Proceedings of the Edinburgh Mathematical Society (2002) **45**, 557–563 ©
DOI:10.1017/S0013091500000420 Printed in the United Kingdom

SETS OF UNIQUENESS OF SERIES OF STOCHASTICALLY INDEPENDENT FUNCTIONS

FRANCISCO J. FRENICHE AND RICARDO RÍOS-COLLANTES-DE-TERÁN

Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Sevilla, Apdo. 1160, Sevilla 41080, Spain
(freniche@us.es; rori@us.es)

(Received 4 April 2000)

Abstract It is shown that, for every sequence (f_n) of stochastically independent functions defined on $[0,1]$—of mean zero and variance one, uniformly bounded by M—if the series $\sum_{n=1}^{\infty} a_n f_n$ converges to some constant on a set of positive measure, then there are only finitely many non-null coefficients a_n, extending similar results by Stechkin and Ul'yanov on the Rademacher system. The best constant C_M is computed such that for every such sequence (f_n) any set of measure strictly less than C_M is a set of uniqueness for (f_n).

Keywords: set of uniqueness; stochastically independent system; set of constancy

AMS 2000 *Mathematics subject classification:* Primary 42C25
Secondary 60G50

1. Introduction

First of all, let us recall the concept of a set of uniqueness of a system of complex functions. Let (f_n) be an orthonormal sequence of functions defined on the unit interval $[0,1]$. It is said that $E \subset [0,1]$ is a set of uniqueness for (f_n) if the unique series $\sum_{n=1}^{\infty} a_n f_n$, which converges to zero on the complement of E, is the null series.

Sets of uniqueness for trigonometrical series have been studied deeply since Cantor, who proved that finite sets are sets of uniqueness. Sets of Lebesgue positive measure $m(E) > 0$ are never sets of uniqueness for the trigonometric system, but there exist sets of measure zero which are not of uniqueness, a result of Menshov. The behaviour of sets of uniqueness for lacunary trigonometrical series is quite different: it was shown by Zygmund that if $m(E) < 1$, then E is of uniqueness [**3**].

For the Rademacher system and any of its permutations, Stechkin and Ul'yanov [**2**] proved that if $m(E) < \frac{1}{2}$, then E is of uniqueness, as well as proving that a weaker uniqueness theorem is true for sets with $m(E) < 1$: namely, that any series which converges to zero on the complement of E is actually a finite sum.

We consider a fixed constant $M \geqslant 1$. We shall denote by $\mathcal{F}_M$ the class of all measurable functions defined on $[0,1]$ with mean zero, variance one and bounded by M, that is to say, such that

$$\int_0^1 f(x)\,\mathrm{d}x = 0, \quad \int_0^1 |f(x)|^2\,\mathrm{d}x = 1, \quad |f(x)| \leqslant M \text{ almost everywhere.}$$

We shall extend the Stechkin and Ul'yanov theorems for systems of functions (f_n) in the class $\mathcal{F}_M$ which are stochastically independent, that is, those systems for which

$$m(f_1^{-1}(B_1) \cap \cdots \cap f_n^{-1}(B_n)) = m(f_1^{-1}(B_1)) \cdots m(f_n^{-1}(B_n))$$

for every finite family of Borel sets $B_1, \ldots, B_n$. We prove in Theorem 2.2, for these systems, that a non-null series can only be constant on a set of measure at most $k_M = M^2/(M^2+1)$—in particular, if $m(E) < 1 - k_M$, then E is of uniqueness—and we construct a system for which this bound is attained. In Theorem 2.3 it is shown that, if a series converges to a constant on a set of positive measure, then only finitely many of the coefficients are non-zero.

We next recall the result of Kashin and Saakyan [**1**, p. 30, Theorem 7]: there exists a constant C_M such that $m(\{x \in [0,1] : |P(x)| \geqslant \frac{1}{2}\|P\|_2\}) \geqslant C_M$ for every system (f_n) of independent functions in $\mathcal{F}_M$ and every polynomial $P = \sum_{n=1}^{p} a_n f_n$ in (f_n).

Let us remark that the constant C_M satisfies the property that if $m(E) < C_M$, then E is of uniqueness for every system of independent functions in $\mathcal{F}_M$. This fact follows from our Theorem 2.2 by taking $C_M = 1 - k_M$. We point out that the proof of this theorem is more elementary than that of the inequality of Kashin and Saakyan.

The last section of the paper is devoted to computing the best constant C_M satisfying this property, which is obtained in Theorem 3.6 (see also the remark following it). In order to compute it we introduce and study the coefficients $\alpha(M,p)$ which bound the measures of the sets of constancy of polynomials $P = \sum_{n=1}^{p+1} a_n f_n$ with every $a_n \neq 0$ and $f_1, \ldots, f_{p+1} \in \mathcal{F}_M$ stochastically independent.

It is worth remarking that, although we assume the functions f_n to be complex valued, the proofs apply to functions with values in any finite-dimensional normed space, without any change in the constants appearing in the statements of the theorems.

2. Sets of constancy

We begin with the study of the measure of the sets $F \subset [0,1]$ on which a series $\sum_{n=1}^{\infty} a_n f_n$ is constant, for some sequence (f_n) of stochastically independent functions in $\mathcal{F}_M$, without all the coefficients a_n being zero.

Let us fix such a sequence (f_n), a complex sequence (a_n), and a complex number μ. Write $F = \{x \in [0,1] : \sum_{n=1}^{\infty} a_n f_n(x) = \mu\}$. We shall look for the smallest constant C such that if $m(F) > C$, then $a_1 = 0$.

A natural way to prove that $a_1 = 0$ is, roughly speaking, to find two points $x, y \in F$ such that $\sum_{n=2}^{\infty} a_n f_n(x)$ and $\sum_{n=2}^{\infty} a_n f_n(y)$ are close together, but keeping $f_1(x)$ away from $f_1(y)$.

https://doi.org/10.1017/S0013091500000420 Published online by Cambridge University Press

Assume that $m(F) > C$. Let $\epsilon > 0$. In order to apply the independence hypothesis, we shall work with finitely many functions. Let

$$F_q = \bigcap_{p=q}^{\infty} \biggl\{ x \in [0,1] : \biggl| \sum_{n=1}^{p} a_n f_n(x) - \mu \biggr| < \varepsilon \biggr\}.$$

As the sequence F_q is increasing and $F \subset F_1 \cup F_2 \cup \cdots$, we have that $m(F_q) > C$ for some $q > 1$. It is enough to find $x, y \in F_q$ such that $|f_1(x) - f_1(y)| \geqslant d$ with $d > 0$ independent of ε, and $|a_n|\,|f_n(x) - f_n(y)| \leqslant \varepsilon/2^n$ for $n = 2, \dots, q$. Indeed, we then have

$$\begin{aligned} d|a_1| &\leqslant |a_1 f_1(x) - a_1 f_1(y)| \\ &\leqslant \sum_{n=2}^{q} |a_n|\,|f_n(x) - f_n(y)| + \biggl| \sum_{n=1}^{q} a_n f_n(x) - \sum_{n=1}^{q} a_n f_n(y) \biggr| \\ &\leqslant \sum_{n=2}^{q} \frac{\varepsilon}{2^n} + 2\varepsilon \\ &\leqslant 3\varepsilon. \end{aligned}$$

We consider sets H which can be written as $f_2^{-1}(B_2) \cap \cdots \cap f_q^{-1}(B_q)$, where B_n is a Borel set of diameter bounded by $\varepsilon/(2^n|a_n|)$. Let us observe that $|a_n|\,|f_n(x) - f_n(y)| \leqslant \varepsilon/2^n$ for $n = 2, \dots, q$, for every $x, y \in H$. As the functions are essentially bounded by M, we can cover $[0,1]$, up to a set of measure zero, with finitely many such sets H; hence there exists such a set H satisfying $m(F_q \cap H) > Cm(H)$.

Let us fix $x \in F_q \cap H$. If we can choose C and d such that $m(F_q \cap H) > m(f_1^{-1}(B) \cap H)$, where B is the disc with centre $f_1(x)$ and radius d, then we will have found $y \in F_q \cap H$ and $y \notin f_1^{-1}(B)$, therefore satisfying $|f_1(x) - f_1(y)| \geqslant d$. On the basis of the independence of the sequence (f_n), it is enough to be able to choose C and d such that $m(F_q) > m(f_1^{-1}(B))$. We estimate the measure of $f_1^{-1}(B)$ in the following lemma.

Lemma 2.1. *Let $0 < d \leqslant 1/2$ and let B be a Borel set contained in a ball of radius d. If $f \in \mathcal{F}_M$, then $m(f^{-1}(B)) \leqslant (M^2/(M^2 + (1-2d)^2))$.*

Proof. We can assume that B is a ball of radius d and that the measure of $E = f^{-1}(B)$ is strictly positive. We define the average $\lambda = (1/m(E)) \int_E f(x)\,\mathrm{d}x$ and the function $g(x) = \lambda$ for $x \in E$ and $g(x) = f(x)$ for $x \notin E$. Since $\lambda \in B$, we have

$$\|f\|_2 - \|g\|_2 \leqslant \|f - g\|_2 = \biggl(\int_E |f(x) - \lambda|^2\,\mathrm{d}x \biggr)^{1/2} \leqslant 2d.$$

As $f \in \mathcal{F}_M$ it follows that

$$(1-2d)^2 \leqslant \|g\|_2^2 \leqslant |\lambda|^2 m(E) + M^2(1 - m(E)).$$

On the other hand,

$$|\lambda| = \frac{1}{m(E)} \biggl| - \int_{[0,1]\setminus E} f(x)\,\mathrm{d}x \biggr| \leqslant \frac{M(1 - m(E))}{m(E)}.$$

The statement follows from these two inequalities. □

In this way we obtain the next theorem.

Theorem 2.2. *Let (f_n) be a sequence of stochastically independent functions in $\mathcal{F}_M$. Let (a_n) be a sequence of complex numbers. If the series $\sum_{n=1}^{\infty} a_n f_n$ converges to a constant on a set of measure strictly greater than $k_M = M^2/(M^2+1)$, then $a_n = 0$ for every n.*

Proof. Let $F = \{x \in [0,1] : \sum_{n=1}^{\infty} a_n f_n(x) = \mu\}$. We take $0 < d < 1/2$ and C such that

$$m(F) > C > \frac{M^2}{M^2 + (1-2d)^2} > k_M.$$

As we explained before, we obtain that $a_1 = 0$. Then, by induction, we get $a_n = 0$ for every n. □

The estimate obtained in Lemma 2.1 gives us the best bound for the measure of sets of constancy of functions f in $\mathcal{F}_M$, namely $m(f = \mu) \leqslant k_M$. Indeed, the function f_1 defined as $f_1(x) = 1/M$ on $[0, k_M]$ and $f_1(x) = -M$ on $(k_M, 1]$ is clearly in $\mathcal{F}_M$ and $m(f_1 = 1/M) = k_M$.

This fact points out to us that the constant k_M in Theorem 2.2 is the best possible. To see this, we construct the following sequence of independent functions (f_n) which are piecewise constant (according to [**1**, p. 18]): we start with f_1 as defined above, and, assuming that f_n has been defined, we then define f_{n+1} on each interval $[a, b)$, where f_n is constant as $f_{n+1}(x) = 1/M$ for $x \in [a, a + (b-a)k_M)$ and $f_{n+1}(x) = -M$ for $x \in [a + (b-a)k_M, b)$. It is clear that $f_n \in \mathcal{F}_M$ and, taking $a_1 = 1$ and $a_n = 0$ for $n \geqslant 2$, we obtain that the series $\sum_{n=1}^{\infty} a_n f_n$ is constant on the interval $[0, k_M]$ whose length is k_M.

We remark that, if $M = 1$, then the sequence we have constructed is the Rademacher system. Moreover, as $k_1 = 1/2$, we obtain the result by Stechkin and Ul'yanov.

In the previous example, only finitely many coefficients are non-null: this is not accidental, because, as we shall see in Theorem 3.6 below, this must happen for every series which is constant on a set of positive measure. Why is this so? Assume that (f_n) is the Rademacher sequence and that F is a set of constancy of $\sum_{n=1}^{\infty} a_n f_n$. Then there exists a dyadic interval H, let us say of length 2^{-p}, such that $m(F \cap H) > k_1 m(H)$. As $f_1, \dots, f_p$ are constants on H we can apply Theorem 2.2 to the tail subsequence $f_{p+1}, f_{p+2}, \dots$ on H, obtaining that $a_n = 0$ for every $n > p$. In the general case we can find a set H acting as the dyadic interval, but $\sum_{n=1}^{p} a_n f_n$ is not necessarily constant on $F \cap H$, hence Theorem 2.2 cannot be applied directly and we need to modify its proof.

Theorem 2.3. *Let (f_n) be a sequence of stochastically independent functions in the class $\mathcal{F}_M$. Let (a_n) be a sequence of complex numbers. Assume that the series $\sum_{n=1}^{\infty} a_n f_n(x)$ converges to a constant on a set of positive measure. Then there are only finitely many a_n with $a_n \neq 0$.*

Proof. Let F be the set where the series $\sum_{n=1}^{\infty} a_n f_n$ converges to a constant, let us say μ. As F is measurable with respect to the σ-algebra generated by the sequence (f_n),

https://doi.org/10.1017/S0013091500000420 Published online by Cambridge University Press

there exists a set $H = f_1^{-1}(B_1) \cap \cdots \cap f_p^{-1}(B_p)$, where $B_1, \ldots, B_p$ are Borel sets, such that $m(F \cap H) > k_M m(H)$.

We take $0 < d < 1/2$ and C such that

$$\frac{m(F \cap H)}{m(H)} > C > \frac{M^2}{M^2 + (1-2d)^2} > k_M.$$

To show that $a_{p+1} = 0$ we fix $\varepsilon > 0$ and we consider the sets F_q defined as above. Let $q > p+1$ such that $m(F_q \cap H) > Cm(H)$. There exist Borel sets $A_1, \ldots, A_p, A_{p+2}, \ldots, A_q$ satisfying that the diameter of every A_n is bounded by $\varepsilon/(2^n|a_n|)$ and $m(F_q \cap G) > Cm(G)$, where $G = f_1^{-1}(A_1) \cap \cdots f_p^{-1}(A_p) \cap f_{p+2}^{-1}(A_{p+2}) \cap \cdots \cap f_q^{-1}(A_q)$, and such that $A_n \subset B_n$ for $n = 1, \ldots, p$.

As in the proof of Theorem 2.2 we can find $x, y \in F_q \cap G$ such that $|f_{p+1}(x) - f_{p+1}(y)| \geqslant d$. It follows that $a_{p+1} = 0$ and, therefore, $a_n = 0$ for every $n > p$. □

3. Sets of uniqueness

In this section we shall determine the best constant for sets of uniqueness of series of uniformly bounded independent functions. We introduce the following definition: given a non-negative integer $p \geqslant 0$, let us define $\alpha(M, p)$ as the smallest upper bound of the measures of the sets A such that there exist $f_1, \ldots, f_{p+1}$, stochastically independent, in the class $\mathcal{F}_M$, satisfying that $\sum_{n=1}^{p+1} a_n f_n$ is constant on A, for some $a_1 \neq 0, \ldots, a_{p+1} \neq 0$.

Remark 3.1. Let us notice that $\alpha(M, 0)$ was computed in the paragraphs following Theorem 2.2, where we showed that $\alpha(M, 0) = M^2/(M^2 + 1)$.

Our first task is to show that $\alpha(M, p)$ is a decreasing function of p. We need the following lemma.

Lemma 3.2. *Let f be integrable on $[0, 1]$ and let $\alpha_f = \sup\{m(f = \mu) : \mu \in \mathbb{C}\}$. For every $\varepsilon > 0$ there exists $\delta > 0$ such that $m(|f - \mu| < \delta) \leqslant \alpha_f + \varepsilon$ for every complex number μ.*

Proof. Assume by contradiction that there exist $\varepsilon > 0$, $\delta_n \in (0, 1)$, $\delta_n \to 0$ and a sequence (μ_n) such that $m(|f - \mu_n| < \delta_n) > \alpha_f + \varepsilon$. As f is integrable, there is N such that $m(f \geqslant N) < \varepsilon$. Hence (μ_n) is bounded by $N + 1$, so we can assume as well that μ_n converges to some μ.

Given $\delta > 0$ we choose n satisfying both $|\mu_n - \mu| < \delta/2$ and $\delta_n < \delta/2$. We have that

$$m(|f - \mu| < \delta) \geqslant m(|f - \mu_n| < \delta_n) > \alpha_f + \varepsilon.$$

If $\delta \to 0$ we obtain that $m(f = \mu) \geqslant \alpha_f + \varepsilon > \alpha_f$, contradicting the definition of α_f. □

Proposition 3.3. $\alpha(M, p+1) \leqslant \alpha(M, p)$ *for every* $p \geqslant 0$.

https://doi.org/10.1017/S0013091500000420 Published online by Cambridge University Press

Proof. Let $\varepsilon > 0$. Let $a_1 \neq 0, \dots, a_{p+2} \neq 0$, $f_1, \dots, f_{p+2} \in \mathcal{F}_M$ stochastically independent. We apply Lemma 3.2 to the function $f = a_1 f_1 + \cdots + a_{p+1} f_{p+1}$, obtaining $\delta > 0$ such that $m(|f - \lambda| < \delta) \leqslant \alpha_f + \varepsilon \leqslant \alpha(M, p) + \varepsilon$ for every $\lambda \in \mathbb{C}$.

Let $\mathcal{P}$ be a countable partition of the complex plane, such that every $B \in \mathcal{P}$ is contained in some ball of radius δ, thus satisfying $m(f \in B) \leqslant \alpha(M, p) + \varepsilon$. For every $\mu \in \mathbb{C}$, since f and f_{p+2} are independent, we have

$$\begin{aligned} m(a_1 f_1 + \cdots + a_{p+2} f_{p+2} = \mu) &\leqslant \sum_{B \in \mathcal{P}} m((f \in B) \cap (a_{p+2} f_{p+2} \in \mu - B)) \\ &= \sum_{B \in \mathcal{P}} m(f \in B) m(a_{p+2} f_{p+2} \in \mu - B) \\ &\leqslant \sum_{B \in \mathcal{P}} (\alpha(M, p) + \varepsilon) m(a_{p+2} f_{p+2} \in \mu - B) \\ &= \alpha(M, p) + \varepsilon. \end{aligned}$$

It follows by definition that $\alpha(M, p + 1) \leqslant \alpha(M, p) + \varepsilon$. □

We remark that, if (f_n) is an independent sequence in $\mathcal{F}_M$ and, for some μ, $m(\sum_{n=1}^{\infty} a_n f_n = \mu) > \alpha(M, p)$, then $a_n = 0$ for all n except for at most p coefficients. Indeed, Theorem 2.3 implies that there are only finitely many $a_n \neq 0$; let us assume that just q of them are non-null. By definition we have $\alpha(M, q) \geqslant m(\sum_{n=1}^{\infty} a_n f_n = \mu)$ and Proposition 3.3 above gives $q < p$.

In order to compute $\alpha(M, 1)$ we need the following lemma.

Lemma 3.4. *Let $0 < R < 1$. Let $b_j \geqslant 0$, $c_j \geqslant 0$ such that $b_j \leqslant R$, $c_j \leqslant R$ for $j = 1, \dots, p$. If $\sum_{j=1}^{p} b_j \leqslant 1$ and $\sum_{j=1}^{p} c_j \leqslant 1$, then $\sum_{j=1}^{p} b_j c_j \leqslant R^2 + (1 - R)^2$.*

Proof. Replacing R by $1 - R$ if necessary we can assume that $R \geqslant 1/2$. It is easy to check that the extreme points of the convex subset of $\mathbb{R}^p$ defined by the inequalities $0 \leqslant b_j \leqslant R$, $j = 1, \dots, p$, and $\sum_{j=1}^{p} b_j \leqslant 1$ are of the following types: (a) the origin, (b) points with some $b_j = R$ and $b_k = 0$ for $k \neq j$, and (c) points with some $b_j = R$, some $b_k = 1 - R$ and $b_i = 0$ for $i \neq j$, $i \neq k$.

For fixed c_j, $j = 1, \dots, p$, the linearity of $\sum_{j=1}^{p} b_j c_j$ implies that the maximum must be attained at some extreme point of type (c), since $c_j \geqslant 0$. It follows that there must exist k and j such that $\sum_{j=1}^{p} b_j c_j \leqslant R c_j + (1 - R) c_k$.

A similar argument applies to the linear function $R c_j + (1 - R) c_k$, showing that $\sum_{j=1}^{p} b_j c_j \leqslant R^2 + (1 - R)^2$, since $R \geqslant 1/2$. □

Proposition 3.5. $\alpha(M, 1) = (M^4 + 1)/(M^2 + 1)^2$.

Proof. Let $a_1 \neq 0$, $a_2 \neq 0$, $f_1, f_2 \in \mathcal{F}_M$ independent, $0 < d < 1$. For every $\mu \in \mathbb{C}$, we consider a countable partition $\mathcal{P}$ of the complex plane such that both B and $(\mu - a_1 B)/a_2$ have diameter bounded by d for all $B \in \mathcal{P}$. We have

$$m(a_1 f_1 + a_2 f_2 = \mu) \leqslant \sum_{B \in \mathcal{P}} m(f_1 \in B) m(f_2 \in (\mu - a_1 B)/a_2).$$

https://doi.org/10.1017/S0013091500000420 Published online by Cambridge University Press

By Lemma 2.1, we can apply Lemma 3.4 with $R = M^2/(M^2 + (1-d))^2$, obtaining

$$m(a_1 f_1 + a_2 f_2 = \mu) \leqslant \frac{M^4 + (1-d)^4}{(M^2 + (1-d)^2)^2}.$$

Letting $d \to 0$ it follows that $\alpha(M, 1) \leqslant (M^4 + 1)/(M^2 + 1)^2$.

This bound is attained with the first and the second terms of the sequence (f_n) constructed below Theorem 2.2, taking $a_1 = 1$, $a_2 = -1$ and $\mu = 0$. □

Finally, we determine the best constant that we are looking for as follows.

Theorem 3.6. *If $m(E) < (2M^2/(M^2 + 1)^2)$, then E is a set of uniqueness for every sequence of stochastically independent functions in $\mathcal{F}_M$ which are not null on any set of positive measure.*

Proof. Let (a_n) be such that $\sum_{n=1}^{\infty} a_n f_n(x) = 0$ for every $x \in F = [0, 1] \setminus E$. As $m(F) > \alpha(M, 1)$, then all the coefficients are null except, perhaps, one of them, say a_n, which also satisfies $a_n = 0$ because $m(f_n = 0) = 0$. □

We point out that the example in the proof of Proposition 3.5 gives a set of measure $2M^2/(M^2 + 1)^2$ which is not a set of uniqueness. Therefore, the constant obtained in Theorem 3.6 is the best possible.

Remark 3.7. Let us observe that in the proof of Theorem 3.6, if we assume that some f_n can be null on some set of positive measure, in order to obtain that $a_n = 0$ as well, it is enough to have $m(F) > m(f_n = 0)$. Thus, since for every $f \in \mathcal{F}_M$, $m(f = 0) \leqslant (M^2 - 1)/M^2$, it follows that if $m(E) < C_M = \inf\{(1/M^2), (2M^2/(M^2 + 1)^2)\}$, then E is a set of uniqueness for every sequence of stochastically independent functions in $\mathcal{F}_M$.

Let us observe that if $M^2 < 1 + \sqrt{2}$, then the infimum is $2M^2/(M^2 + 1)^2$, whereas if $M^2 \geqslant 1 + \sqrt{2}$, then it is $1/M^2$. Nevertheless, sets E with

$$(1/M^2) \leqslant m(E) < (2M^2/(M^2 + 1)^2),$$

which are not of uniqueness, are in some sense trivial, because their complements must be contained in the set where some function of the system is null.

Acknowledgements. Research supported partly by DGES grant BFM2000-0514 and the Patronato Fundación Cámara Universidad de Sevilla.

References

1. B. S. Kashin and A. A. Saakyan, *Orthogonal series*, Translations of Mathematical Monographs, vol. 75 (American Mathematical Society, Providence, RI, 1989).
2. S. B. Stechkin and P. L. Ul'yanov, On sets of uniqueness, *Izv. Akad. Nauk SSSR Ser. Mat.* **26** (1962), 211–222.
3. A. Zygmund, *Trigonometric series*, 2nd edn (Cambridge University Press, 1968).