{x,x_n})) = e^{\beta(h(x_n)-h(x))}\varphi(e_{x_n,x_n}).$$

As $(\varphi(e_{x_n,x_n}))_n$ is bounded and $\beta < 0$, we conclude that $\varphi(e_{x,x}) = 0$ by letting n go to infinity. ■

Proof of Theorem 1.6 Suppose φ is a strongly continuous (σ_h,β)-KMS state on $\mathrm{C}_u^*(X)$. Fix $x_0 \in X$ (this can be thought of as the "center" of X). Since all maps $(f_x\colon \{x_0\} \to \{x\})_{x\in X}$ are partial translations, the KMS condition gives us that

$$\varphi(e_{x,x}) = e^{-\beta(h(x)-h(x_0))}\varphi(e_{x_0,x_0})$$

for all $x \in X$ (see Theorem 2.5). As φ is strongly continuous,

$$1 = \varphi(\chi_X) = \sum_{x\in X}\varphi(e_{x,x}) = e^{\beta h(x_0)}\varphi(e_{x_0,x_0})\sum_{x\in X}e^{-\beta h(x)}.$$

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So, $\varphi(e_{x_0,x_0}) \neq 0$ and

$$Z(\beta) = \sum_{x\in X} e^{-\beta h(x)} = \frac{1}{e^{\beta h(x_0)}\varphi(e_{x_0,x_0})}$$

must be finite (as well as independent on x_0). The formula for φ in the statement of the theorem then follows immediately from the strong continuity of φ.

Suppose now $Z(\beta)$ is finite and φ is given as in the statement of the theorem. Clearly, φ is a strongly continuous state on $\mathrm{C}^*_u(X)$. Moreover, if $f\colon A \to B$ is a partial translation of X, then, by the formula of φ, we have

$$\begin{aligned}\varphi(\chi_{f(A)}) &= \frac{1}{Z(\beta)} \sum_{x\in f(A)} e^{-\beta h(x)}\\ &= \frac{1}{Z(\beta)} \sum_{x\in A} e^{-\beta h(f(x))}\\ &= \varphi(\chi_A e^{\beta(\overline{h-h\circ f})}).\end{aligned}$$

So, by Theorem 2.5, φ is a (σ_h,β)-KMS state on $\mathrm{C}^*_u(X)$. ∎

The following is a straightforward consequence of Proposition 2.9 and Theorem 1.6.

Corollary 2.10 *Let X be a u.l.f. metric space and $h\colon X \to [0,\infty)$ be an unbounded coarse map. The subset of all $\beta \in \mathbb{R}$ for which there are strongly continuous (σ_h,β)-KMS states on $\mathrm{C}^*_u(X)$ is either of the form (t,∞) or $[t,\infty)$ for some $t \geq 0$.*

Remark 2.11 Throughout these notes, we will see many examples for which the set of β's admitting are strongly continuous (σ_h,β)-KMS states on $\mathrm{C}^*_u(X)$ are of the form (β_0,∞) for some $\beta_0 > 0$. This could give the impression this must always be the case, however, this is not so. For instance, let $X = \{n \in \mathbb{N} \mid n \geq 3\}$ and let $h(x) = \log(x\log^2(x))$ for all $x \in X$ (the restriction of $x \geq 3$ is simply so that h is well defined). In this case,

$$\sum_{n=3}^{\infty} e^{-\beta h(x)} = \sum_{n=3}^{\infty} \frac{1}{x^\beta \log^{2\beta}(x)}$$

and this series converges if and only if $\beta \geq 1$.

2.5 The simplest coarse space

Under the optics of coarse geometry, the simplest infinite metric space is the *coarse disjoint union of singletons*; that is, any metric space which is bijectively coarsely equivalent to

$$X_0 = \left\{n^2 \in \mathbb{N} \mid n \in \mathbb{N}\right\},$$

where X_0 is endowed with the usual metric d on the natural numbers. In this subsection, we study KMS states on X_0. The simplicity of the geometry of X_0 makes

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any map $h\colon X_0 \to Y$, where Y is another metric space, be automatically coarse. Also, given any $r > 0$, there is a finite $F \subseteq X_0 \times X_0$ such that

$$\Big\{(x,y) \in X_0 \times X_0 \mid d(x,y) < r\Big\} = \Big\{(x,x) \in X_0 \times X_0 \mid x \in X_0\Big\} \cup F.$$

Therefore, it follows that

$$\mathrm{C}^*_u(X) = \ell_\infty(X) + \mathcal{K}(\ell_2(X)).$$

Proposition 2.12 *Let (X_0, d) be the coarse disjoint union of singletons described above. If φ is a state on $\ell_\infty(X_0)$ such that $\varphi \restriction c_0(X_0) = 0$, then $\varphi \circ E$ is a (σ_h, β)-KMS state on $\mathrm{C}^*_u(X_0)$ for all $h\colon X_0 \to \mathbb{R}$ and all $\beta \in \mathbb{R}$; where $E\colon \mathrm{C}^*_u(X_0) \to \ell_\infty(X_0)$ denotes the canonical conditional expectation.*

Proof Let $f\colon A \subseteq X_0 \to B \subseteq X_0$ be a partial translation. Then, there must be a partition $A = A_1 \sqcup A_2$ such that $f(x) = x$ for all $x \in A_1$ and $|A_2| < \infty$. As $\varphi \restriction c_0(X_0) = 0$, we have that

$$\varphi(\chi_{f(A)}) = \varphi(\chi_{f(A_1)} + \chi_{f(A_2)}) = \varphi(\chi_{f(A_1)}) = \varphi(\chi_{A_1}).$$

Similarly, we have

$$\varphi\Big(\chi_A e^{\beta(\overline{h - h\circ f})}\Big) = \varphi\Big(\chi_{A_1} e^{\beta(\overline{h - h\circ f})}\Big) = \varphi(\chi_{A_1}).$$

The result then follows from Theorem 2.5. ∎

Remark 2.13 Here is a more conceptual way of obtaining Proposition 2.12: notice that since $\mathrm{C}^*_u(X_0) = \ell_\infty(X_0) + \mathcal{K}(\ell_2(X_0))$, we must have $\mathrm{Q}^*_u(X_0) \cong \ell_\infty/c_0$; so, $\mathrm{Q}^*_u(X_0)$ is abelian. Moreover, as σ_h is the identity on $\ell_\infty(X_0)$, the flow σ_h^∞ induced by σ_h on $\mathrm{Q}^*_u(X_0)$ is trivial (see Section 1 for the definition of σ_h^∞). In particular, any state on $\mathrm{Q}^*_u(X_0)$ is KMS for any β. The result is then a corollary of Proposition 1.8.

We now restrict our study of KMS states on X_0 to a specific map h. This will allow us to find all KMS states on $\mathrm{C}^*_u(X_0)$ for the corresponding flow. For the sake of generality, we first isolate a result which does not depend on X being the coarse disjoint union of singletons per se.

Corollary 2.14 *Let d be any u.l.f. metric on $\mathbb{N}$ for which the map $h(x) = \log(x)$ is coarse and let $\beta \in \mathbb{R}$. If φ is a strongly continuous (σ_h, β)-KMS state on $\mathrm{C}^*_u(\mathbb{N}, d)$, then $\beta > 1$ and*

$$\varphi([a_{x,y}]) = \frac{1}{\sum_{n=1}^\infty \frac{1}{n^\beta}} \sum_{n=1}^\infty \frac{a_{x,x}}{n^\beta}, \tag{2.9}$$

for all $[a_{x,y}] \in \mathrm{C}^*_u(\mathbb{N}, d)$.

Proof This is a straightforward consequence of Theorem 1.6. ∎

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We can now describe the KMS states on X_0 completely with $h = \log$. Precisely:

Corollary 2.15 *Let $X_0 = \{n^2 \mid n \in \mathbb{N}\}$ be the coarse disjoint union of singletons described above, $\beta \in \mathbb{R}$, and $h\colon X \to \mathbb{R}$ be given by $h(x) = \log(\sqrt{x})$ for all $x \in X_0$. The (σ_h, β)-KMS states of $\mathrm{C}^*_u(X_0)$ are precisely the following:*

1. *Any state on $\mathrm{C}^*_u(X_0)$ which vanishes on $c_0(X_0)$,*
2. *If $\beta > 1$, then $\mathrm{C}^*_u(X_0)$ has a unique strongly continuous (σ_h, β)-KMS state and this state is given by*

$$\varphi([a_{x,y}]) = \frac{1}{\sum_{n=1}^{\infty} \frac{1}{n^\beta}} \sum_{n=1}^{\infty} \frac{a_{n^2,n^2}}{n^\beta},$$

   for all $[a_{x,y}] \in \mathrm{C}^*_u(X_0)$, and
3. *for $\beta > 1$, any convex combination of the states above.*

Proof This follows immediately from Propositions 2.12 and Corollary 2.14. ∎

3 Intermission

As seen in Theorem 1.6, strongly continuous KMS states on uniform Roe algebras are completely understood; so we are left to understand the strongly discontinuous case. In this section, before explicitly perusing this goal, we take a short break from uniform Roe algebras per se, and present some results about KMS states on arbitrary C*-algebras with respect to arbitrary flows. The technical results herein will be essential in the analysis to follow of KMS states on uniform Roe algebras which are strongly discontinuous.

We start by properly stating the settings of this section. But firstly, we recall some standard notation: if A is a C*-algebra, then $\mathcal{Z}(A)$ denotes the *center of* A, that is,

$$\mathcal{Z}(A) = \{b \in A \mid ab = ba,\ \forall a \in A\}.$$

Moreover, if K is a compact Hausdorff space, then $C(K)$ denotes the C*-algebra of all continuous functions $K \to \mathbb{C}$.

Assumption 3.1 Throughout this section, we fix a unital C*-algebra A, a flow σ on A, and $\beta \in \mathbb{R}$. Moreover, we fix a unital C*-subalgebra $C \subseteq A$ contained in $\mathcal{Z}(A)$, and identify C with $C(\Omega(C))$ via the Gelfand transform; here $\Omega(C)$ denotes the spectrum of C.

Proposition 3.2 *In the setting of Assumption 3.1: If φ is a (σ, β)-KMS state on A and $c \in A$ is a positive element in the center of A with $\varphi(c) \neq 0$, then the state φ_c on A defined by*

$$\varphi_c(a) = \frac{\varphi(ac)}{\varphi(c)}, \text{ for all } a \in A,$$

is a (σ, β)-KMS state on A.

Proof First notice that, as $c \in \mathcal{Z}(A)$, then ac is positive for all positive $a \in A$. Therefore, φ_c is indeed a state. Given $a, b \in A$, with b analytic, we have

$$\varphi_c(a\sigma_{i\beta}(b)) = \frac{\varphi(a\sigma_{i\beta}(b)c)}{\varphi(c)} = \frac{\varphi(ac\sigma_{i\beta}(b))}{\varphi(c)} = \frac{\varphi(bac)}{\varphi(c)} = \varphi_c(ba).$$

So, φ_c is a (σ, β)-KMS state on A. ∎

Proposition 3.3 *In the setting of Assumption 3.1: If φ is an extreme (σ, β)-KMS state on A, then there is $x \in \Omega(C)$ such that*

$$\varphi(a) = a(x) \ \textit{for all} \ \ a \in C = C(\Omega(C)).$$

In particular, letting

$$J_x = \{a \in C(\Omega(C)) \mid a(x) = 0\},$$

we have that $\varphi \restriction J_x = 0$.

Proof By Riesz representation theorem, there is a probability measure μ on $\Omega(C)$ such that

$$\varphi(a) = \int_{\Omega(C)} a d\mu \ \text{ for all } a \in C.$$

Let $K \subseteq \Omega(C)$ be the support of μ. Let us show that K is a singleton. In order to prove this, suppose by contradiction that there are two distinct points $x, y \in K$. By Urysohn's lemma, we can pick a positive $k \in C(\Omega(C))$ with $\|k\| \leq 1$ and such that $k(x) = 1$ and $k(y) = 0$. Setting $\ell = 1 - k$, we have that both k and ℓ are not identically zero on K, so both $\varphi(k)$ and $\varphi(\ell)$ are nonzero. By Proposition 3.2, φ_k and φ_ℓ are (σ, β)-KMS states on A, and it is clear that

$$\varphi = \lambda\varphi_k + (1 - \lambda)\varphi_\ell,$$

where $\lambda = \varphi(k)$. Since $\varphi_k \neq \varphi_\ell$, this contradicts the assumption that φ is an extreme (σ, β)-KMS state. So, K contains only one point, say $K = \{x\}$. Therefore, μ must be the dirac measure on $\{x\}$, which gives that

$$\varphi(a) = a(x), \text{ for all } a \in C(\Omega(C)).$$

The last claim follows straightforwardly from the above. ∎

Definition 3.4 In the setting of Assumption 3.1:

1. We denote the set of all (σ, β)-KMS states on A by K_β.
2. For each $x \in \Omega(C)$, let

$$J_x = \{a \in C = C(\Omega(C)) \mid a(x) = 0\} \text{ and } K_\beta^x = \{\varphi \in K_\beta \mid \varphi \restriction J_x = 0\}.$$

It is plainly clear that each K_β^x is a weak*-closed convex subset of K_β.

Recall that if C is a convex subset in a vector space, and $E \subseteq C$ is convex, then E is said to be an *extreme subset* of C if, for any pair of points x and y in C, such that

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$\lambda x + (1-\lambda)y \in E$, with $0 < \lambda < 1$, one has that both x and y lie in E. For example, if x is an extreme point of C, then $\{x\}$ is an extreme subset of C.

Proposition 3.5 *In the setting of Assumption 3.1: For all $x \in \Omega(C)$, one has that K_β^x is an extreme subset of K_β.*

Proof Pick $\varphi \in K_\beta^x$ and assume that

$$\varphi = \lambda\varphi_1 + (1-\lambda)\varphi_2,$$

where $\varphi_1, \varphi_2 \in K_\beta$ and $\lambda \in (0,1)$. Denoting by ψ, ψ_1, and ψ_2 the restrictions of φ, φ_1, and φ_2 to C, respectively, it is apparent that

$$\psi = \lambda\psi_1 + (1-\lambda)\psi_2.$$

By Proposition 3.3, ψ is a character of $C = C(\Omega(C))$. Hence, ψ is an extreme point of the unit ball of the dual of C. This shows that $\psi = \psi_1 = \psi_2$, which in turn implies that both φ_1 and φ_2 vanish on J_x. Therefore, $\varphi_1, \varphi_2 \in K_\beta^x$ as desired. ∎

We can now present the main result of this section. In it, $\mathrm{Ext}(K_\beta)$ (resp. $\mathrm{Ext}(K_\beta^x)$) denotes the subset of all extreme elements of K_β (resp. $\mathrm{Ext}(K_\beta^x)$).

Theorem 3.6 *In the setting of Assumption 3.1: We have*

$$\mathrm{Ext}(K_\beta) = \bigsqcup_{x\in\Omega(C)} \mathrm{Ext}(K_\beta^x).$$

Moreover, if there is a (σ,β)-KMS state on A whose restriction to C is faithful, then $K_\beta^x \neq \varnothing$ for all $x \in \Omega(C)$. In particular, if such KMS state exists, we have that

$$|\mathrm{Ext}(K_\beta)| \geq |\Omega(C)|.$$

Proof By Proposition 3.3, every extreme point φ of K_β lies in some K_β^x and, in this case, φ is evidently an extreme point of K_β^x. Conversely, as each K_β^x is an extreme subset of K_β (Proposition 3.5), every extreme point of any K_β^x is an extreme point of K_β.

Suppose now that there is a (σ,β)-KMS state φ on A whose restriction to C is faithful. Fix $x \in \Omega(C)$ and let us show $K_\beta^x \neq \varnothing$. Let $\mathcal{V}$ be the family of all open subsets of $\Omega(C)$ which contain x and, for each $V \in \mathcal{V}$, let $h_V\colon \Omega(C) \to [0,1]$ be a continuous function such that $h_V(x) = 1$ and $h_V(y) = 0$ for all $y \notin V$. By the faithfulness of φ, $\varphi(h_V) \neq 0$ for all $V \in \mathcal{V}$. Therefore, by Proposition 3.2, each $\varphi_V = \varphi_{h_V}$ is a (σ,β)-KMS state on A.

Consider $\mathcal{V}$ as a directed set with the usual reverse containment order. By Banach-Alaoglu theorem, K_β is weak*-compact. Hence, by passing to a subset if necessary, we can assume that $(\varphi_V)_{V\in\mathcal{V}}$ converges to some $\psi \in K_\beta$ in the weak*-topology. As ψ is a limit of $(\varphi_V)_{V\in\mathcal{V}}$ and as $\lim_{V,\mathcal{V}} \|ah_V\| = 0$, for all $a \in J_x$, the state ψ must vanish on J_x. This shows that $\varphi \in K_\beta^x$ and K_β^x cannot be empty as desired.

The last claim is a straightforward consequence of the above. ∎

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4 Factoring KMS states through the uniform Roe corona

In this section, we return to the setting of uniform Roe algebras and study strongly discontinuous KMS states (the strongly continuous case was already completely treated in Theorem 1.6). We start noticing that, in order to study such states, it is enough to study the KMS states which vanish on the ideal of compact operators. Precisely:

Proposition 4.1 *Let X be a u.l.f. metric space, $h\colon X \to \mathbb{R}$ be coarse, and $\beta \in \mathbb{R}$. Suppose φ is a (σ_h, β)-KMS state on $\mathrm{C}^*_u(X)$ and define a positive functional ψ on $\mathrm{C}^*_u(X)$ by letting*

$$\psi(a) = \lim_{F,\mathcal{F}} \sum_{x \in F} a_{x,x}\varphi(e_{x,x}) \text{ for all } a = [a_{x,y}] \in \mathrm{C}^*_u(X),$$

where $\mathcal{F}$ is the net of all finite subsets of X ordered by reverse inclusion. Then, ψ is well defined and

1. *ψ is strongly continuous and satisfies the (σ_h, β)-KMS condition, and*
2. *$\varphi - \psi$ is a positive functional which satisfies the (σ_h, β)-KMS condition and vanishes on $\mathcal{K}(\ell_2(X))$.*

Proof The fact that ψ is well defined follows straightforwardly from the fact that φ is positive and factors through $\ell_\infty(X)$ (Theorem 1.5). Positivity and strong continuity of ψ are then completely immediate. It is also immediate that $\psi \le \varphi$, so $\varphi - \psi$ is also positive. Since $\psi \restriction \mathcal{K}(\ell_2(X)) = \varphi \restriction \mathcal{K}(\ell_2(X))$, $\varphi - \psi$ vanishes on the compacts. We are only left to show that both ψ and $\varphi - \psi$ satisfy the (σ, β)-KMS condition. But this is an immediate consequence of Theorem 2.5 and the formula of ψ. ∎

Theorem 1.6 and Proposition 4.1 show that, in order to understand the KMS states on uniform Roe algebras, we only need to focus of the states which vanish on the ideal of compact operators. For the remainder of this section, this will be our focus. Since the compacts form an ideal, we can factor those states through the quotient algebra. For that, recall that the *uniform Roe corona of X* is

$$\mathrm{Q}^*_u(X) = \mathrm{C}^*_u(X)/\mathcal{K}(\ell_2(X))$$

(see Definition 1.7). If φ is a state on $\mathrm{C}^*_u(X)$ which vanishes on $\mathcal{K}(\ell_2(X))$, then φ gives rise to a well-defined state ψ on $\mathrm{Q}^*_u(X)$ determined by

$$\psi(\pi(a)) = \varphi(a), \text{ for all } a \in \mathrm{C}^*_u(X).$$

Moreover, given a coarse map $h\colon X \to \mathbb{R}$, the flow σ_h induces a flow σ_h^∞ on $\mathrm{Q}^*_u(X)$ by letting

$$\sigma^\infty_{h,t}(\pi(a)) = \pi(\sigma_{h,t}(a)) \text{ for all } a \in \mathrm{C}^*_u(X) \text{ and all } t \in \mathbb{R}$$

(see Section 1.3 for more details).

Proposition 1.8 highlights the relations between φ and ψ, and σ_h and σ_h^∞ defined above.

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Proof of Proposition 1.8. Notice that if b is an analytic element in $\mathrm{C}^*_u(X)$ for σ_h, then $\pi(b)$ is analytic for σ_h^∞ and, moreover,

$$\pi(\sigma_{h,z}(b)) = \sigma^\infty_{h,z}(\pi(b)) \text{ for all } z \in \mathbb{C}.$$

Therefore, the image of the set of all analytic elements in $\mathrm{C}^*_u(X)$ under π forms a dense set of analytic elements in $\mathrm{Q}^*_u(X)$. Consequently, in order to check that a state ψ on $\mathrm{Q}^*_u(X)$ is a (σ^∞_h, β)-KMS state, it suffices to prove that

$$\psi(\pi(a)\sigma^\infty_{h,i\beta}(\pi(b))) = \psi(\pi(b)\pi(a)),$$

for all $a, b \in A$ with b analytic. Observing that the left-hand-side above coincides with $(\psi \circ \pi)(a\sigma_{h,i\beta}(b))$ and that the right-hand-side equals $(\psi \circ \pi)(ba)$, the first statement of the proposition follows. The second statement in turn follows from the first one immediately. ∎

Proposition 1.8 then reduces our problem to the one of understanding the KMS states on the uniform Roe corona $\mathrm{Q}^*_u(X)$. In view of Section 3, it will be useful to study the center $\mathrm{Q}^*_u(X)$ as well as its C^*-subalgebras. This brings up a seemingly unexpected link between KMS states and the *Higson corona*. Recall:

Definition 4.2 Let X be a u.l.f. metric space.

1. A bounded function $f\colon X \to \mathbb{C}$ is a *Higson function* if for all $\varepsilon > 0$ and all $R > 0$ there is a finite $F \subseteq X$ such that

   $$\forall x, y \in X \smallsetminus F,\ d(x, y) < R \text{ implies } |f(x) - f(y)| < \varepsilon.$$

   The set of all Higson functions on X forms a C^*-subalgebra of $\ell_\infty(X)$ which we denote by $C_h(X)$.
2. The spectrum of $C_h(X)$, denoted by hX, is called the *Higson compactification* of X. So, the Gelfand transform gives us the identification $C(hX) = C_h(X)$.
3. The boundary $\nu X = hX \smallsetminus X$ is called the *Higson corona* and we have the identification $C(\nu X) = C_h(X)/c_0(X)$.

Notice that, as $C_h(X) \subseteq \ell_\infty(X)$, we may canonically view $C_h(X)/c_0(X)$ as a C^*-subalgebra of $\mathrm{Q}^*_u(X)$; so, by the identification $C(\nu X) = C_h(X)/c_0(X)$, we have

$$C(\nu X) \subseteq \mathrm{Q}^*_u(X).$$

It has been recently observed that the center of $\mathrm{Q}^*_u(X)$ is precisely the Higson corona of X. Indeed, the following was proven in [BBF+22a, Proposition 3.6] as a consequence of [ŠZ20, Theorem 3.3].

Proposition 4.3 *Given a u.l.f. metric space X, we have that*

$$C(\nu X) = \mathcal{Z}(\mathrm{Q}^*_u(X)).$$

We now apply our results of Section 3 to our coarse setting. In what follows, if C is a unital C^*-algebra, $\Omega(C)$ denotes the spectrum of C. So, $\Omega(C)$ is a compact Hausdoff

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topological space and we use the identification $C = C(\Omega(C))$ given by the Gelfand transform.

Theorem 4.4 *Let X be a u.l.f. metric space, $h\colon X \to \mathbb{R}$ be a coarse map, and $\beta \in \mathbb{R}$. Let C be a unital C^*-subalgebra of $C(\nu X)$.*

1. *For any extreme (σ_h^∞, β)-KMS state ψ on $\mathrm{Q}_u^*(X)$, there is $x \in \Omega(C)$ such that*
$$\psi(a) = a(x) \ \textit{for all} \ a \in C = C(\Omega(C)).$$
2. *Suppose there is a (σ_h^∞, β)-KMS state on $\mathrm{Q}_u^*(X)$ whose restriction to C is faithful. Then, for any $x \in \Omega(C)$, there is an extreme (σ_h^∞, β)-KMS state ψ on $\mathrm{Q}_u^*(X)$ such that*
$$\psi(a) = a(x) \ \textit{for all} \ a \in C = C(\Omega(C)).$$

Proof This is a mere corollary of Theorem 3.6. ■

Proof of Theorem 1.9. This is a particular case of Theorem 4.4 with $C = C(\nu X)$. ■

We now obtain Theorem 1.11 by proving a more general version of it. For that, we first generalize Definition 1.10.

Definition 4.5 Let X be a u.l.f. metric space and $\bar{X}$ be a compactification of X.

1. We call $\bar{X}$ *Higson compatible* if
$$f \restriction X \in C_h(X) \ \text{for all} \ f \in C(\bar{X}).$$
2. If $\bar{X}$ is Higson compatible and $x \in \bar{X}$, we say that a state φ on $\mathrm{C}_u^*(X)$ is $\bar{X}$-*supported on* x if for all neighborhoods $U \subseteq \bar{X}$ of x, we have $\varphi(\chi_{U\cap X}) = 1$.

Notice that if $\bar{X}$ is a Higson compatible compactification of X, then $C(\bar{X})$ can be canonically identified with a C^*-subalgebra of $C_h(X)$, which in turn allows us to identify $C(\bar{X})/c_0(X)$ with a C^*-subalgebra of $C(\nu X) \subseteq \mathrm{Q}_u^*(X)$.[7]

Theorem 4.6 *Let X be a u.l.f. metric space, $h\colon X \to \mathbb{R}$ be a coarse map, and $\beta \in \mathbb{R}$. Let $\bar{X}$ be a Higson compatible compactification of X. The following holds:*

1. *Any extreme (σ_h, β)-KMS state on $\mathrm{C}_u^*(X)$ which vanishes on the compacts is $\bar{X}$-supported at some element of $\bar{X}$.*
2. *If there is a (σ_h, β)-KMS state on $\mathrm{C}_u^*(X)$ which vanishes on the compacts and such that its induced state on $\mathrm{Q}_u^*(X)$ is faithful on $C(\bar{X})/c_0(X)$, then for every $x \in \bar{X}$ there is a (σ_h, β)-KMS state on $\mathrm{C}_u^*(X)$ which is $\bar{X}$-supported on x.*

Proof This is a mere corollary of Proposition 1.8 and Theorem 4.4. ■

Proof of Theorem 1.11. This is a particular case of Theorem 4.6 with $\bar{X} = hX$. ■

[7] This should be compared with the *coarse compactification* of a coarse space; see [Roe03, Definition 2.38 and Proposition 2.39].

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4.1 The size of the Higson corona

We show that the Higson corona of an infinite u.l.f. metric space must always have $2^{2^{\aleph_0}}$ many elements (Theorem 4.14). Together with the previous results in this section, this will give us a very strong control of the cardinality of KMS states on $C^*_u(X)$.

In this subsection, we work a lot with partial bijections f of X and it will be useful to be able to write "$f(A)$" regardless of whether $A \subseteq \mathrm{Dom}(f)$. We then define: given any set X, a partial bijection $f\colon \mathrm{Dom}(f) \to \mathrm{Im}(f)$ of X, and $A \subseteq X$, we let

$$f[A] = f(A \cap \mathrm{Dom}(f)).$$

Also, given partial bijections f and g of X, we let $g \circ f$ be the partial bijection from $f^{-1}[\mathrm{Dom}(g)]$ to $g[\mathrm{Im}(f)]$ defined by $g \circ f(x) = g(f(x))$ for all $x \in f^{-1}[\mathrm{Dom}(g)]$.

The following lemma is an easy exercise and we leave the details to the reader.

Lemma 4.7 *Let f and g be partial bijections of X. Then*

$$f[A] \cap g[B] = g((g^{-1} \circ f)[A] \cap B)$$

for all $A, B \subseteq X$.

Definition 4.8 Let X be a u.l.f. metric space. A subset $A \subseteq X$ is *thin* if $f[A] \cap A$ is finite for all partial translations f of X which *do not fix points*, that is, such that $f(x) \neq x$ for all $x \in \mathrm{Dom}(f)$.

Lemma 4.9 *Every infinite u.l.f. metric space contains an infinite thin subset.*

Proof If (X, d) is infinite and u.l.f., then X is unbounded. Hence, we can inductively pick a sequence $(x_i)_{i\in\mathbb{N}}$ in X such that

$$d(x_k, x_\ell) \geq \max_{i,j<\ell} d(x_i, x_j) + \ell$$

for all $\ell > k$ in $\mathbb{N}$. The set $A = \{x_i \mid i \in \mathbb{N}\}$ is clearly thin.[8] ■

Proposition 4.10 *Let X be a u.l.f. metric space, $C \subseteq X$ be thin, and let $C = A \sqcup B$ be a partition of C. If f and g are partial translations of X, then $f[A] \cap g[B]$ is finite.*

Proof By Lemma 4.7, it is enough to show that $(g^{-1} \circ f)[A] \cap B$ is finite. As the composition of partial translations is still a partial translation, it is enough to show that $f[A] \cap B$ is finite for any partial translation f of X. Fix such f and, replacing A with $A \cap \mathrm{Dom}(f)$, we also assume that $A \subseteq \mathrm{Dom}(f)$. Let us show that $f(A) \cap B$ is finite. Set

$$A_0 = \{x \in A \mid f(x) = x\} \text{ and } A_1 = A \setminus A_0.$$

Then, as $A \cap B = \emptyset$, we have

$$f(A) \cap B = f(A_0 \cup A_1) \cap B = f(A_1) \cap B.$$

[8] Equivalently, if $A \subseteq X$ is the image of a coarse embedding of $\{n^2 \mid n \in \mathbb{N}\}$ in X, then A is thin.

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Let $f_1 = f \restriction A_1$. Then f_1 has no fixed points and

$$f_1(A) \cap B \subseteq f_1[C] \cap C.$$

Since C is thin, $f_1[C] \cap C$ must be finite. So, $f(A) \cap B$ is finite. ∎

Given a u.l.f. metric space X, let βX denote the *Stone–Čech compactification of* X.[9] Since X is discrete, βX can be identified with the space of ultrafilters on X endowed with the *Stone topology*, that is, the topology generated by open sets of the form

$$U_A = \{\omega \in \beta X \mid A \in \omega\}$$

for all $A \subseteq X$. Given $A \subseteq X$, we let $\bar{A}$ denote the closure of A in βX and let $\hat{A} = \bar{A} \setminus A$. By the defining property of βX, any element in $\ell_\infty(X)$ extends to one in $C(\beta X)$. This defines a canonical isomorphism between $\ell_\infty(X)$ and $C(\beta X)$, and we identify those algebras under this isomorphism. We identify $C(\hat{X})$ with $C(\beta X)/c_0(X)$ via Gelfand transform. Hence, under these identifications, we have

$$C(\hat{X}) = \ell_\infty(X)/c_0(X) \subseteq \mathrm{Q}^*_u(X).$$

We now define invariant subsets of the Stone–Čech compactification. For that, recall that, by the defining property of βX, any partially defined map $f\colon \mathrm{Dom}(f) \subseteq X \to \mathrm{Im}(f) \subseteq X$ can be continuously extended to a (necessarily surjective) map $\overline{\mathrm{Dom}(f)} \to \overline{\mathrm{Im}(f)}$. By abuse of notation, we still denote this map by f.

Definition 4.11 Let X be a u.l.f. metric space and $A \subseteq \beta X$. We say that A is *invariant* if $f[A] \subseteq A$ for all partial translations f of X.

For the next lemma, notice that if $L \subseteq \hat{X}$ is a clopen subset, then $\chi_L \in C(\hat{X})$. Hence, it makes sense to wonder whether χ_L can also be in $C(\nu X) \subseteq C(\hat{X})$.

Lemma 4.12 *Let X be a u.l.f. metric space and $L \subseteq \hat{X}$ be an invariant clopen subset. Then $\chi_L \in C(\nu X)$.*

Proof By Proposition 4.3, it is enough to notice that χ_L is in the center of $\mathrm{Q}^*_u(X)$. Hence, since $\mathrm{C}^*_u[X]$ is dense in $\mathrm{C}^*_u(X)$ and spanned by av_f, where $a \in \ell_\infty(X)$ and f is a partial translation of X, we only need to show that χ_L commutes with $w_f = \pi(v_f)$ for all partial translations f of X. Fix such partial translation f and let $A = \mathrm{Dom}(f)$ and $B = \mathrm{Im}(f)$. Then, $w_f = \chi_{\hat{B}} w_f \chi_{\hat{A}}$ and

$$w_f \chi_L = w_f \chi_{\hat{A} \cap L} = \chi_{\hat{B} \cap f[L]} w_f = \chi_{f[L]} w_f \tag{4.1}$$

notice that $f[L] = \hat{B} \cap L$. Indeed, since L is invariant and f is a partial translation, $f[L] \subset \hat{B} \cap L$. On the other hand, as f^{-1} is also a partial translation, we have $f^{-1}[L] \subseteq L$. Hence, as $\hat{B} \cap L \subseteq f[f^{-1}[L]]$, we also have $\hat{B} \cap L \subseteq f[L]$. We can then conclude from (4.1) that $w_f \chi_L = \chi_L w_f$. As the partial translation f was arbitrary, we conclude that $\chi_L \in C(\nu X)$ as desired. ∎

[9] Please be careful not to mistake this β for the inverse temperature!

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Lemma 4.13 *Let X be a u.l.f. metric space and $C \subseteq X$ be thin. If $\omega, \omega' \in \hat{C}$ are distinct, then there are disjoint invariant open subsets $U, V \subseteq \hat{X}$ such that $\omega \in U$ and $\omega' \in V$.*

Proof Since $\omega, \omega' \in \bar{C}$, it follows that $C \in \omega$ and $C \in \omega'$. As $\omega \neq \omega'$, there is $D \subseteq X$ such that $D \in \omega$ and $D \notin \omega'$. Hence,

$$A = C \cap D \in \omega \ \text{ and } \ B = C \setminus D \in \omega'.$$

Therefore, $\omega \in \hat{A}$ and $\omega' \in \hat{B}$. Let $\mathcal{PT}$ denote the set of all partial translations of X and define

$$U = \bigcup_{f \in \mathcal{PT}} \widehat{f[A]} \ \text{ and } \ V = \bigcup_{f \in \mathcal{PT}} \widehat{f[B]}.$$

Clearly, U and V are open, invariant and contain ω and ω', respectively. We only need to notice they are also disjoint. For that, notice that Proposition 4.10 implies that $f[A] \cap g[B]$ is finite for all $f, g \in \mathcal{PT}$. But then $\widehat{f[A]} \cap \widehat{g[B]} = \varnothing$ for all $f, g \in \mathcal{PT}$, which in turn implies that $U \cap V = \varnothing$. ∎

Theorem 4.14 *Let X be an infinite u.l.f. metric space. Then, νX has at least $2^{2^{\aleph_0}}$ elements.*

Proof Let $p: \hat{X} \to \nu X$ be the continuous surjection such that the canonical identification of $C(\nu X)$ with a C*-subalgebra of $C(\hat{X})$ is given by the map

$$a \in C(\nu X) \mapsto a \circ p \in C(\hat{X}).$$

Let $C \subseteq X$ be an infinite thin subset given by Lemma 4.9. As $\hat{C}$ is the set of all nonprincipal ultrafilters on C and C is countable, we have that $|\hat{C}| = 2^{2^{\aleph_0}}$. Therefore, in order to obtain that νX has $2^{2^{\aleph_0}}$ elements, it is enough to show that p is injective on $\hat{C}$.

Let $\omega, \omega' \in \hat{C}$ be distinct. By Lemma 4.13, there are disjoint invariant open subsets $U, V \subseteq \hat{X}$ containing ω and ω', respectively. As βX is extremely disconnected, $\bar{U}$ is clopen in $\hat{X}$ which implies that the characteristic function of $\bar{U}$, $\chi_{\bar{U}}$, is a continuous function in $C(\hat{X})$. As $\bar{U}$ is invariant, Lemma 4.12 shows that $\chi_{\bar{U}} \in C(\nu X)$. Therefore, since we clearly have $\chi_{\bar{U}}(\omega) = 1$ and $\chi_{\bar{U}}(\omega') = 0$, this shows that $p(\omega) \neq p(\omega')$. ∎

Remark 4.15 We would like to observe that Theorem 4.14 is only valid for *metric* u.l.f. spaces. Precisely, Higson coronas can be defined more generally for *coarse spaces* — for brevity, we do not define coarse spaces here, the reader can find the precise definition in [Roe03] or [BBF+22b, Section 5]. It is known that every perfectly normal compact Hausdorff space is homeomorphic to the Higson corona of some u.l.f. coarse space (see [BP20, p. 2]). It is however not surprising that the Higson corona of nonmetrizable u.l.f. coarse spaces can be much smaller since there will be fewer Higson functions in this case. The proof of Theorem 4.14 cannot hold outside the metrizable world since thin sets may not exist. For instance, if $\mathcal{E}_{\max}$ is the maximal u.l.f. coarse structure on an infinite set X (see [BBF+22b, Subsection 1.3] for the precise definition), then it is clear that $(X, \mathcal{E}_{\max})$ has no infinite thin subsets.

Corollary 4.16 *Let X be an infinite u.l.f. metric space, $h\colon X \to \mathbb{R}$ be a coarse map, and $\beta \in \mathbb{R}$. If there is a (σ_h^∞, β)-KMS state on $\mathrm{Q}_u^*(X)$ whose restriction to $C(\nu X)$ is faithful, then there are $2^{2^{\aleph_0}}$ extreme (σ_h^∞, β)-KMS states on $\mathrm{Q}_u^*(X)$. In particular, there are $2^{2^{\aleph_0}}$ extreme (σ_h, β)-KMS states on $\mathrm{C}_u^*(X)$ which vanish on $\mathcal{K}(\ell_2(X))$.*

Proof The statement for the uniform Roe corona follows from Theorems 4.4 and 4.14. The statement for the uniform Roe algebra is then a consequence of Proposition 1.8. ∎

5 Applications: Branching trees

In this section, we apply the theory of KMS states on uniform Roe algebras developed above to n-branching trees. Recall that, as mentioned in the introduction, the choice for those spaces are, in a sense, very natural. Precisely, as explained in Section 1.4, as long as $h\colon X \to \mathbb{R}$ is such that $h(x)$ is bounded below by an affine map in terms of $d(x, x_0)$ for a given $x_0 \in X$, there will always be (σ_h, β)-KMS states on $\mathrm{C}_u^*(X)$ for all $\beta > 0$ as long as X has polynomial growth. Therefore, in order to find more interesting phase transitions, it is natural to look at metric spaces with exponential growth.

5.1 *n*-branching trees

Given a set Γ, we let $\Gamma^{<\infty}$ be the set of all finite words on Γ, including the empty word; which we denote by $\varnothing$. In other words, if $\gamma \in \Gamma^{<\infty}$, then either $\gamma = \varnothing$ or $\gamma = (\gamma_1, \ldots, \gamma_n)$ for some $n \in \mathbb{N}$ and some $\gamma_1, \ldots, \gamma_n \in \Gamma$. Given $\gamma \in \Gamma^{<\infty}$, if $\gamma = \varnothing$, we say that the *length of* γ is 0, if $\gamma = (\gamma_1, \ldots, \gamma_n)$, we say that the *length of* γ is n; either way, we denote the length of γ by $|\gamma|$ and we write $\gamma = (\gamma_1, \ldots, \gamma_{|\gamma|})$ (here it is understood that if $|\gamma| = 0$, then $\gamma = \varnothing$). Given $\gamma, \gamma' \in \Gamma^{<\infty}$ we denote the *concatenation of* γ *and* γ' by $\gamma^\frown\gamma'$, that is,

$$\gamma^\frown\gamma' = (\gamma_1, \ldots, \gamma_{|\gamma|}, \gamma'_1, \ldots, \gamma'_{|\gamma'|}).$$

Definition 5.1 Let $n \in \mathbb{N}$ and consider $\Gamma = \{1, \ldots, n\}$. We make $\Gamma^{<\infty}$ into a graph by saying that any two distinct elements $\gamma, \gamma' \in \Gamma^{<\infty}$ are adjacent if there is $k \in \Gamma$ such that either $\gamma_1 = \gamma_2^\frown k$ or $\gamma_2 = \gamma_1^\frown k$. This defines a graph structure on $\Gamma^{<\infty}$ making it into a connected (undirected) graph. We can then see $\Gamma^{<\infty}$ as a metric space endowed with the shortest path distance. We call this metric space the *n-branching tree* and denote it by T_n.

For simplicity, we now isolate the setting of this subsection.

Assumption 5.2 Let $n \in \mathbb{N}$ and let T_n be the n-branching tree endowed with the shortest path metric, denoted by d. Let $h\colon T_n \to \mathbb{R}$ be the function given by $h(x) = d(x, \varnothing)$ for all $x \in T_n$

5.2 Strongly continuous KMS states on $\mathrm{C}_u^*(T_n)$

We start with a simple lemma about states on ℓ_∞. In the next lemma, $\ell_\infty = \ell_\infty(\mathbb{N})$ and $c_0 = c_0(\mathbb{N})$.

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Lemma 5.3 *Let φ be a state on ℓ_∞. If $\varphi \restriction c_0$ has norm 1, then φ is strongly continuous.*

Proof For each j, denote by $b_j = \varphi(\chi_{\{j\}})$. Then, as $\varphi \restriction c_0$ has norm 1,

$$\sum_{j=1}^{\infty} b_j = \lim_k \sum_{j=1}^{k} b_j = \lim_k \varphi(\chi_{\{1,\dots,k\}}) = 1.$$

Let a be a positive element in ℓ_∞ with norm at most 1. Then, as φ is positive, we have that

$$\sum_{j=1}^{k} a_j b_j = \varphi(a\chi_{\{1,\dots,k\}}) \leq \varphi(a) \text{ for all } k \in \mathbb{N}.$$

So, upon taking the limit as $k \to \infty$, we get

$$\sum_{j=1}^{\infty} a_j b_j \leq \varphi(a).$$

Applying the same reasoning to $1 - a$ we deduce that

$$\varphi(a) = 1 - \varphi(1-a) \leq 1 - \sum_{j=1}^{\infty}(1-a_j)b_j = \sum_{j=1}^{\infty} a_j b_j.$$

Hence,

$$\varphi(a) = \sum_{j=1}^{\infty} a_j b_j.$$

Now, for an arbitrary $a \in \ell_\infty$, splitting a into its real and imaginary parts, and splitting each such parts into their positive and negative parts, the previous paragraph imply that $\varphi(a) = \sum_{j=1}^{\infty} a_j b_j$, so the lemma follows. ■

The next result is a partial version of Theorem 1.12.

Theorem 5.4 *In the setting of Assumption 5.2: Given $\beta \in \mathbb{R}$, there is a (σ_h, β)-KMS state on $\mathrm{C}^*_u(T_n)$ if and only if $\beta \geq \log(n)$. Moreover,*

1. *For $\beta > \log(n)$, there is a unique (σ_h, β)-KMS state φ_β on $\mathrm{C}^*_u(T_n)$ and φ_β is given by*
$$\varphi_\beta([a_{x,y}]) = \sum_{y \in T_n} a_{y,y}\left(e^{-\beta|y|} - ne^{-\beta(|y|+1)}\right)$$
for all $[a_{x,y}] \in \mathrm{C}^*_u(T_n)$.
2. *For $\beta = \log(n)$, the (σ_h, β)-KMS states on $\mathrm{C}^*_u(T_n)$ all vanish on $\mathcal{K}(\ell_2(T_n))$.*

Proof Suppose φ is a (σ_h, β)-KMS state on $\mathrm{C}^*_u(T_n)$. Notice that, for each $y \in T_n$, the map $f\colon T_n \to T_n$ given by $f(x) = x^\frown y$, for all $x \in T_n$, is a partial translation; indeed, $d(x, f(x)) = |y|$ for all $x \in T_n$. So, each v_f belongs to $\mathrm{C}^*_u[T_n]$. Then, for each $y \in T_n$, we have

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$$\begin{aligned}\sigma_{h,i\beta}(v_f^*) &= e^{-\beta\bar{h}}v_f^*e^{\beta\bar{h}}\\ &= e^{-\beta\bar{h}}\Big(\text{SOT-}\sum_{x\in X}e_{x,x^\frown y}\Big)e^{\beta\bar{h}}\\ &= e^{\beta|y|}v_f^*.\end{aligned}$$

For each $y\in T_n$, set

$$T_n^\frown y=\{x\in T_n\mid x=z^\frown y\text{ for some }z\in T_n\}.$$

Hence, as $\chi_{T_n}=v_f^*v_f$ and $\chi_{T_n^\frown y}=v_fv_f^*$, we must have

$$1=\varphi(\chi_{T_n})=\varphi(v_f^*v_f)=\varphi(v_f\sigma_{h,i\beta}(v_f^*))=e^{\beta|y|}\varphi(\chi_{T_n^\frown y})$$

for all $y\in T_n$; which implies

$$\varphi(\chi_{T_n^\frown y})=e^{-\beta|y|}\ \text{ for all }\ y\in T_n. \tag{5.1}$$

Since for each $y\in T_n$, we have

$$\{y\}=T_n^\frown y\smallsetminus\bigsqcup_{k=1}^{n}T_n^\frown k^\frown y,$$

where "$\bigsqcup$" denotes disjoint union, (5.1) implies that

$$\varphi(e_{y,y})=\varphi\Big(\chi_{T_n^\frown y}-\sum_{k=1}^{n}\chi_{T_n^\frown k^\frown y}\Big)=e^{-\beta|y|}-ne^{-\beta(|y|+1)} \tag{5.2}$$

for all $y\in T_n$.

As φ is positive, each $\varphi(e_{y,y})$ must be positive. So, (5.2) gives that

$$e^{-\beta|y|}\geq ne^{-\beta(|y|+1)}\ \text{ for all }\ y\in T_n.$$

Solving for β, this implies $\beta\geq\log(n)$. Moreover, as (5.2) must hold regardless of β, this also shows that the $(\sigma_h,\log(n))$-KMS states on $\mathrm{C}_u^*(T_n)$ all vanish on $c_0(T_n)$. Since such states factors through $\ell_\infty(T_n)$ (Theorem 1.5), (2) follows.

We must now show that if $\beta\geq\log(n)$, then (σ_h,β)-KMS states exist. This will however be an immediate consequence of (1). Indeed, the set of all β's for with (σ_h,β)-KMS states exist is always a closed set (see [BR97, Proposition 5.3.25]).

We now show (1) holds. For this, suppose $\beta>\log(n)$ and let us show that any given (σ_h,β)-KMS state φ must have the required form. Notice that $\varphi\restriction\ell_\infty(T_n)$ is a state on $\ell_\infty(T_n)$. Moreover, the computations above show that

$$\varphi(a)=\sum_{y\in T_n}a_y\Big(e^{-\beta|y|}-ne^{-\beta(|y|+1)}\Big) \tag{5.3}$$

for all $a=(a_y)_y\in c_0(T_n)$. Hence, an easy computation gives

$$\lim_{F,\mathcal{F}}\varphi(\chi_F)=1,$$

where $\mathcal{F}$ is the net of all finite subsets of T_n ordered by reverse inclusion. Therefore, it follows that $\|\varphi\restriction c_0(T_n)\|=1$ and, by Lemma 5.3, $\varphi\restriction\ell_\infty$ is strongly continuous. This implies that (5.3) holds for all $a=(a_y)_y\in\ell_\infty(T_n)$. In order to notice that this

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holds for arbitrary elements of $\mathrm{C}_u^*(T_n)$, let $E\colon \mathrm{C}_u^*(X) \to \ell_\infty(X)$ be the canonical conditional expectation and recall that, by Theorem 1.5, we have $\varphi = \varphi \circ E$. This proves the uniqueness part of (1).

We are left to notice that a φ given by the formula above is indeed a (σ_h, β)-KMS state on $\mathrm{C}_u^*(T_n)$. This will be done by using Theorem 2.5.[10] So, let $f\colon A \to f(A)$ be a partial translation on X. On one hand, we have that

$$\varphi(\chi_{f(A)}) = \sum_{y \in f(A)} \left(e^{-\beta|y|} - n e^{-\beta(|y|+1)}\right).$$

On the other hand,

$$\begin{aligned}\varphi\left(\chi_A e^{\beta(\overline{h - h\circ f})}\right) &= \sum_{x\in A} e^{\beta(|x|-|f(x)|)}\left(e^{-\beta|x|} - n e^{-\beta(|x|+1)}\right)\\ &= \sum_{x\in A}\left(e^{-\beta|f(x)|} - n e^{-\beta(|f(x)|+1)}\right).\end{aligned}$$

The change of variable $y = f(x)$ give us

$$\varphi(\chi_{f(A)}) = \varphi\left(\chi_A e^{\beta(\overline{h-h\circ f})}\right).$$

As $\varphi = \varphi \circ E$, Theorem 2.5 gives us that φ is a (σ_h, β)-KMS state on $\mathrm{C}_u^*(X)$. ■

5.3 KMS states on $\mathrm{C}_u^*(T_n)$ vanishing on compacts

In order to complete the proof of Theorem 1.12, we must further analyze the case $\beta = \log(n)$. According to Theorem 5.4, the KMS states for this inverse temperature will all vanish on the ideal of compact operators and we can then make use of the material of Section 4. Moreover, ideas in [Cho69, Lemma 3] will also be extremely useful in order to compute to the precise cardinality of the set of extreme (σ_h, β)-KMS states on $\mathrm{C}_u^*(T_n)$.

5.3.1 Precise cardinality of the set of KMS states on $\mathrm{C}_u^*(T_n)$ for $\beta = \log(n)$

We start by setting up some notation. Given $y \in T_n$, consider the map

$$\begin{aligned}\tilde{y}\colon T_n &\to \beta T_n\\ x &\mapsto x^\frown y.\end{aligned}$$

Then, by the defining property of βT_n, $\tilde{y}$ can be extended to a continuous map $\beta T_n \to \beta T_n$ which, by abuse of notation, we still denote by $\tilde{y}$. Notice that

$$\overline{\tilde{y}(A)} = \tilde{y}(\bar{A}) \text{ for all } A \subseteq T_n,$$

where the closures above are taken in βT_n (see Lemma [Cho69, Lemma 2.1]). We call a subset $A \subseteq \beta T_n$ *right-invariant*[11] if

[10] Equivalently, this could also be done using Theorem 1.6, but the computations would not be shorter.

[11] The reader is invited to compare this notion with Definition 4.11 above. Notice that this notion is weaker since we only consider partial translations of T_n given by adding a letter to the right, but not by deleting one.

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$$\tilde{y}(A) \subseteq A \text{ for all } y \in T_n.$$

The following is a particular case of [Cho69, Lemma 2 and Proposition 4.1], except that Chou prefers to work with left translations.[12]

Lemma 5.5 *Given $n \in \mathbb{N}$, βT_n contains at least $2^{2^{\aleph_0}}$ nonempty, mutually disjoint, closed, invariant subsets.*[13]

Theorem 5.6 *In the setting of Assumption 5.2: If $\beta = \log(n)$, then there are $2^{2^{\aleph_0}}$ extreme (σ_h, β)-KMS states on $C^*_u(T_n)$.*

Before proving Theorem 5.6, let us isolate an easy lemma for further reference. Precisely, the next result is simply a more specialized version of Theorem 2.5.

Lemma 5.7 *In the setting of Assumption 5.2: Suppose φ is a state on $\ell_\infty(T_n)$ such that*

$$\varphi(\chi_{\tilde{y}(A)}) = \varphi\Big(\chi_A e^{\beta(\overline{h - h\circ\tilde{y}})}\Big) \tag{5.4}$$

for all $y \in T_n$ and all $A \subseteq T_n$. Then, $\varphi \circ E$ is a (σ_h, β)-KMS state on $C^*_u(T_n)$; where $E\colon C^*_u(T_n) \to \ell_\infty(T_n)$ is the canonical conditional expectation.

Proof Any partial translation f of T_n can be written as a disjoint union $f = \bigcup_{i=1}^k f_i$, where each f_i is a composition of partial isometries of the form

$$x \in A \to \tilde{y}(x) \in \tilde{y}(A)$$

for some $y \in T_n$ and $A \subseteq T_n$, and partial isometries of the form

$$\tilde{y}(x) \in \tilde{y}(A) \to x \in A,$$

for some $y \in T_n$ and $A \subseteq T_n$. Therefore, by Theorem 2.5, it is enough to notice that (5.4) holds for partial isometries of the second kind. For that, fix $y \in T_n$ and $A \subseteq T_n$, and let $g\colon \tilde{y}(A) \to A$ be the partial translation given by $g(\tilde{y}(x)) = x$ for all $x \in A$. Then, since

$$h(x) - h(\tilde{y}(x)) = |y| \text{ for all } x \in T_n,$$

our assumption on φ implies that

$$\begin{aligned} \varphi(\chi_A) &= e^{-\beta|y|}\varphi\Big(\chi_A e^{\beta(\overline{h-h\circ\tilde{y}})}\Big) \\ &= e^{-\beta|y|}\varphi(\chi_{\tilde{y}(A)}) \\ &= \varphi\Big(\chi_{\tilde{y}(A)} e^{\beta(\overline{h-h\circ g})}\Big). \end{aligned}$$

So, we are done. ∎

[12] Equivalently, this could be obtained as in Lemma 4.13 above.

[13] In [Cho69], Chou works with semigroups, but this is precisely what T_n is endowed with the products $x * y = x^\frown y$.

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Proof of Theorem 5.6. We start establishing some convention. Firstly, recall that $\ell_\infty(T_n)$ is canonically isomorphic to $C(\beta T_n)$. In order to keep track of notation, if $a \in \ell_\infty(T_n)$, we write $\bar{a}$ to denote a as an element of $C(\beta T_n)$. Notice that, if $a = \chi_A$ for some $A \subseteq T_n$, then

$$\overline{\chi_A} = \chi_{\bar{A}},$$

where the closure $\bar{A}$ is taken in βT_n. Therefore, if φ is a state on $\ell_\infty(T_n)$, we can view it as a state on $C(\beta T_n)$, that is, φ is a Borel measure on βT_n and

$$\varphi(a) = \int_{\beta T_n} \bar{a} d\varphi \ \text{ for all } \ a \in \ell_\infty(T_n).$$

With this in mind, we define the *support of* φ as the support of φ as a Borel measure on βT_n and denote it by $\mathrm{supp}(\varphi) \subseteq \beta T_n$. Suppose now that φ is a state on $\mathrm{C}^*_u(T_n)$. Then, $\varphi \restriction \ell_\infty(T_n)$ is a state on $\ell_\infty(T_n)$ and, by abuse of notation, we write

$$\mathrm{supp}(\varphi) = \mathrm{supp}(\varphi \restriction \ell_\infty(T_n)).$$

We now start the proof. By Lemma 5.5, there is a family $(L_j)_{j \in J}$ of nonempty, mutually disjoint, closed, invariant subsets of βT_n such that $|J| = 2^{2^{\aleph_0}}$. Fix $j \in J$ and, for simplicity, let $L = L_j$. Denote the subset of all (σ_h, β)-KMS states on $\mathrm{C}^*_u(T_n)$ which vanish on the compacts by K_β and define

$$K^L_\beta = \{\varphi \in K_\beta \mid \mathrm{supp}(\varphi) \subseteq L\}.$$

Clearly, K^L_β is convex and weak*-compact. Let us show K^L_β is nonempty.

By Theorem 5.4, $K_\beta \neq \emptyset$. From now on, we fix $\varphi \in K_\beta$. As L is nonempty, fix also $\omega \in L$. We define a state ψ on $\ell_\infty(T_n)$ as follows: for each $a \in \ell_\infty(T_n)$, let $\tilde{a} \in \ell_\infty(T_n)$ be given by

$$\tilde{a}(y) = \bar{a}(\tilde{y}(\omega)) \ \text{ for all } \ y \in T_n.$$

We then let ψ be the state on $\ell_\infty(T_n)$ given by

$$\psi(a) = \varphi(\tilde{a}) \ \text{ for all } \ a \in \ell_\infty(T_n).$$

We extend ψ to the whole $\mathrm{C}^*_u(T_n)$ in the usual way, that is, we let $\psi = \psi \circ E$ where $E\colon \mathrm{C}^*_u(T_n) \to \ell_\infty(T_n)$ is the canonical conditional expectation. Since it is immediate that ψ is indeed a state on $\mathrm{C}^*_u(T_n)$, we only need to show that ψ satisfies the required KMS condition and that $\mathrm{supp}(\psi) \subseteq L$.

For the KMS conditions, let $y \in T_n$ and $A \subseteq T_n$; so, $\tilde{y} \restriction A\colon A \to \tilde{y}(A)$ is a partial translation on T_n. Notice that

$$\widetilde{\chi_{\tilde{y}(A)}}(x) = \overline{\chi_{\tilde{y}(A)}}(\tilde{x}(\omega)) = \chi_{\overline{\tilde{y}(A)}}(\tilde{x}(\omega)) = \chi_{\tilde{y}(\bar{A})}(\tilde{x}(\omega)) \tag{5.5}$$

for all $x \in T_n$. In order to understand $\chi_{\tilde{y}(\bar{A})}(\tilde{x}(\omega))$, notice that

$$\tilde{y}(\{x \in T_n \mid \tilde{x}(\omega) \in \bar{A}\}) \subseteq \{x \in T_n \mid \tilde{x}(\omega) \in \tilde{y}(\bar{A})\}$$

and

$$\{x \in T_n \mid \tilde{x}(\omega) \in \tilde{y}(\bar{A}) \text{ and } |x| \geq |y|\} \subseteq \tilde{y}(\{x \in T_n \mid \tilde{x}(\omega) \in \bar{A}\}).$$

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Therefore, as $\{x \in T_n \mid |x| < |y|\}$ is finite and as φ vanishes on compacts, letting

$$B = \{x \in T_n \mid \tilde{x}(\omega) \in \bar{A}\} \text{ and } C = \{x \in T_n \mid \tilde{x}(\omega) \in \tilde{y}(\bar{A})\},$$

we have that $\varphi(\chi_{\tilde{y}(B)}) = \varphi(\chi_C)$. By (5.5), we have $\widetilde{\chi_{\tilde{y}(A)}} = \chi_C$ and our discussion gives

$$(5.6) \qquad \begin{aligned} \psi(\chi_{\tilde{y}(A)}) &= \varphi(\widetilde{\chi_{\tilde{y}(A)}}) \\ &= \varphi(\chi_{\tilde{y}(B)}) \\ &= \varphi\Big(\chi_B e^{\beta(\overline{h - h\circ\tilde{y}})}\Big). \end{aligned}$$

As $h - h\circ\tilde{y}$ is bounded, $(h - h\circ\tilde{y})^{\sim}$ is well defined. Let $(z_j)_j$ be a net of elements of T_n converging to ω. Notice that

$$h(x^\frown y) = h(x) + h(y) \text{ for all } x, y \in T_n.$$

Therefore,

$$\begin{aligned} (h - h\circ\tilde{y})^{\sim}(x) &= \overline{(h - h\circ\tilde{y})}(\tilde{x}(\omega)) \\ &= \lim_i (h(z_i^\frown x) - h(z_i^\frown x^\frown y)) \\ &= \lim_i (h(x) - h(x^\frown y)) \\ &= (h - h\circ\tilde{y})(x) \end{aligned}$$

for all $x \in T_n$. By the definition of B, it is clear that $\chi_B = \widetilde{\chi_A}$. Therefore,

$$(5.7) \qquad \begin{aligned} \psi\Big(\chi_A e^{\beta(\overline{h - h\circ\tilde{y}})}\Big) &= \varphi\Big(\Big(\chi_A e^{\beta(\overline{h - h\circ\tilde{y}})}\Big)^{\sim}\Big) \\ &= \varphi\Big(\widetilde{\chi_A}\Big(e^{\beta(\overline{h - h\circ\tilde{y}})}\Big)^{\sim}\Big) \\ &= \varphi\Big(\chi_B e^{\beta(\overline{h - h\circ\tilde{y}})}\Big). \end{aligned}$$

By (5.6) and (5.7), we conclude that

$$\psi(\chi_{\tilde{y}(A)}) = \psi\Big(\chi_A e^{\beta(\overline{h - h\circ\tilde{y}})}\Big).$$

As $y \in T_n$ and $A \subseteq T_n$ were arbitrary, this shows that ψ is a (σ_h, β)-KMS state on $C^*_u(T_n)$.

Let us notice that $\operatorname{supp}(\psi) \subseteq L$. Suppose $\omega' \notin L$. Then there is $A \subseteq T_n$ such that $\omega' \in \bar{A}$ and $\bar{A} \cap L = \varnothing$. As $\omega \in L$ and L is invariant, $\tilde{x}(\omega) \in L$ for all $x \in T_n$. Hence,

$$\widetilde{\chi_A}(x) = \chi_{\bar{A}}(\tilde{x}(\omega)) = 0$$

for all $x \in T_n$. Then, thinking of ψ as being defined on $C(\beta T_n)$ as described above, we have that $\psi(\chi_A) = 0$. This shows that $\operatorname{supp}(\psi) \subseteq L$ and we concluded our proof that $K^L_\beta \neq \varnothing$.

Since $j \in J$ was arbitrary, we have that each $K^{L_j}_\beta$ is convex, weak* compact, and nonempty. Hence, Krein–Milman theorem implies that each of them contains

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extreme points. Since $(L_j)_{j\in J}$ are disjoint, this implies that there are $2^{2^{\aleph_0}}$ many extreme points and we are done. ■

5.3.2 Localization of KMS states on $\mathrm{C}^*_u(T_n)$ for $\beta = \log(n)$.

We are left to notice that a version of Theorem 5.6 holds along every branch of T_n. For that, we must further analyze the Higson corona of T_n. More precisely, we must identify a C*-subalgebra of $C(\nu T_n)$ which will help us to locate the KMS states on $\mathrm{C}^*_u(T_n)$ for inverse temperature $\beta = \log(n)$ better.

We first introduce some notation. Firstly, let $[T_n]$ denote the *branches* of T_n, that is,

$$[T_n] = \{1, \dots, n\}^{\mathbb{N}}.$$

Given $\bar{x} = (x_j)_{j=1}^{\infty} \in [T_n]$ and $k \in \mathbb{N}$, we let $\bar{x}|k$ be the initial segment of $\bar{x}$ with k letters, that is,

$$\bar{x}|k = (x_1, \dots, x_k).$$

We now set

$$\mathcal{T}_n = T_n \cup [T_n]$$

and endow $\mathcal{T}_n$ with an appropriate topology. For that, we first extend the concatenation operation: for $y \in T_n$ and $\bar{x} \in [T_n]$, we let

$$y^\frown \bar{x} = (y_1, \dots, y_{|y|}, x_1, x_2, \dots) \in [T_n].$$

Given any $y \in T_n$, we let

$$y^\frown \mathcal{T}_n = \{x \in \mathcal{T}_n \mid \exists z \in \mathcal{T}_n \text{ with } x = y^\frown z\},$$

that is, $y^\frown \mathcal{T}_n$ denotes the set of words, finite or not, which "start" with y. We define $y^\frown T_n$ and $y^\frown [T_n]$ analogously, that is,

$$y^\frown T_n = (y^\frown \mathcal{T}_n) \cap T_n \ \text{ and } \ y^\frown [T_n] = (y^\frown \mathcal{T}_n) \cap [T_n].$$

We endow $\mathcal{T}_n$ with the topology generated by

$$\mathcal{P}(T_n) \cup \{y^\frown \mathcal{T}_n \mid y \in T_n\}.$$

So, T_n is an open subset of $\mathcal{T}_n$ and the inclusion

$$T_n \hookrightarrow \mathcal{T}_n$$

is a homeomorphic embedding with dense range. Moreover, it is easy to see that $\mathcal{T}_n$ is a compact space. Hence, $\mathcal{T}_n$ is a *compactification* of T_n.

As T_n is dense in $\mathcal{T}_n$, this allow us to see $C(\mathcal{T}_n)$ as a C*-subalgebra of $\ell_\infty(T_n)$ in a canonical way. Precisely, we identify $C(\mathcal{T}_n)$ with the image of the following injective *-homomorphism

$$f \in C(\mathcal{T}_n) \mapsto f \restriction T_n \in \ell_\infty(T_n).$$

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Lemma 5.8 *Let $n \in \mathbb{N}$ and consider the n-branching tree T_n. Then:*

1. *For all $y \in T_n$, the projection $\chi_{y^\frown T_n}$ is a Higson function.*
2. *The Banach space*

$$C_n = \overline{\operatorname{span}}\{\chi_{y^\frown T_n} \mid y \in T_n\}$$

is a C^*-algebra contained in $C_h(T_n)$.

3. *Under the identification of $C(\mathcal{T}_n)$ with the C^*-subalgebra of $\ell_\infty(T_n)$ described above, we have $C_n = C(\mathcal{T}_n)$. In particular, the compactification $\mathcal{T}_n$ is Higson compatible.*

In particular, identifying $C([T_n]) = C(\mathcal{T}_n)/c_0(T_n)$ via Gelfand transfom, we have that $C([T_n]) \subseteq Q_u^*(T_n)$.

Proof (1) Fix $y \in T_n$. Let $\varepsilon > 0$ and $R > 0$. Let

$$F = \{x \in T_n \mid |x| \le |y| + R\}.$$

Then, if $x, z \in T_n \setminus F$ and $d(x,z) < R$, we must have that either both x and z are in $y^\frown T_n$, or neither of them are. In either case, we have

$$|\chi_{y^\frown T_n}(x) - \chi_{y^\frown T_n}(z)| = 0,$$

so $\chi_{y^\frown T_n}$ is a Higson function.

(2) It is evident that C_n is closed under the adjoint operator. So, we only need to show that C_n is also closed under product. If $x, z \in T_n$, we write $x \le z$ if $|x| \le |z|$ and $x_i = z_i$ for all $i \in \{1, \dots, |x|\}$. The fact that C is a C^*-algebra follows from the straightforward fact that, for all $x, z \in T_n$, we have

$$\chi_{y^\frown T_n}\chi_{z^\frown T_n} = \begin{cases} \chi_{z^\frown T_n}, & \text{if } y \le z, \\ \chi_{y^\frown T_n}, & \text{if } z \le y, \\ 0, & \text{otherwise.} \end{cases}$$

So, C_n is closed under multiplication. The fact that $C_n \subseteq C_h(T_n)$ follows from (1).

(3) We start noticing that

$$a \in C_n \Leftrightarrow a = \sum_{y \in T_n} a_y \chi_{y^\frown T_n} \text{ for some } (a_y)_{y \in T_n} \in \ell_\infty(T_n) \text{ such that}$$

$$\text{the sums } \Big(\sum_{k \in \mathbb{N}} a_{\bar{x}|k}\Big)_{\bar{x} \in [T_n]} \text{ are equi-convergent.}$$

In particular, if $\bar{x} \in [T_n]$ and $a = \sum_{y \in T_n} a_y \chi_{y^\frown T_n}$ is as above, the limit

$$\lim_{k \to \infty} a(\bar{x}|k) = \sum_{k \in \mathbb{N}} a_{\bar{x}|k}$$

exists. We can then define an *-isomorphic embedding $\Phi \colon C_n \to C(\mathcal{T}_n)$ by letting

$$\Phi(a)(w) = \begin{cases} a(w), & \text{if } w \in T_n, \\ \lim_{k \to \infty} a(w|k), & \text{if } w \in [T_n]. \end{cases}$$

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It is straightforward to show that Φ is indeed well-defined, that is, $\Phi(a)$ is a continuous function on $\mathcal{T}_n$ for all $a \in C_n$. Moreover, it is also clear Φ is an injective *-homomorphism and that

$$\Phi(a) \restriction T_n = a.$$

We are left to notice that the Φ is subjective. For that, we show that the image of

$$\mathrm{span}\{\chi_{y^\frown T_n} \mid y \in T_n\}$$

under Φ is dense in $C(\mathcal{T}_n)$. Fix $f \in C(\mathcal{T}_n)$ and $\varepsilon > 0$. As f is continuous and $[T_n]$ is compact, we can pick $y_1, \dots, y_k \in T_n$ such that

$$[T_n] \subseteq \bigcup_{j=1}^{k} y_j^\frown \mathcal{T}_n \tag{5.8}$$

and

$$|f(x) - f(z)| < \varepsilon \text{ for all } j \in \{1, \dots, k\} \text{ and all } x, z \in y_j^\frown T_n.$$

By (5.8), there is a finite set $F \subseteq T_n$ such that

$$T_n \subseteq F \cup \bigcup_{j=1}^{k} y_j^\frown T_n.$$

For simplicity, assume $F \cap y_j^\frown T_n = \varnothing$ for all $j \in \{1, \dots, k\}$ and let $a \in \ell_\infty(X)$ be given by

$$a(x) = \begin{cases} f(x), & \text{if } x \in F, \\ f(y_j), & \text{if } j \in \{1, \dots, k\} \text{ and } x \in y_j^\frown T_n. \end{cases}$$

It is straightforward to check that

$$a \in \mathrm{span}\{\chi_{y^\frown T_n} \mid y \in T_n\}$$

and that $\|\Phi(a) - f\| \leq \varepsilon$. ■

The next couple of results will focus more on KMS states on $\mathrm{Q}^*_u(T_n)$ and will not be necessary for the main result of this section per se (Theorem 1.12). The reader interested only in Theorem 1.12 can safely skip to Lemma 5.12.

Definition 5.9 In the setting of Assumption 5.2: For each $\beta > \log(n)$, let φ_β be the (σ_h, β)-KMS state in Theorem 5.4. If $(\beta_k)_k \subseteq (\log(n), \infty)$ is a sequence converging to $\log(n)$ and $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$, then

$$\varphi = w^*\text{-}\lim_{k,\mathcal{U}} \varphi_{\beta_k}$$

is a $(\sigma_h, \log(n))$-KMS state on $\mathrm{C}^*_u(T_n)$. We call any such KMS states a *limiting KMS state*. By Theorem 5.4, those states always vanish on $\mathcal{K}(\ell_2(T_n))$.

Corollary 5.10 *In the setting of Assumption 5.2: Let $\beta = \log(n)$ and φ be a limiting (σ_h, β)-KMS state on $\mathrm{C}^*_u(T_n)$. Let ψ be the (σ_h^∞, β)-KMS on $\mathrm{Q}^*_u(T_n)$ such that $\varphi = \psi \circ \pi$. Then, the restriction of ψ to $C([T_n])$ is faithful.*

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Proof Let μ be the probability measure on $[T_n]$ given by Riesz representation theorem by restricting ψ to $C([T_n])$, that is,

$$\psi(a) = \int_{[T_n]} a d\mu \text{ for all } a \in C([T_n]).$$

Since φ is a limiting $(\sigma_h, \log(n))$-KMS state, let $(\beta_k)_k \subseteq (\log(n), \infty)$ be a sequence converging to $\log(n)$ and $\mathcal{U}$ be a nonprincipal ultrafilter such that

$$\varphi = w^*\text{-}\lim_{k,\mathcal{U}} \varphi_{\beta_k}.$$

By the formula of each φ_{β_k} given by Theorem 5.4, it follows that

$$\varphi_{\beta_k}(\chi_{y^\frown T_n}) = e^{-\beta_k|y|} \text{ for all } y \in T_n.$$

Hence, by the formula of φ, we have

$$\varphi(\chi_{y^\frown T_n}) = \lim_{k\to\infty} e^{-\beta_k|y|} = \frac{1}{n^{|y|}} \text{ for all } y \in T_n.$$

This shows that μ is the Bernoulli measure on $[T_n] = \{1,\ldots,n\}^{\mathbb{N}}$. Since the support of the Bernoulli measure is the whole $[T_n]$, this shows that φ is faithful on $C([T_n])$. This completes the proof. ■

Corollary 5.11 *In the setting of Assumption 5.2: If $\beta = \log(n)$, then for all $\bar{x} \in [T_n]$ there is an extreme (σ_h^∞, β)-KMS state ψ on $\mathrm{Q}_u^*(T_n)$ such that*

$$\varphi(a) = a(x) \text{ for all } a \in C([T_n]).$$

Moreover, if φ is a limiting (σ_h, β)-KMS state on $\mathrm{C}_u^*(T_n)$, then the (σ_h^∞, β)-KMS state ψ on $\mathrm{Q}_u^*(T_n)$ determined by $\varphi = \psi \circ \pi$ is not extreme.

Proof The first assertion follows from Theorem 4.4 and Corollary 5.10. For the second assertion, notice that if $\psi = \varphi \circ \pi$ were extreme, then there would be $\bar{x} \in [T_n]$ such that ψ vanishes on the ideal

$$J_x = \{a \in C([T_n]) \mid a(\bar{x}) = 0\}$$

(see Proposition 3.3). However, it was shown in the proof of Corollary 5.10 that ψ is faithful on $C([T_n])$; contradiction. ■

We now return to the proof of Theorem 1.12. The following lemma is trivial and we isolate it for further reference.

Lemma 5.12 *Let $n \in \mathbb{N}$ and T_n be the n-branching tree. Given any $\bar{x}, \bar{y} \in [T_n]$ there is an isometry $f\colon T_n \to T_n$ such that $f(\bar{x}|k) = \bar{y}|k$.*

Given a metric space X and an isometry $f\colon X \to X$, we let $u_f\colon \ell_2(X) \to \ell_2(X)$ be the (linear) isometry determined by

$$u_f(\delta_x) = \delta_{f(x)} \text{ for all } x \in X.$$

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Lemma 5.13 *Let $n \in \mathbb{N}$ and T_n be the n-branching tree. Let $f\colon T_n \to T_n$ be an isometry and consider the (linear) isometry $u_f\colon \ell_2(T_n) \to \ell_2(T_n)$ defined above. Then, the map*

$$\varphi \to \varphi \circ \mathrm{Ad}(u_f)$$

is an affine isometry of the set of (σ_h, β)-KMS states on $\mathrm{C}^*_u(T_n)$ to itself.

Proof It is enough to notice that $\varphi \circ \mathrm{Ad}(u_f)$ is a (σ_h, β)-KMS state on $\mathrm{C}^*_u(T_n)$ given that the same holds for φ. Indeed, once this is done the result follows since this map will clearly be an affine isometry with inverse $\varphi \to \varphi \circ \mathrm{Ad}(u_f^*)$.

Notice that

$$h(f(x)) = h(x) \text{ for all } x \in X.$$

Indeed, any isometry of the tree T_n must satisfy $f(\varnothing) = \varnothing$. Therefore, for each $x \in T_n$, we have

$$h(f(x)) = d(f(x), \varnothing) = d(f(x), f(\varnothing)) = d(x, \varnothing) = h(x).$$

Using this, an immediate computation gives us that

$$\langle u_f^* e^{it\bar{h}} a e^{-it\bar{h}} u_f \delta_x, \delta_y \rangle = \langle e^{it\bar{h}} u_f^* a u_f e^{-it\bar{h}} \delta_x, \delta_y \rangle$$

for all $t \in \mathbb{R}$, all $a \in \mathrm{C}^*_u(T_n)$, and all $x, y \in T_n$. In other words, the flow σ_h is invariant under $\mathrm{Ad}(u_f)$. This shows that $\varphi \circ \mathrm{Ad}(u_f)$ must be a (σ_h, β)-KMS state on $\mathrm{C}^*_u(T_n)$ given that φ is one (equivalently, this could also be shown with the help of Theorem 2.5). ∎

Theorem 5.14 *In the setting of Assumption 5.2: If $\beta = \log(n)$, then for each $\bar{x} \in [T_n]$ there are $2^{2^{\aleph_0}}$ extreme (σ_h, β)-KMS states φ on $\mathrm{C}^*_u(T_n)$ such that*

$$\varphi(\chi_{\bar{x}|k^\frown T_n}) = 1 \textit{ for all } k \in \mathbb{N}.$$

Proof Fix $\beta = \log(n)$. By Theorem 5.6, there are $2^{2^{\aleph_0}}$ extreme (σ_h, β)-KMS states on $\mathrm{C}^*_u(T_n)$. By Lemma 5.8, $\mathcal{T}_n$ is a Higson compatible compactification of T_n. Therefore, Theorem 4.6 implies that for any extreme (σ_h, β)-KMS state φ on $\mathrm{C}^*_u(T_n)$, there is $\bar{x} \in [T_n]$ such that

$$\varphi(\chi_{\bar{x}|k^\frown T_n}) = 1 \text{ for all } k \in \mathbb{N}. \tag{5.9}$$

Therefore, since $|[T_n]| = 2^{\aleph_0}$, a pigeonhole argument implies that there is at least one $\bar{x} \in [T_n]$ for which there are $2^{2^{\aleph_0}}$ extreme (σ_h, β)-KMS states on $\mathrm{C}^*_u(T_n)$ satisfying (5.9) for $\bar{x}$. Fix such $\bar{x} \in [T_n]$.

Let now $\bar{y} \in [T_n]$ be arbitrary and let $f\colon T_n \to T_n$ be an isometry such that

$$f(\bar{x}|k) = \bar{y}|k \text{ for all } k \in \mathbb{N}$$

(Lemma 5.12). Clearly, we must have that

$$f\big(\bar{x}|k^\frown T_n\big) = \bar{y}|k^\frown T_n \text{ for all } k \in \mathbb{N}.$$

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Hence, $\mathrm{Ad}(u_f)(\chi_{\bar{y}|k^\frown T_n}) = \chi_{\bar{x}|k^\frown T_n}$ for all $k \in \mathbb{N}$ and, if φ satisfies (5.9) for $\bar{x}$, it follows that

$$\big(\varphi \circ \mathrm{Ad}(u_f)\big)(\chi_{\bar{y}|k^\frown T_n}) = 1 \text{ for all } k \in \mathbb{N}.$$

The result then follows from Lemma 5.13. ■

Proof of Theorem 1.12. Theorem 5.4 gives that there is a (σ_h, β)-KMS state on $\mathrm{C}^*_u(T_n)$ if and only if $\beta \geq \log(n)$. Moreover, item (1) and the first claim of item (2) of Theorem 1.12 also follow from Theorem 5.4.

We are left to notice that the second and third claim of Theorem 1.12(2) hold. From now on, let $\beta = \log(n)$. By Lemma 5.8, $\mathcal{T}_n$ is a Higson compatible compactification of T_n. Therefore, Theorem 4.6 implies that any extreme (σ_h, β)-KMS state φ on $\mathrm{C}^*_u(T_n)$ must have the required form, that is, there must be $\bar{x} \in [T_n]$ such that

$$\varphi(\chi_{\bar{x}|k^\frown T_n}) = 1 \text{ for all } k \in \mathbb{N}.$$

Finally, the fact that for each $\bar{x} \in [T_n]$, there are $2^{2^{\aleph_0}}$ extreme (σ_h, β)-KMS states on $\mathrm{C}^*_u(T_n)$ satisfying the above is now simply Theorem 5.14. ■

5.4 Obtaining distinct KMS states on $\mathrm{C}^*_u(T_n)$ for $\beta = \log(n)$

We finish the article presenting a more concrete way of obtaining distinct KMS states for inverse temperature $\beta = \log(n)$. Precisely, if $(\beta_n)_n$ is a sequence converging to $\log(n)$ from the right and $(\varphi_{\beta_n})_n$ is a sequence of states such that each φ_{β_n} is a $(\sigma_h, \beta_n)_n$-KMS state on $\mathrm{C}^*_u(T_n)$, then $w^*\text{-}\lim_{n,\mathcal{U}} \varphi_{\beta_n}$ is a $(\sigma_h, \log(n))$-KMS state, where $\mathcal{U}$ is an arbitrary nonprincipal ultrafilter on $\mathbb{N}$. The next theorem shows that, picking different sequences (β_n) as above, this procedure may give us distinct $(\sigma_h, \log(n))$-KMS states. As mentioned at the end of Section 1.4, this kind of behavior is unusual (see [vER07]) and known as *chaotic behavior* of *chaotic convergence of KMS states.*

Theorem 5.15 *In the setting of Assumption 5.2: different sequences $(\beta_n)_n$ converging to $\log(n)$ may converge to distinct $(\sigma_h, \log(n))$-KMS states on $\mathrm{C}^*_u(T_n)$.*

Proof Let $\beta = \log(n)$. For each $\beta' > \log(n)$, let $\varphi_{\beta'}$ be the (σ_h, β')-KMS state on $\mathrm{C}^*_u(T_n)$ given by Theorem 5.4(1), that is,

$$\varphi_{\beta'}([a_{x,y}]) = \sum_{y \in T_n} a_{y,y}\Big(e^{-\beta|y|} - ne^{-\beta(|y|+1)}\Big)$$

for all $[a_{x,y}] \in \mathrm{C}^*_u(T_n)$. Given any nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and any sequence $(\beta_n)_n$ converging to β from the right, we know that $w^*\text{-}\lim_{n,\mathcal{U}} \varphi_{\beta_n}$ is a (σ_h, β)-KMS state on $\mathrm{C}^*_u(T_n)$. Our strategy will be to construct different sequences $(\beta_n)_n$ as above which give us different (σ_h, β)-KMS states on $\mathrm{C}^*_u(T_n)$. For that, some manipulations with the formula of $\varphi_{\beta'}$ will be useful. Firstly, given $E \subseteq T_n$ and $k \in \mathbb{N}$, write

$$E_k = \{y \in E \mid |y| = k\}$$

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and notice that $|E_k| \leq n^k$. Then, given an arbitrary $\beta' > \beta$, we have

$$\begin{aligned}\varphi_{\beta'}(\chi_E) &= \sum_{y\in E}(e^{-\beta'|y|} - ne^{-\beta'(|y|+1)}) \\ &= \sum_{k=0}^{\infty}|E_k|(e^{-\beta' k} - ne^{-\beta'(k+1)}) \\ &= (1 - ne^{-\beta'})\sum_{k=0}^{\infty}|E_k|e^{-\beta' k}.\end{aligned}$$

Applying the change of variables $\tau = ne^{-\beta'}$ and letting $a_k = |E_k|/n^k$ for each $k \geq 0$, we have that each a_k is in $[0,1]$ and

$$\varphi_{\beta'}(\chi_E) = (1-\tau)\sum_{k=0}^{\infty}a_k\tau^k.$$

Moreover, $\beta' \to \log(n)$ from the right if and only if $\tau \to 1$ from the left. At last, notice that if E is such that there are $p < q \in \mathbb{N}$ with

$$a_k = \begin{cases}1, & k \in [p,q] \cap \mathbb{N}, \\ 0, & k \notin [p,q] \cap \mathbb{N},\end{cases}$$

then

$$\varphi_{\beta'}(\chi_E) = \tau^p - \tau^{q+1}. \tag{5.10}$$

This finishes the manipulations in the formula of $\varphi_{\beta'}$ that we will need.

We now construct increasing sequences $(\tau_k)_k$ and $(\theta_k)_k$ converging to 1, and sequences $(p_k)_k$ and $(q_k)_k$ of natural numbers by induction for which the following holds:

- $p_k < q_k < p_{k+1} - 1$ for all $k \in \mathbb{N}$,
- $\tau_k^{p_k} - \tau_k^{q_k+1} > 1/2$ for all $k \in \mathbb{N}$, and
- $\theta_k^{p_m} - \theta_k^{q_m+1} < 2^{-m-2}$ for all $k, m \in \mathbb{N}$.

This can be easily done as follows: let $k \geq 2$ and suppose $(\tau_m)_{m=1}^{k-1}$, $(\theta_m)_{m=1}^{k-1}$, $(p_m)_{m=1}^{k-1}$, and $(q_m)_{m=1}^{k-1}$ where chosen appropriately; step 1 of the induction can clearly be done. Step k of the induction goes as follows. Pick $p_k > q_{k-1} + 1$ such that $\theta_m^{p_k} < 2^{-k-2}$ for all $m \leq k-1$. Then pick $\tau_k \in (\tau_{k-1}, 1)$ such that $\tau_k^{p_k} > 3/4$ and $q_k > p_k$ with $\tau_k^{q_k+1} < 1/4$. Choose now $\theta_k \in (\theta_{k-1}, 1)$ with $\theta_k^{p_m} - \theta_k^{q_m+1} < 2^{-m-2}$ for all $m \leq k$. This finishes the induction.

We now use the sequences constructed in the previous paragraph to finish the proof. Precisely, we show that if $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$, then

$$w^*\text{-}\lim_{n,\mathcal{U}}\varphi_{\tau_n} \neq w^*\text{-}\lim_{n,\mathcal{U}}\varphi_{\theta_n}.$$

For this, let $E \subseteq T_n$ be given by

$$E = \left\{x \in T_n \mid |x| \in \bigcup_{m=1}^{\infty}[p_m, q_m]\right\}.$$

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Then $|E_k| = n^k$ if $k \in \bigcup_{m=1}^{\infty}[p_m, q_m]$ and $|E_k| = 0$ otherwise. Hence, letting $a_k = |E_k|/n^k$ as above, we have that

$$a_k = \begin{cases} 1, & k \in \bigcup_{m=1}^{\infty}[p_m, q_m] \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, using (5.10) above, we have

$$\varphi_{\tau_k}(\chi_E) = \sum_{m=1}^{\infty}(\tau_k^{p_m} - \tau_k^{q_m+1}) \geq \tau_k^{p_k} - \tau_k^{q_k+1} > \frac{1}{2}$$

for all $k \in \mathbb{N}$. On the other hand,

$$\varphi_{\theta_k}(\chi_E) = \sum_{m=1}^{\infty}(\theta_k^{p_m} - \theta_k^{q_m+1}) < \sum_{m=1}^{\infty} 2^{-m-2} = 1/4$$

for all $k \in \mathbb{N}$. Therefore, we conclude that

$$\Big(w^*\text{-}\lim_{n,\mathcal{U}} \varphi_{\tau_n}\Big)(\chi_E) \geq 1/2 > 1/4 \geq \Big(w^*\text{-}\lim_{n,\mathcal{U}} \varphi_{\theta_n}\Big)(\chi_E).$$

This finishes the proof. ∎

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IMPA, Estrada Dona Castorina 110, Rio de Janeiro 22460-320, Brazil
e-mail: demendoncabraga@gmail.com
URL: https://sites.google.com/site/demendoncabraga

Department of Mathematics, Universidade Federal de Santa Catarina, Florianopolis SC 88040-970, Brazil
e-mail: ruyexel@gmail.com
URL: http://www.mtm.ufsc.br/~exel/

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