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On the associativity of the torsion functor

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Let R be a commutative ring with identity. We say that tor *is associative over* R if for all R-modules A, B, C there is an isomorphism

$$\operatorname{tor}_1^R\left(A, \operatorname{tor}_1^R(B, C)\right) \simeq \operatorname{tor}_1^R\left(\operatorname{tor}_1^R(A, B), C\right) .$$

Our main results are that

(1) tor is associative over a noetherian ring R if and only if R is the direct sum of a finite number of Dedekind rings and uniserial rings, and

(2) tor is associative over an integral domain R if and only if R is a Prüfer ring.

1. Introduction

In Cartan and Eilenberg [2] it is proved that any commutative semi-hereditary ring R has the property that there is an R-module isomorphism

$$\operatorname{tor}_1^R\left(A, \operatorname{tor}_1^R(B, C)\right) \simeq \operatorname{tor}_1^R\left(\operatorname{tor}_1^R(A, B), C\right)$$

for all R-modules A, B, C. It is the purpose of this paper to examine rings having this isomorphism property.

All the rings that we consider are commutative with identity. For

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simplicity we write $\operatorname{tor}(A, B)$ instead of $\operatorname{tor}_1^R(A, B)$ for any two modules A and B over the ring R, except when it is necessary to specify the ring. Also given a ring R we say that tor *is associative over* R if there is an R-module isomorphism

$$\operatorname{tor}(A, \operatorname{tor}(B, C)) \simeq \operatorname{tor}(\operatorname{tor}(A, B), C) ,$$

not necessarily natural, for all R-modules A, B, C.

Our main result is that if R is a noetherian ring then tor is associative over R if and only if R is the direct sum of finitely many Dedekind rings and uniserial rings. We also show that tor is associative over an integral domain R if and only if R is Prüfer.

2. The associativity of tor over local noetherian semi-prime rings

We prove in this section that tor is associative over a local noetherian semi-prime ring R if and only if R is a discrete valuation ring.

In the following result (and throughout the paper) the term "local ring" simply means that the ring has precisely one maximal ideal.

PROPOSITION 2.1. *Let R be a local ring with maximal ideal M and let x be an element of R which is not a zero-divisor. If* tor *is associative over R then, for any element y of R, either y divides x or x divides y.*

Proof. Suppose that y is any element of R and that y does not divide x and x does not divide y. Since x is not a zero-divisor, for any ideal a of R we have

$$\operatorname{tor}(R/xR, a) = \ker(xR \otimes a \to xa) = 0 .$$

In particular, $\operatorname{tor}(R/xR, xR+yR) = 0$. Thus, since tor is associative over R, we have

$$\begin{aligned}\operatorname{tor}\big(\operatorname{tor}(xR+yR, R/(xR+yR)), R/xR\big) &= \operatorname{tor}\big(R/(xR+yR), \operatorname{tor}(xR+yR, R/xR)\big) \\ &= \operatorname{tor}\big(R/(xR+yR), 0\big) = 0 .\end{aligned}$$

Now,

$$\operatorname{tor}\big(xR+yR, R/(xR+yR)\big) = \ker\big(g : (xR+yR) \otimes (xR+yR) \to xR+yR\big) ,$$

where $g\big((ax+by) \otimes (cx+dy)\big) = (ax+by)(cx+dy)$ for all a, b, c, d in R. Thus $g(x \otimes y - y \otimes x) = xy - yx = 0$, so that

$$x \otimes y - y \otimes x \in \operatorname{tor}\big(xR+yR, R/(xR+yR)\big) .$$

We will now show that $x \otimes y - y \otimes x$ is non-zero.

We define $f : (xR+yR) \times (xR+yR) \to R/M$ by $f\big((ax+by, cx+dy)\big) = ad + M$ for all a, b, c, d in R. Suppose $px + qy = 0$ for p, q in R. Then p is not a unit, since otherwise y divides x, contradicting our assumptions. Similarly, q is not a unit. Thus

$$f\big((px+qy, cx+dy)\big) = pd + M = 0$$

since pd is not a unit and so must belong to the maximal ideal M. This shows that f is well-defined on $(xR+yR) \times (xR+yR)$. Also, easily, f is bilinear. Thus, there is a homomorphism $\bar{f} : (xR+yR) \otimes (xR+yR) \to R/M$ such that $\bar{f}\big((ax+by) \otimes (cx+dy)\big) = ad + M$ for all a, b, c, d in R. Now

$$\bar{f}(x \otimes y - y \otimes x) = \bar{f}(x \otimes y) - \bar{f}(y \otimes x) = (1+M) + (0+M) = 1 + M .$$

Thus $x \otimes y - y \otimes x \neq 0$. Thus $\operatorname{tor}\big(xR+yR, R/(xR+yR)\big) \neq 0$.

We now show that $\operatorname{tor}\big(R/xR, \operatorname{tor}\big(xR+yR, R/(xR+yR)\big)\big) \neq 0$, thus obtaining a contradiction to our initial assumptions. In fact

$$\begin{aligned}&\operatorname{tor}\big(R/xR, \operatorname{tor}\big(xR+yR, R/(xR+yR)\big)\big)\\ &\qquad = \ker\big(h : xR \otimes \operatorname{tor}\big(xR+yR, R/(xR+yR)\big) \to \operatorname{tor}\big(xR+yR, R/(xR+yR)\big)\big) ,\end{aligned}$$

where $h(tx \otimes a) = txa$ for every a in $\operatorname{tor}\big(xR+yR, R/(xR+yR)\big)$ and every t in R. Thus, in particular,

$$\begin{aligned}h\big(x \otimes (x \otimes y - y \otimes x)\big) &= x(x \otimes y - y \otimes x)\\ &= x \otimes yx - xy \otimes x = x \otimes yx - x \otimes yx = 0 ,\end{aligned}$$

so that $x \otimes (x \otimes y - y \otimes x)$ is an element of

$$\operatorname{tor}\big(R/xR, \operatorname{tor}\big(xR+yR, R/(xR+yR)\big)\big) .$$

We now define $k : xR \times \operatorname{tor}\big(xR+yR, R/(xR+yR)\big) \to R/M$ by

$$k\big(\big(tx, (rx+sy) \otimes (ux+vy)\big)\big) = t\bar{f}\big((rx+sy) \otimes (ux+vy)\big)$$

for all t in R and all suitable r, s, u, v. Since $\bar{f}$ is a homomorphism and x is not a zero-divisor, k is well-defined and clearly a bilinear mapping. Thus there is induced a homomorphism

$$\bar{k} : xR \otimes \operatorname{tor}\big(xR+yR, R/(xR+yR)\big) \to R/M$$

such that

$$\overline{k}\big(tx \otimes ((rx+sy) \otimes (ux+vy))\big) = t\overline{f}\big((rx+sy) \otimes (ux+vy)\big)$$

for all suitable r, s, t, u, v in R. Now $x \otimes y - y \otimes x$ is an element of $\operatorname{tor}\big(xR+yR,\ R/(xR+yR)\big)$ and, moreover,

$$\overline{k}\big(x \otimes (x \otimes y - y \otimes x)\big) = 1.\overline{f}(x \otimes y - y \otimes x) = 1.(1+M) = 1 + M .$$

Thus $x \otimes (x \otimes y - y \otimes x)$ is a non-zero element in $\operatorname{tor}\big(R/xR,\ \operatorname{tor}\big(xR+yR,\ R/(xR+yR)\big)\big)$. This gives a contradiction. Thus either x divides y or y divides x, as required.

PROPOSITION 2.2. *Let R be a local ring with maximal ideal M and suppose that* tor *is associative over R. If $\bigcap_{n=1}^{\infty} M^n = 0$ and if there exists an element in R which is neither a unit nor a zero-divisor then R is a valuation ring.*

Proof. We shall first of all show that R is an integral domain. By hypothesis, there exists an element x of R which is neither a unit nor a zero-divisor. Let y be any zero-divisor of R. Then, by Proposition 2.1, either x divides y or y divides x. Clearly the latter is impossible and so x must divide y. In other words, there exists an element c_1 of R such that $y = c_1x$. Since c_1 is then a zero-divisor, by repeating the argument there exists a non-zero element c_2 of R such that $c_1 = c_2x$. Then $y = c_1x = c_2x^2$ and so c_2 is a zero-divisor. Repeating this procedure a suitable number of times we get, for any positive integer n, that $y = c_nx^n$ where c_n is a zero-divisor in R. Thus, since x is an element of M, we have $y \in \bigcap_{n=1}^{\infty} M^n$. Hence $y = 0$. It follows that R is an integral domain. Thus, by Proposition 2.1, since tor is associative over R, given any two elements a and b of R either a divides b or b divides a. This means that R is a valuation ring, as required.

It can also be proved that the valuation ring of Proposition 2.2 is in fact a discrete valuation ring by using, for example, Theorem 14.3

and Theorem 14.5 of Gilmer [5]. We now come to the main result of this section.

THEOREM 2.3. *Let* R *be a local noetherian ring with maximal ideal* M *such that either* $\operatorname{ann} M$ *, the annihilator of* M *, is zero or* R *is semi-prime. Then* tor *is associative over* R *if and only if* R *is a discrete valuation ring.*

Proof. By Theorem 3, p. 50 of Northcott [13], $\bigcap_{n=1}^{\infty} M^n = 0$. Thus, if there exists an element of R which is neither a unit nor a zero-divisor, then, by Proposition 2.2, R is a valuation ring. Since R is noetherian, R must be a discrete valuation ring.

Suppose, on the other hand, that every non-unit of R is also a zero-divisor. Then, by Theorem 80, p. 55 of Kaplansky [10], M is the annihilator of one of its elements. If $\operatorname{ann} M = 0$ this is impossible. If R is semi-prime then $M = 0$ so that R is a field and therefore, trivially, a discrete valuation ring.

3. Uniserial rings

If R is an artinian principal ideal ring then we say that R is a *uniserial ring*. In this section we shall show that tor is associative over any uniserial ring. We shall use the following proposition. Part (1) has been proved by Kaplansky [9, Theorem 13.3] and part (2) is a result of Köthe [11] (see also Cohen and Kaplansky [3]).

PROPOSITION 3.1. (1) *Any uniserial ring is the direct sum of finitely many local uniserial rings.*

(2) *Any module over a uniserial ring is the direct sum of cyclic submodules.*

Let R be a local uniserial ring with maximal ideal M and suppose that R is not a field. Then, since R is a principal ideal ring, $M = pR$ for some non-unit p of R . Moreover, since R is artinian, $M^n = 0$ for some least positive integer n . Also, any proper ideal of R is of the form p^tR for some integer t such that $0 < t < n$. Thus any proper cyclic R-module is of the form R/p^tR for some integer t such

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that $0 < t < n$. For convenience we will call this integer t the *order* of the cyclic module R/p^tR. Any cyclic R-module is completely determined by its order.

PROPOSITION 3.2. *Let R be a local uniserial ring with maximal ideal pR such that $p^n = 0$ for some integer $n > 1$ but $p^{n-1} \neq 0$. If s, t are integers such that $0 < s < n$ and $0 < t < n$ then $\operatorname{tor}(R/p^sR,\ R/p^tR)$ is a cyclic R-module of order $\min(n,\ s+t) - \max(s,\ t)$.*

Proof. Suppose first that $\min(n,\ s+t) = n$. Then

$$\operatorname{tor}(R/p^sR,\ R/p^tR) = p^{\max(s,t)}R/p^{s+t}R = p^{\max(s,t)}R$$

since $p^{s+t}R = 0$. Now define $f : R \to p^{\max(s,t)}R$ by $f(1) = p^{\max(s,t)}$. Then $\ker f = p^{n-\max(s,t)}R$. Thus, since f is an epimorphism, $p^{\max(s,t)}R = R/\ker f = R/p^{n-\max(s,t)}R$. Hence $\operatorname{tor}(R/p^sR,\ R/p^rR)$ is a cyclic R-module of order $n - \max(s,\ t) = \min(n,\ s+t) - \max(s,\ t)$, as required.

Now suppose that $\min(n,\ s+t) = s + t$. Then

$$\operatorname{tor}(R/p^sR,\ R/p^tR) = p^{\max(s,t)}R/p^{s+t}R .$$

Define $g : R \to p^{\max(s,t)}R/p^{s+t}R$ by $g(1) = p^{\max(s,t)} + p^{s+t}R$. Then g is an epimorphism with kernel $p^{s+t-\max(s,t)}R$. It follows that $\operatorname{tor}(R/p^sR,\ R/p^tR)$ is a cyclic R-module of order

$$s + t - \max(s,\ t) = \min(n,\ s+t) - \max(s,\ t) .$$

This completes the proof.

COROLLARY 3.3. *Let R be a local uniserial ring with maximal ideal pR such that $p^n = 0$ for some integer $n > 1$ but $p^{n-1} \neq 0$. If r, s, t are positive integers all less than n then*

$$\operatorname{tor}\left(R/p^rR,\ \operatorname{tor}(R/p^sR,\ R/p^tR)\right) = \operatorname{tor}\left(\operatorname{tor}(R/p^sR,\ R/p^tR),\ R/p^rR\right)$$

is a cyclic R-module of order

$$\min\{n,\ \min(s+t,\ n)-\max(s,\ t)+r\} - \max\{r,\ \min(s+t,\ n)-\max(s,\ t)\} .$$

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THEOREM 3.4. tor *is associative over any uniserial ring* R .

Proof. By part (1) of Proposition 3.1, any uniserial ring is the direct sum of finitely many local uniserial rings. Thus, since tor commutes with finite direct sums of rings, it is sufficient to assume that R is a local uniserial ring.

Suppose that R has maximal ideal pR where $p^n = 0$ for some integer $n > 1$ but $p^{n-1} \neq 0$ (in the case of $n = 1$ the result is obvious since R is a field). Let A, B, C be R-modules. Then, by part (2) of Proposition 3.1 we may write $A = \bigoplus_I A_i$, $B = \bigoplus_J B_j$, $C = \bigoplus_K C_k$ where each A_i, B_j, C_k is a cyclic R-module. Moreover, since tor commutes with direct sums of R-modules, we have

$$\operatorname{tor}\left(\operatorname{tor}(A, B), C\right) = \bigoplus_I \bigoplus_J \bigoplus_K \operatorname{tor}\left(\operatorname{tor}(A_i, B_j), C_k\right)$$

and

$$\operatorname{tor}\left(A, \operatorname{tor}(B, C)\right) = \bigoplus_I \bigoplus_J \bigoplus_K \operatorname{tor}\left(A_i, \operatorname{tor}(B_j, C_k)\right) .$$

Thus to prove the theorem it is sufficient to show that

$$\operatorname{tor}\left(A, \operatorname{tor}(B, C)\right) \simeq \operatorname{tor}\left(\operatorname{tor}(A, B), C\right)$$

for any cyclic R-modules A, B, C .

Let r, s, t be positive integers all less than n . We must show that

$$\operatorname{tor}\left(\operatorname{tor}\left(R/p^rR, R/p^sR\right), R/p^tR\right) \simeq \operatorname{tor}\left(R/p^rR, \operatorname{tor}\left(R/p^sR, R/p^tR\right)\right) .$$

By Corollary 3.3 these two modules are cyclic and so it is sufficient to show that they have the same order, in other words that

$$\begin{aligned} &\min\{n, \min(s+t, n)-\max(s, t)+r\} - \max\{r, \min(s+t, n)-\max(s, t)\} \\ &\quad = \min\{n, \min(r+s, n)-\max(r, s)+t\} - \max\{t, \min(r+s, n)-\max(r, s)\} . \end{aligned}$$

This can be proved by either reducing one side of the equation to an expression symmetric in r, s, t or by considering various cases - we omit this verification.

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4. Local noetherian rings with $\operatorname{ann}M \neq 0$

Our objective now is to show that given a local noetherian ring R with maximal ideal M then R is uniserial if $\operatorname{ann}M \neq 0$ and tor is associative over R. We require the following.

PROPOSITION 4.1. *Let R be a local ring with non-trivial maximal ideal M. If* tor *is associative over R then* $\operatorname{ann}M$ *is contained in every non-zero ideal of R.*

Proof. Let a and b be non-zero elements of $\operatorname{ann}M$ and M respectively. Then $ab = 0$ and $a^2 = 0$. Thus

$$\operatorname{tor}(R/aR,\ R/bR) = \frac{aR \cap bR}{abR} = aR \cap bR$$

and

$$\operatorname{tor}\big(R/aR,\ R/(aR+bR)\big) = \frac{(aR+bR) \cap aR}{aR.(aR+bR)} = aR\ .$$

Hence

$$\begin{aligned}\operatorname{tor}\big(\operatorname{tor}\big(R/aR,\ R/(aR+bR)\big),\ R/bR\big) &= \operatorname{tor}(aR,\ R/bR)\\ &= \ker(aR \otimes bR \to aR) = aR \otimes bR \simeq R/\operatorname{ann}a \otimes R/\operatorname{ann}b\\ &= R/M \otimes R/\operatorname{ann}b \simeq \frac{R}{M+\operatorname{ann}b} = R/M \neq 0\ .\end{aligned}$$

However, since tor is associative over R,

$$\operatorname{tor}\big(\operatorname{tor}\big(R/aR,\ R/(aR+bR)\big),\ R/bR\big) \simeq \operatorname{tor}\big(\operatorname{tor}(R/aR,\ R/bR),\ R/(aR+bR)\big)$$

and so $\operatorname{tor}(R/aR,\ R/bR) \neq 0$. Hence, by above, $aR \cap bR \neq 0$. Thus there exists $s \in R$ such that $as \neq 0$ and $as \in bR$. Hence, since $a \in \operatorname{ann}M$, s must be a unit and so $a = (as)s^{-1} \in bR$. It follows that $\operatorname{ann}M \subseteq bR$ and so that $\operatorname{ann}M$ is contained in every non-zero ideal of R, as required.

COROLLARY 4.2. *Let R be a local ring with non-trivial maximal ideal M. If* tor *is associative over R then R has either no minimal ideals or precisely one, namely* $\operatorname{ann}M$.

Proof. Let $a = xR$ be a minimal ideal of R. Then either $aM = a$ or $aM = 0$. If $aM = a$ then $x = xm$ for some $m \in M$. Thus $x(1-m) = 0$, which is impossible since $x \neq 0$ and $1 - m$ is a unit. Thus

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$aM = 0$ and so $\operatorname{ann}M \neq 0$. The result now follows from Proposition 4.1.

Recall that an ideal a of R is called *large* if for every non-zero ideal b of R we have $a \cap b \neq 0$. Also, $Z(R)$, the *singular ideal* of R , is the set of all elements of R which annihilate large ideals of R .

THEOREM 4.3. *Let R be a local noetherian ring with non-trivial maximal ideal M such that $\operatorname{ann}M \neq 0$. If* tor *is associative over R then R is uniserial.*

Proof. By Proposition 4.1, $\operatorname{ann}M$ is contained in every non-zero ideal of R and so it is large. Thus $M = \operatorname{ann}(\operatorname{ann}M) \subseteq Z(R)$ and so $M = Z(R)$. Hence, by Proposition 3, p. 107 of Lambek [12] there is a natural number n such that $M^{n+1} = 0$ but $M^n \neq 0$. Moreover $M^n \subseteq \operatorname{ann}M$ since $M^n.M = 0$ and so $M^n = \operatorname{ann}M$. Since $\operatorname{ann}M$ is a simple module it follows that $M^n \simeq R/M$.

Now let k be an integer such that $0 < k \leq n$ and suppose that $y \in M^k$ but $y \notin M^{k+1}$. Then

$$\operatorname{tor}(\operatorname{tor}(R/yR,\ R/M^k),\ R/M^n) = \operatorname{tor}\left(\frac{yR\cap M^k}{yM^k},\ R/M^n\right) = \operatorname{tor}\left(\frac{yR}{yM^k},\ R/M^n\right)$$

$$= \ker\left(M^n \otimes \frac{yR}{yM^k} \to \frac{yR}{yM^k}\right) = M^n \otimes \frac{yR}{yM^k} \simeq R/M \otimes \frac{yR}{yM^k} \simeq R/M \otimes R/M^k \otimes yR$$

$$\simeq R/(M+M^k) \otimes yR \simeq R/M \otimes R/\operatorname{ann}y \simeq R/(M+\operatorname{ann}y) = R/M .$$

However, since tor is associative over R ,

$$\operatorname{tor}(\operatorname{tor}(R/yR,\ R/M^k),\ R/M^n) \simeq \operatorname{tor}(\operatorname{tor}(R/yR,\ R/M^n),\ R/M^k)$$

$$= \operatorname{tor}\left(\frac{yR\cap M^n}{yM^n},\ R/M^k\right) = \operatorname{tor}(M^n,\ R/M^k) \simeq \operatorname{tor}(R/M,\ R/M^k) = \frac{M\cap M^k}{M.M^k} = M^k/M^{k+1} .$$

Thus $M^k/M^{k+1} \simeq R/M$. Hence for any k , $0 < k \leq n$, M^k/M^{k+1} is a simple module. Thus $R \supset M \supset M^2 \supset \ldots \supset M^n \supset M^{n+1} = 0$ is a composition series for R . It follows, since R is local, that $M = xR$ for some $x \in M$, each ideal of R is of the form x^kR for some k , $0 \leq k \leq n+1$, and so R is uniserial, as required.

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5. Arithmetical rings

Following Fuchs [4] we call a ring R *arithmetical* if

$$a \cap (b+c) = a \cap b + a \cap c$$

for all ideals a, b, c of R. Jensen [8] has shown that a ring R is arithmetical if and only if, for any maximal ideal M, the ideals of the local ring R_M are totally ordered by set inclusion. Using this we are able to summarise the main results proved so far in the following theorem.

THEOREM 5.1. *Let R be a noetherian ring. Then the following statements are equivalent:*

(1) *R is arithmetical;*

(2) *R is the direct sum of finitely many Dedekind rings and uniserial rings;*

(3) tor *is associative over R.*

Proof. The equivalence of (1) and (2) has been proved by Asano [1].

By Theorem 3.4, tor is associative over any uniserial ring. Also, since every Dedekind ring is semi-hereditary, tor is associative over any Dedekind ring, by Proposition 3.5, p. 115 of Cartan and Eilenberg [2]. Thus, since tor commutes with direct sums of rings, (2) implies (3).

Now suppose tor is associative over R. Let S denote the localization of R with respect to a particular maximal ideal of R. Then, by Theorem 7, p. 171 of Northcott [14], tor is associative over S. Let M denote the maximal ideal of S. Then, if $\operatorname{ann} M = 0$, by Theorem 2.3, S is a valuation ring. If $\operatorname{ann} M \neq 0$, by Theorem 4.3, S is a local uniserial ring. Since in both valuation rings and local uniserial rings, the ideals are totally ordered by set inclusion, it follows that R is arithmetical. Thus (3) implies (1).

Because of Theorem 5.1 it seems reasonable to conjecture that given any ring R then tor is associative over R if and only if R is arithmetical. We now give further evidence to support this.

A semi-hereditary integral domain is called a Prüfer *ring*. A well-known result of Jensen [7] characterizes Prüfer rings as those integral domains which are arithmetical. Also Hattori [6] has shown that an integral domain R is a Prüfer ring if and only if every torsion-free R-module is flat.

Let R be an integral domain and suppose that tor is associative over R. Let M be any maximal ideal of R. Then tor is associative over the integral domain R_M and it then follows from Proposition 2.1 that R_M is a valuation ring. Hence R is arithmetical. It may be of interest to present here an alternative proof of this, independent of Proposition 2.1.

THEOREM 5.2. *Let* R *be an integral domain. Then* tor *is associative over* R *if and only if* R *is arithmetical.*

Proof. Suppose R is arithmetical. Then, by the remarks above, R is semi-hereditary and so tor is associative over R.

Conversely, suppose tor is associative over R. Let M be any torsion-free R-module. Let a be any non-zero ideal of R and choose $r \in a$, $r \neq 0$. Then, for any R-module A we have $\operatorname{tor}(R/rR, A) = \ker(r : A \to A)$ where the mapping r is multiplication by r (see, for example, Cartan and Eilenberg [2], p. 129). Thus

$$\begin{aligned}\operatorname{tor}(M, R/a) &= \operatorname{tor}\big(M, \ker(r : R/a \to R/a)\big) = \operatorname{tor}\big(M, \operatorname{tor}(R/rR, R/a)\big)\\ &\simeq \operatorname{tor}\big(\operatorname{tor}(R/rR, M), R/a\big) = \operatorname{tor}\big(\ker(r : M \to M), R/a\big) = \operatorname{tor}(0, R/a),\end{aligned}$$

since M is torsion-free. Thus, for any non-zero ideal a of R, $\operatorname{tor}(M, R/a) = 0$. Hence M is flat. It follows now, from Hattori's result mentioned above, that R is Prüfer and so arithmetical, as required.

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