A NOTE ON THE ASSOCIATED LEGENDRE POLYNOMIALS

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1. This paper gives what appears to be a new Rodrigues' formula for the Associated Legendre Polynomials defined by [5, p. 122]

$$P_n^m(x) = (x^2-1)^{\frac{1}{2}m}\left(\frac{d}{dx}\right)^m P_n(x), \tag{1.1}$$

with the restriction that m is an even positive integer, which helps to evaluate some integrals.

Putting $m = 2k$ in (1.1) and replacing $P_n(x)$ by the Gegenbauer Polynomial $C_n^{\frac{1}{2}}(x)$ and using [3, p. 176]

$$\left(\frac{d}{dx}\right)^m C_n^\lambda(x) = 2^m(\lambda)_m C_{n-m}^{\lambda+m}(x), \tag{1.2}$$

we obtain

$$P_n^{2k}(x) = 2^{2k}(-1)^k(\tfrac{1}{2})_{2k}(1-x^2)^k C_{n-2k}^{2k+\frac{1}{2}}(x). \tag{1.3}$$

Putting $\alpha = \nu - \frac{1}{2}$ in the relation [4, p. 283]

$$\int_{-1}^{1}(1-x)^\alpha(1+x)^{\nu-\frac{1}{2}}C_m^\mu(x)C_n^\nu(x)\,dx = \frac{2^{\alpha+\nu+\frac{1}{2}}\Gamma(\alpha+1)\Gamma(\nu+\frac{1}{2})\Gamma(\nu-\alpha+n-\frac{1}{2})\Gamma(2\mu+m)\Gamma(2\nu+n)}{m!\,n!\,\Gamma(\nu-\alpha-\frac{1}{2})\Gamma(\nu-\alpha+n+\frac{3}{2})\Gamma(2\mu)\Gamma(2\nu)}$$
$$\times {}_4F_3\left[\begin{matrix}-m,\ m+2\mu,\ \alpha+1,\ \alpha-\nu+\frac{3}{2};\ 1\\ \mu+\frac{1}{2},\ \nu+\alpha+n+\frac{3}{2},\ \alpha-\nu-n+\frac{3}{2}\end{matrix}\right],$$

we get

$$\int_{-1}^{1}(1-x^2)^{\nu-\frac{1}{2}}C_n^\nu(x)C_m^\mu(x)\,dx$$
$$= \frac{(\mu)_m(2\nu)_n\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(1-\mu-m)\Gamma(1-\mu+\nu)}{n!\left(\frac{m-n}{2}\right)!\,\Gamma\left(\frac{m}{2}+\frac{n}{2}+\nu+1\right)\Gamma\left(1-\mu+\nu-\frac{m}{2}+\frac{n}{2}\right)\Gamma\left(1-\mu-\frac{m}{2}-\frac{n}{2}\right)}$$
$$(m \geqq n,\ m+n \text{ even}). \tag{1.4}$$

Consider

$$(1-x^2)^k C_{n-2k}^{2k+\frac{1}{2}}(x) = \sum A_l C_{n-2l}^{k+\frac{1}{2}}(x). \tag{1.5}$$

Multiplying both sides of (1.5) by

$$(1-x^2)^k C_{n-2l}^{k+\frac{1}{2}}(x),$$

integrating with respect to x between the limits $(-1, 1)$, and using the result (1.4) and the

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orthogonal property of the Gegenbauer Polynomials, we find A_l, and substitution in (1.5) then gives

$$(1-x^2)^k C_{n-2k}^{2k+\frac{1}{2}}(x) = \frac{(n+2k)!\,\{(2k)!\}^2}{(4k)!\,(n-2k)!} \sum_{l=0}^{k} \frac{\Gamma(n+k+\frac{3}{2}-2l)}{l!\,(k-l)!\,\Gamma(n+k-l+\frac{3}{2})} \times \frac{\Gamma(\frac{1}{2}-k-n+2l)(n-2l)!}{\Gamma(\frac{1}{2}-n+l)\Gamma(n+2k+1-2l)} C_{n-2l}^{k+\frac{1}{2}}(x). \tag{1.6}$$

On expanding $C_{n-2l}^{k+\frac{1}{2}}(x)$ in powers of $(1-x)/2$ with the help of [3, p. 176]

$$n!\, C_n^{\lambda}(x) = (2\lambda)_n F(-n, n+2\lambda;\ \lambda+\tfrac{1}{2};\ \tfrac{1}{2}-\tfrac{1}{2}x), \tag{1.7}$$

(1.6) becomes

$$(1-x^2)^k C_{n-2k}^{2k+\frac{1}{2}}(x) = \frac{(2k)!\,(-1)^k}{(4k)!\,(n-2k)!\,k!\,(n+\frac{1}{2})_k} \sum_{s=k}^{n} \frac{(-n)_s(n+2k+s)!}{s!\,(1+k)_s}$$

$$\times {}_7F_6\left[\begin{matrix} -n-k-\frac{1}{2},\ \frac{3}{4}-\frac{1}{2}k-\frac{1}{2}n,\ \frac{1}{2}-k-\frac{1}{2}n,\ -k-\frac{1}{2}n,\ -\frac{1}{2}n+\frac{1}{2}s,\ -\frac{1}{2}n+\frac{1}{2}s+\frac{1}{2},\ -k;\ 1 \\ -\frac{1}{2}n-\frac{1}{2}k-\frac{1}{4},\ -\frac{1}{2}n,\ -\frac{1}{2}n+\frac{1}{2},\ -\frac{1}{2}n-\frac{1}{2}s-k+\frac{1}{2},\ -\frac{1}{2}n-\frac{1}{2}s-k,\ \frac{1}{2}-n \end{matrix}\right]\left(\frac{1-x}{2}\right)^s,$$

and because [1, p. 26]

$${}_7F_6\left[\begin{matrix} a,\ 1+\frac{1}{2}a,\ b,\ c,\ d,\ e,\ -m;\ 1 \\ \frac{1}{2}a,\ 1+a-b,\ 1+a-c,\ 1+a-d,\ 1+a-e,\ 1+a+m \end{matrix}\right] = \frac{(1+a)_m(1+a-d-e)_m}{(1+a-d)_m(1+a-e)_m}\, {}_4F_3\left[\begin{matrix} 1+a-b-c,\ d,\ e,\ -m;\ 1 \\ 1+a-b,\ 1+a-c,\ d+e-a-m \end{matrix}\right],$$

(1.3) reduces to

$$P_n^{2k}(x) = \frac{n!}{(n-2k)!} \sum_{s=k}^{n} \frac{(-n)_s(n+1)_s}{s!\,s!}\, {}_4F_3\left[\begin{matrix} -k,\ k,\ -\frac{1}{2}n+\frac{1}{2}s,\ -\frac{1}{2}n+\frac{1}{2}s+\frac{1}{2};\ 1 \\ -\frac{1}{2}n,\ -\frac{1}{2}n+\frac{1}{2},\ 1+s \end{matrix}\right]\left(\frac{1-x}{2}\right)^s. \tag{1.8}$$

2. Let

$$P_n^{2k}(x) = \sum A_{k,r} C_{n-2r}^{r+\frac{1}{2}}(x). \tag{2.1}$$

Giving particular values for $k = 1, 2, 3, \dots$ we deduce that

$$P_n^{2k}(x) = \frac{1}{(n-2k)!} \sum_{r=0}^{k} \frac{(-k)_r(k)_r 2^{2r}(n-2r)!\,(\frac{1}{2})_r}{r!} C_{n-2r}^{r+\frac{1}{2}}(x). \tag{2.2}$$

To prove (2.2), we expand $C_{n-2r}^{r+\frac{1}{2}}(x)$ with the help of (1.7) and get the result (1.8). Hence the result (2.2) is true.

Substituting the value of $C_{n-2r}^{r+\frac{1}{2}}(x)$ and using [3, p. 180]

$$\left(\frac{d}{dx}\right)^{m+n}(x^2-1)^n = 2^{m+n}(\tfrac{1}{2})_m n!\, C_{n-m}^{m+\frac{1}{2}}(x) \tag{2.3}$$

in (2.2), we see that the Rodrigues formula for $P_n^{2k}(x)$ is given by

$$P_n^{2k}(x) = \frac{1}{2^n(n-2k)!}\left(\frac{d}{dx}\right)^n\left\{(x^2-1)^n\,{}_3F_2\left[\begin{matrix}-k,\ -n,\ k; & 1/(1-x^2)\\ -\tfrac{1}{2}n,\ -\tfrac{1}{2}n+\tfrac{1}{2} & \end{matrix}\right]\right\}. \tag{2.4}$$

It can be easily verified that $\phi_i(x)$ given by [2] is $\dfrac{1}{n(n+1)}P_{n+1}^2(x)$ satisfying (2.4).

3. Taking m, p and q as positive integers and l any integer, consider

$$\left[\left(\frac{d}{dx}\right)^q\left\{(x^2-1)^{-l}P_m^{2p}(x)\right\}\right]_{x=\pm 1} \quad (p \geqq l,\ q = 0, 1, 2, \ldots). \tag{3.1}$$

Substituting the value of $P_m^{2p}(x)$ and using (1.1) in (3.1), applying Demoivre's theorem of differentiation and using the relations (2.3) and

$$\left[\left(\frac{d}{dx}\right)^s P_n(x)\right]_{x=\pm 1} = (\pm 1)^{s+n}\frac{(s+n)!}{s!\,2^s(n-s)!}$$

in (3.1), we obtain

$$\left[\left(\frac{d}{dx}\right)^q\left\{(x^2-1)^{-l}P_m^{2p}(x)\right\}\right]_{x=\pm 1} = 0 \quad \text{for } q = 0, 1, 2, \ldots, p-l-1, \tag{3.2}$$

$$= (\pm 1)^{q+m}\frac{(m+p+l+q)!\,q!}{2^{q+2l}(q-p+l)!\,(p+q+l)!\,(m-p-q-l)!}$$

$$\times\,{}_3F_2\left[\begin{matrix}-p+l,\ -q+p-l,\ -p-q-l; & 1\\ -m-p-q-l,\ 1+m-p-q-l & \end{matrix}\right] \tag{3.3}$$

for $q = p-l,\ p-l+1, \ldots, (m-p-l)$, and

$$= (\pm 1)^{q+m}\frac{q!\,(2m)!\,(p-l)!}{2^{q+2l}(m-2p)!\,(p+l-m+q)!\,(m-2l-q)!\,m!}$$

$$\times\,{}_3F_2\left[\begin{matrix}-m,\ -m+2p,\ -m+2l+q; & 1\\ -2m,\ 1+p+l+q-m & \end{matrix}\right] \tag{3.4}$$

for $q = m-p-l+1,\ m-p-l+2, \ldots, m-2l$.

We have, from [3, p. 175],

$$\left(\frac{d}{dx}\right)^n\left[(1-x^2)^{n+\lambda-\frac{1}{2}}\right] = \frac{2^n n!\,(\lambda+\frac{1}{2})_n(1-x^2)^{\lambda-\frac{1}{2}}C_n^\lambda(x)}{(-1)^n(2\lambda)_n}. \tag{3.5}$$

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Now consider

$$A_{n-q}(x) \equiv \left(\frac{d}{dx}\right)^{n-q}\left[(x^2-1)^n\, {}_3F_2\left\{\begin{matrix}-k,\ -n,\ k; & 1/(1-x^2)\\ -\frac{1}{2}n,\ -\frac{1}{2}n+\frac{1}{2} & \end{matrix}\right\}\right]. \tag{3.6}$$

From this, with the help of (3.5) and (2.3), we get, for $q \leqq k$,

$$A_{n-q}(x) = \sum_{r=0}^{q-1} \frac{(-k)_r(k)_r(n-q)!\,(-1)^{q+n}(1-x^2)^{q-r}(q-r+1)_{n-q}2^{2r+n-q}}{r!\,(n-r)!\,(2q-2r+1)_{n-q}} C_{n-q}^{q-r+\frac{1}{2}}(x)$$

$$+ \sum_{r=q}^{k} \frac{(-k)_r(k)_r 2^{2r+n-q}(n-2r)!\,(\frac{1}{2})_{r-q}}{r!} C_{n+q-2r}^{r-q+\frac{1}{2}}(x),$$

and as

$$C_n^{\lambda}(\pm 1) = (\pm 1)^n \frac{(2\lambda)_n}{n!},$$

we have

$$A_{n-q}(\pm 1) = 0, \quad \text{for } q = k+1, k+2, \ldots, n; \tag{3.7}$$

$$A_{n-q}(\pm 1) = (\pm 1)^{n+q} \frac{(-k)_q(k)_q 2^{n+q}(n-2q)!}{q!} \times {}_4F_3\left[\begin{matrix}-k+q,\ k+q,\ -\frac{1}{2}n+\frac{1}{2}q,\ -\frac{1}{2}n+\frac{1}{2}q+\frac{1}{2}; & 1\\ -\frac{1}{2}n+q,\ -\frac{1}{2}n+q+\frac{1}{2},\ 1+q & \end{matrix}\right], \tag{3.8}$$

for $q = 0, 1, 2, \ldots, k$.

4. Consider the integral

$$\int_{-1}^{1} (1-x^2)^{-l} P_m^{2p}(x) P_n^{2k}(x)\, dx \quad (m+n \text{ even}). \tag{4.1}$$

Without loss of generality we can take $m \leqq n$. Substituting the value of $P_n^{2k}(x)$ from (2.4) in (4.1) and integrating by parts as many times as necessary, using the results (3.2), (3.3), (3.4) and (3.7), (3.8), we obtain:

(i) When $p-l \geqq k$ and $m-2l < n$,

$$\int_{-1}^{1} (1-x^2)^{-l} P_m^{2p}(x) P_n^{2k}(x)\, dx = 0; \tag{4.2}$$

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(ii) When $p-l<k\leqq m-p-l$,

$$\int_{-1}^{1}(1-x^2)^{-l}P_m^{2p}(x)P_n^{2k}(x)\,dx$$

$$=\frac{(-1)^l}{2^{2l-2}(n-2k)!}\sum_{q=p-l}^{k-1}\frac{(-1)^q(m+p+q+l)!\,(-k)_{q+1}(k)_{q+1}(n-2q-2)!}{(q-p+l)!\,(p+q+l)!\,(m-p-q-l)!\,(q+1)}$$

$$\times\,{}_3F_2\left[\begin{matrix}-p+l,\ -q+p-l,\ -p-q-l;\ 1\\ -m-p-q-l,\ 1+m-p-q-l\end{matrix}\right]$$

$$\times\,{}_4F_3\left[\begin{matrix}-k+q+1,\ k+q+1,\ -\tfrac{1}{2}n+\tfrac{1}{2}q+\tfrac{1}{2},\ -\tfrac{1}{2}n+\tfrac{1}{2}q+1;\ 1\\ -\tfrac{1}{2}n+q+1,\ -\tfrac{1}{2}n+q+\tfrac{3}{2},\ 2+q\end{matrix}\right]. \tag{4.3}$$

Thus under different conditions we can find the value of the integral (4.1) with the help of (3.2), (3.3), (3.4), (3.7) and (3.8).

On the same lines it can be proved that

$$\int_{-1}^{1}(1\pm x)^{-l}P_m^{2p}(x)P_n^{2k}(x)\,dx=0, \tag{4.4}$$

if $m\leqq n$, $p-l\geqq k$ and l satisfies the condition

$$-\frac{n-m}{2}<l\leqq p;$$

and generally

$$\int_{-1}^{1}f(x)P_m^{2p}(x)P_n^{2k}(x)\,dx=0, \tag{4.5}$$

where $f(x)$ is any polynomial of degree l, under the conditions $m+l<n$ and $p\geqq k$.

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