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ON THE PERTURBATION CLASSES OF CONTINUOUS SEMI-FREDHOLM OPERATORS*

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Abstract. We prove that the perturbation class of the upper semi-Fredholm operators from X into Y is the class of the strictly singular operators, whenever X is separable and Y contains a complemented copy of $C[0, 1]$. We also prove that the perturbation class of the lower semi-Fredholm operators from X into Y is the class of the strictly cosingular operators, whenever X contains a complemented copy of ℓ_1 and Y is separable. We can remove the separability requirements by taking suitable spaces instead of $C[0, 1]$ or ℓ_1.

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1. Introduction. The *perturbation class* $P\mathcal{S}$ of a class $\mathcal{S}$ of continuous operators between Banach spaces is defined by its components:

$$P\mathcal{S}(X, Y) := \{K \in L(X, Y) : K + A \in \mathcal{S}(X, Y), \quad \text{for every } A \in \mathcal{S}(X, Y)\},$$

where X and Y are Banach spaces such that $\mathcal{S}(X, Y)$ is non-empty.

The concept of perturbation class has been considered in other situations. For example, it is well known that the perturbation class of the group G of invertible operators in a Banach algebra A is the radical of A [**6**]. Hence

$$P(G) = \{x \in A : e + ax \in G \quad \text{for all } a \in G\}.$$

Here we consider the perturbation class $P\mathcal{S}$ in the cases $\mathcal{S} = \Phi$, the Fredholm operators, $\mathcal{S} = \Phi_+$ the upper semi-Fredholm operators, and $\mathcal{S} = \Phi_-$, the lower semi-Fredholm operators. It is well known that $P\Phi = \mathcal{I}n$, the inessential operators [**6**, **3**]. However, the perturbation classes for Φ_+ and Φ_- are not well known. In [**7**, 26.6.12]

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it is stated as an open problem whether $P\Phi_+ = \mathcal{SS}$, the strictly singular operators, or $P\Phi_- = \mathcal{SC}$, the strictly cosingular operators.

Positive solutions of the above-mentioned problems are interesting because they provide intrinsic characterizations of the operators in $P\Phi_+$ and $P\Phi_-$. For example, the definition of $\mathcal{SS}$ is intrinsic because $K \in \mathcal{SS}(X, Y)$ only depends on the action of K over the subspaces of X. However, the fact that $K \in P\Phi_+(X, Y)$ depends on the properties of the sums of K with all the operators in $\Phi_+(X, Y)$.

It is known that $P\Phi_+(X, Y) = \mathcal{SS}(X, Y)$ in the following cases:

1. Y subprojective [**11**, **3**];
2. $X = Y = L_p(\mu)$, $1 \leq p \leq \infty$ [**9**]; and
3. X hereditarily indecomposable [**3**, Theorem 3.14].

Note that $P\Phi_+(X, Y) = \mathcal{SS}(X, Y) = \mathcal{I}n(X, Y)$ in the first two cases. Also it is known that $P\Phi_-(X, Y) = \mathcal{SC}(X, Y)$ in the following cases:

1. X superprojective [**11**, **3**];
2. $X = Y = L_p(\mu)$, $1 \leq p \leq \infty$ [**9**]; and
3. X quotient indecomposable [**3**, Theorem 3.14].

Note again that $P\Phi_-(X, Y) = \mathcal{SC}(X, Y) = \mathcal{I}n(X, Y)$ in the first two cases. However, the problem remains unsolved in general.

We show that $P\Phi_+(X, Y) = \mathcal{SS}(X, Y)$ whenever X is separable and Y contains a complemented copy of $C[0, 1]$, and that $P\Phi_-(X, Y) = \mathcal{SC}(X, Y)$ whenever X contains a complemented copy of ℓ_1 and Y is separable. Moreover, the separability requirements can be removed by taking suitable spaces $\ell_\infty(I)$ and $\ell_1(I)$ instead of $C[0, 1]$ and ℓ_1, respectively.

Our results provide new examples of pairs X, Y of Banach spaces for which the problem of the perturbation classes for semi-Fredholm operators has a positive answer. Indeed, it is well known that every separable Banach space X is isomorphic to a subspace of $C[0, 1]$. Hence $\Phi_+(X, C[0, 1]) \neq \emptyset$. Moreover, if X contains no complemented copies of c_0, then $\mathcal{I}n(X, C[0, 1]) = L(X, C[0, 1])$ [**4**]. Thus for every infinite dimensional, separable Banach space X containing no copy of c_0,

$$P\Phi_+(X, C[0, 1]) = \mathcal{SS}(X, C[0, 1]) \neq \mathcal{I}n(X, C[0, 1]).$$

Analogously, if Y is separable, then Y is isomorphic to a quotient of ℓ_1. Thus $\Phi_-(\ell_1, Y) \neq \emptyset$. Moreover, if Y contains no complemented copies of ℓ_1, then $\mathcal{I}n(\ell_1, Y) = L(\ell_1, Y)$. Thus for every infinite dimensional, separable Banach space Y containing no complemented copies of ℓ_1, we have

$$P\Phi_-(\ell_1, Y) = \mathcal{SC}(\ell_1, Y) \neq \mathcal{I}n(\ell_1, Y).$$

We observe that the perturbation classes studied in [**5**, **10**] correspond to not necessarily bounded operators. These classes are smaller than those that we consider here and so most of the results in [**10**] are not relevant for us.

In relation to the questions we tackle here, it has been open for some time whether $P\Phi$, the inessential operators, coincide with the *improjective operators*, introduced by Tarafdar [**8**]. The definition of these operators is intrinsic and similar to that of the strictly singular operators. There are many classes of spaces for which these classes of operators coincide [**1**], but recently it has been proved that the problem has a negative answer in general [**2**].

Throughout the paper, X, Y, Z, W are Banach spaces and I_X is the identity operator on X. For a closed subspace M of X, J_M is the inclusion of M into X and Q_M is the

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quotient map onto X/M. An operator $A \in L(X, Y)$ is *upper semi-Fredholm* if its range is closed and its null space is finite dimensional; it is *lower semi-Fredholm* if its range is finite codimensional and so closed. Also it is *Fredholm* if it is upper semi-Fredholm and lower semi-Fredholm. We denote by $\Phi_+(X, Y)$, $\Phi_-(X, Y)$ and $\Phi(X, Y)$ the classes of upper semi-Fredholm, lower semi-Fredholm and Fredholm operators, respectively.

An operator $T \in L(X, Y)$ is *inessential* if $I_X - ST \in \Phi(X, X)$, for every $S \in L(X, Y)$; it is *strictly singular* if no restriction TJ_M of T to a closed infinite dimensional subspace M of X is an isomorphism; and it is *strictly cosingular* if there is no closed infinite codimensional subspace N of Y such that $Q_N T$ is surjective. We denote by $\mathcal{I}n(X, Y)$, $\mathcal{SS}(X, Y)$ and $\mathcal{SC}(X, Y)$ the inessential, strictly singular and strictly cosingular operators, respectively.

We shall need the next Lemma, which was proved in [**3**]. We give a proof for the convenience of the reader.

Lemma 1. [See **3**, Lemma 3.3]. *Let $\mathcal{S}$ be Φ_+, Φ_- or Φ. Assume that $\mathcal{S}(X, Y) \neq \emptyset$, and let $K \in P\mathcal{S}(X, Y)$.*

(1) *If A is an isomorphism from W onto X and B is an isomorphism from Y onto Z, then $BKA \in P\mathcal{S}(W, Z)$.*

(2) *If $A \in L(X)$ and $B \in L(Y)$, then $BKA \in P\mathcal{S}(X, Y)$.*

Proof. (1) Note that $\mathcal{S}(W, Z) \neq \emptyset$. Let $T \in \mathcal{S}(W, Z)$. Then

$$T + BKA = B(B^{-1}TA^{-1} + K)A \in \mathcal{S}(W, Z),$$

because of $B^{-1}TA^{-1} \in \mathcal{S}(X, Y)$.

(2) We write $A = A_1 + A_2$ and $B = B_1 + B_2$, where A_1, A_2, B_1, B_2 are invertible operators. Let $T \in \mathcal{S}(X, Y)$. Then

$$T + BKA = T + \sum_{i,j=1}^{2} B_i K A_j \in \mathcal{S}(X, Y),$$

by the first part of this lemma. □

2. Main results. Observe that every separable Banach space is isomorphic to a subspace of $C[0, 1]$. Hence the hypothesis of the following result implies that $\Phi_+(X, Y) \neq \emptyset$.

Theorem 2. *Suppose that X is separable and Y contains a complemented subspace isomorphic to $C[0, 1]$. Then*

$$P\Phi_+(X, Y) = \mathcal{SS}(X, Y).$$

Proof. Since $C[0, 1]$ is isomorphic to $C[0, 1] \times C[0, 1]$, there are closed subspaces W and Z of Y such that W is isomorphic to Y, Z is isomorphic to $C[0, 1]$ and $Y = W \oplus Z$. Let $r > 0$ such that $\|a + b\| \geq r \max\{\|a\|, \|b\|\}$, for every $a \in W$ and $b \in Z$, and let $U \in L(Y)$ be an isomorphism with range equal to W.

Suppose that $K \in L(X, Y)$ is not strictly singular. Let $K_1 := UK \in L(X, Y)$. Without loss of generality, we assume that $\|K_1\| = 1$. Then there exist an infinite dimensional subspace M of X and $c > 0$ so that $\|K_1 m\| \geq c\|m\|$, for every $m \in M$. We denote $d := \min\{c/3, 1/3\}$.

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Since X is separable, there exists an isomorphism V from X/M into Y with range contained in Z. We define $S \in L(X, Y)$ by $S := VQ_M$. Without loss of generality, we assume that $\|Sx\| \geq \|Q_M x\|$, for every $x \in X$.

We shall see that the operator $S + K_1$ is an isomorphism into. Indeed, let $x \in X$ be a norm-one vector. If $\|Q_M x\| \geq d$, then

$$\|(S + K_1)x\| \geq r\|Sx\| \geq rd.$$

Otherwise $\|Q_M x\| < d$, and we can choose $m \in M$ such that $\|x - m\| < d$. Hence $\|m\| > 2/3$, so that

$$\|(S + K_1)x\| \geq r\|K_1 x\| \geq r(\|K_1 m\| - \|x - m\|) \geq r(3d(2/3) - d) = rd.$$

Thus $K_1 \notin P\Phi_+$, because $S + K_1$ is upper semi-Fredholm, but S is not. Hence, $K \notin P\Phi_+$, by Lemma 1. □

REMARK 3. Theorem 2 remains valid if we take as hypothesis either (1) or (2) below.

1. X is separable and Y contains a subspace isomorphic to ℓ_∞.
2. X is non-separable and Y contains a subspace isomorphic to $\ell_\infty(I)$, where the cardinal of the set I is equal to the cardinal of a dense subset of X.

The proofs are similar taking into account the following facts: every Banach space Z (in particular, every quotient of X) is isometric to a subspace of $\ell_\infty(I)$ for some set I with $\operatorname{card}(I) = \operatorname{den}(Z)$ (see [**7**, C.3.3]), $\ell_\infty(I)$ is complemented in every Banach space in which it is contained, and $\ell_\infty(I) \times \ell_\infty(I)$ is isomorphic to $\ell_\infty(I)$.

QUESTION 4. Is it possible to remove the requirement for $C[0, 1]$ to be complemented in Theorem 2?

Observe that every separable Banach space is isomorphic to a quotient of ℓ_1. Hence the hypothesis of the following result implies that $\Phi_-(X, Y) \neq \emptyset$.

THEOREM 5. *Suppose that X contains a complemented subspace isomorphic to ℓ_1 and Y is separable. Then*

$$P\Phi_-(X; Y) = SC(X, Y).$$

Proof. Since ℓ_1 is isomorphic to $\ell_1 \times \ell_1$, there are closed subspaces W and Z of X such that W is isomorphic to X, Z is isomorphic to ℓ_1 and $X = W \oplus Z$. Let $U \in L(X)$ be an operator which is an isomorphism from W onto X, with kernel equal to Z.

Suppose that $K \in L(X, Y)$ is not strictly cosingular. Then there exists a closed infinite codimensional subspace M of Y such that the operator $Q_M K$ is surjective; that is, $M + R(K) = Y$. We consider the operator $K_1 := KU \in L(X, Y)$.

Since Y is separable, there exists an operator $S \in L(X, Y)$ with kernel equal to W and range equal to M. Clearly, $S + K_1$ is surjective, but S is not a lower semi-Fredholm operator. Thus $K_1 \notin P\Phi_-$. Hence, $K \notin P\Phi_-$, by Lemma 1. □

REMARK 6. Theorem 5 remains valid if we take as hypothesis that Y is non-separable and X contains a complemented subspace isomorphic to $\ell_1(I)$, where the cardinal of the set I is equal to the cardinal of a dense subset of Y. The proof is similar taking into account that every Banach space Z (in particular, every subspace of Y) is isometric to a quotient of $\ell_1(I)$, for some set I with $\operatorname{card}(I) = \operatorname{den}(Z)$. (See [**7**, C.3.7].). Also $\ell_1(I) \times \ell_1(I)$ is isomorphic to $\ell_1(I)$.

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Question 7. Is it possible to remove the requirement for ℓ_1 to be complemented in Theorem 5?

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