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LOCAL MINIMAL OVERRINGS

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1. Introduction. Let R be a (commutative integral) domain having quotient field K. A domain S satisfying $R \subseteq S \subseteq K$, is called an *overring* of R. We say R has a *minimal overring* T, in case $R \subsetneqq T \subseteq K$ and there are no domains properly between R and T. The purpose of this paper is the study of certain classes of coherent domains having local minimal overrings; that is, having minimal overrings with unique maximal ideals.

In [**3**], Ferrand and Olivier study the more general notion of minimal homomorphisms of rings, and in [**4**], Gilmer and Heinzer study concepts related to minimal overrings. (Their notion of minimality differs slightly from ours.) We apply their work throughout this note.

Any unexplained terminology is standard as in [**5**] and [**7**].

2. The local case. In this section we analyze those domains R that have minimal overrings which are necessarily local.

As in [**9**], an extension of domains $R \subseteq T$ is said to satisfy *GD* if $R \subseteq T$ satisfies going-down; $R \subseteq T$ is called an *i-extension* if the contraction map Spec $(T) \rightarrow$ Spec (R) is injective. Recall [**1**, pp. 43–44] that a commutative ring is called *coherent* if each finitely generated ideal is finitely presented.

PROPOSITION 2.1. *Let R be local and integrally closed. If R has a minimal overring T, then T is local.*

Proof. Assume T is not local. Since R is integrally closed, [**3**, Théorème 2.2] implies that $R \rightarrow T$ is a flat epimorphism, and [**9**, Remark 2.10] gives that $R \subsetneqq T$ is an i-extension. Let $I = (R : T)_R$. Then I is a prime ideal of T [**3**, Lemme 3.2]. Choose distinct maximal ideals M_1, M_2 of T such that $I \subseteq M_1$. As $R \subseteq T$ is an i-extension, $N_1 = M_1 \cap R \neq M_2 \cap R = N_2$. Also, N_1 and N_2 are contained in I. For if $t \in N_i$, then $t \in M_i$, $i = 1, 2$; and so [**3**, Lemme 2.1] puts $t \in I$. Thus $I = N_1$, and since I is prime in T, we have $N_1 = M_1$. Now by [**8**, 5.D], $R \subseteq T$ satisfies *GD*, and so the following diagram can be completed with some $P \in$ Spec (T):

$$\begin{array}{cccc} T & & \subsetneqq & N_1 = M_1 \\ & \downarrow & & \downarrow \\ R & N_2 & \subsetneqq & N_1 \end{array}$$

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But, $R \subseteq T$ being an i-extension implies $P = M_2$, giving $M_2 \subsetneq M_1$. This contradiction completes the proof.

Remark 2.2. [**4**, Example 4.3] shows that the assumption R integrally closed in Proposition 2.1 is necessary.

LEMMA 2.3. *Let R be local, integrally closed, and assume R has a minimal overring T. If $u \in T \backslash R$, then $u^{-1} \in R$.*

Proof. Suppose $u^{-1} \notin R$ and let M be the maximal ideal of R. Since $T = R[u]$, we apply (u, u^{-1})-lemma [**7**, Theorem 67] to get a nonmaximal prime ideal MT in T. Let N be the unique maximal ideal of T (uniqueness follows from Proposition 2.1). An application of [**4**, Lemma 2.1], shows that $MT = M$ and there are no ideals of T properly between M and N. Hence, $N^2 \subsetneq M$ [**4**, p. 141]. Let $v \in N \backslash M$. Thus $v \notin R$, which gives $T = R[v]$ and $MR[v] = M$ as before. But, $v^2 \in M = MR[v]$. Hence by integrality or [**7**, Theorem 67] we get v or $v^{-1} \in R$. This contradicts the fact that $v \in N \backslash M$.

Remark 2.4. Another proof of Lemma 2.3 can be constructed through the use of [**3**, Théorème 2.2] and [**10**, Theorem 2].

PROPOSITION 2.5. *If R is local and integrally closed, then R has at most one minimal overring* (*and it is necessarily local by Proposition* 2.1).

Proof. Suppose T_1 and T_2 are distinct minimal overrings of R. For $i = 1, 2$, let $u_i \in T_i \backslash R$. Then, by Lemma 2.3, $T_i = R_{u_i^{-1}}$. Note that M does not survive in T_i, since $u_i^{-1} \in M$. Let $N_i T_i$ be the maximal ideals of the T_i. So $N_i \subsetneq M$, and thus $R \subsetneq R_{N_i} \subseteq T_i$. Minimality now gives $R_{N_i} = T_i$. Finally, by prime avoidance, $R \subsetneq R_{R \backslash N_1 \cup N_2} \subseteq R_{N_i}$, which in turn gives $R_{R \backslash N_1 \cup N_2} = R_{N_1} = R_{N_2}$, the desired contradiction.

Example 2.6. This example shows that the assumption that R be integrally closed in Proposition 2.5 is needed. Let Q be the rational numbers and let x be an indeterminate over $Q(\sqrt{2}, \sqrt{3})$. Let $S = Q(\sqrt{2}, \sqrt{3})[[x]]$, $T_1 = Q(\sqrt{2}) + xS$, $T_2 = Q(\sqrt{3}) + xS$ and $R = Q + xS$. Then R is a local domain which is not integrally closed, and T_1, T_2 are local minimal overrings of R. (See [**5**, Theorem A, p. 560] for more details.)

LEMMA 2.7. *Let R be local and integrally closed. If R has a minimal overring T, then there is a prime ideal N of R such that $T = R_N$, $N = NR_N$, and all nonmaximal prime ideals of R are contained in N.*

Proof. From the proof of Proposition 2.5 we get a prime ideal N of R such that $T = R_N$. (One could also deduce the existence of such a prime from Proposition 2.1 and [**10**, Theorem 2].) Minimality gives us that there are no prime ideals properly between N and M, the maximal ideal of R. A direct argument or an application of [**3**, Lemme 2.1] shows that $N = NR_N$ and [**6**, Proposition 1.2, (i)] proves that all the nonmaximal prime ideals of R are contained in N.

THEOREM 2.8. *Let R be coherent, local, and integrally closed. If R has a minimal overring T, then R is a valuation ring.*

Proof. By Lemma 2.7, $T = R_N$, for some $N \in \text{Spec}(R)$. Let M be the maximal ideal of R and choose $a \in M \backslash N$. Then, applying Lemma 2.7 gives that M is minimal over $(a : 1)_R$.

Let I be a finitely generated ideal of R. We wish to show that I is invertible. Let $J = II^{-1}$ and consider a finite presentation of I:

$$R^m \to R^n \to I \to 0.$$

Apply $\text{Hom}_R(-, R)$ to the above exact sequence and obtain the following exact sequence:

$$0 \to \text{Hom}_R(I, R) \to R^n \to R^m.$$

Since $I^{-1} \cong \text{Hom}_R(I, R)$ as R-modules and R is coherent, [**1**, Exercise 11, pp. 43–44] gives that I^{-1} is a finitely generated R-module. Hence, by [**7**, Exercise 39, p. 45], $J^{-1} = R$. Using the minimality of M over $(a : 1)_R$, we apply [**11**, Lemma 3.1] to get $J \not\subseteq M$. Hence $J = R$ and I is invertible.

Remark 2.9. The domain R constructed in Example 2.6 is a Noetherian local domain which is not integrally closed, and so serves to show that the integrally closed assumption is needed in Theorem 2.8. By using the "$D + M$-construction" [**5**, Theorem A, p. 506] and [**2**, Theorem 3] one can construct examples to show that the assumption of coherence is needed in Theorem 2.8.

In general, a valuation ring R with maximal ideal M has a minimal overring if and only if $\bigcup \{P \in \text{Spec}(R) : P \neq M\} \subsetneq M$, or equivalently, if M is minimal over $(a : b)_R$ for some $a, b \in R$.

3. The nonlocal case. In this final section we study nonlocal domains that have local minimal overrings. Throughout this section, R will be a nonlocal domain.

LEMMA 3.1. *If R has a local minimal overring T with maximal ideal M, then $T = R_{M \cap R}$ and $M \cap R$ is a maximal ideal of R.*

Proof. Consider $R \subsetneq T$. By localizing we obtain $R \subsetneq R_{M \cap R} \subseteq T$. So by minimality, $T = R_{M \cap R}$. That $M \cap R$ is a maximal ideal of R, also follows from minimality.

With Lemma 3.1 in mind we state the following:

LEMMA 3.2. *If R has at least one local minimal overring R_M, then R has exactly two maximal ideals M, N and each nonmaximal prime ideal of R contained in N is also contained in M.*

Proof. Suppose R has more than two maximal ideals. Let M, N_1, N_2 be three such maximal ideals. Then $R \subsetneq R_{R \backslash M \cup N_1} \subsetneq R_M$, which contradicts the fact that R_M is a minimal overring of R. Therefore, R has exactly two maximal ideals M and N.

Next assume that there exists a nonmaximal prime ideal $P \subseteq N$ such that $P \not\subseteq M$. Then, as before, $R \subsetneq R_{R \setminus P \cup M} \subsetneq R_M$ contradicting minimality.

LEMMA 3.3. *If R has more than one local minimal overring, then R has exactly two local minimal overrings and each nonmaximal prime ideal of R is contained in the Jacobson radical of R (denoted $J(R)$).*

The proof of this lemma is a straightforward application of Lemmas 3.1 and 3.2.

It would be interesting to know for what classes of domains the conditions forced on the spectrums in Lemmas 3.2 and 3.3 are sufficient. We show that for Prüfer domains, these conditions actually are sufficient.

THEOREM 3.4. *Let R be a Prüfer domain with exactly two maximal ideals M and N.*

(a) *If* $\{P \in \text{Spec }(R): P \subsetneq N\} \subsetneq \{Q \in \text{Spec }(R): Q \subsetneq M\}$ *then R has exactly one local minimal overring, namely R_M.*

(b) *If* $\{P \in \text{Spec }(R): P \subsetneq N\} = \{Q \in \text{Spec }(R): Q \subsetneq M\}$, *then R has exactly two local minimal overrings, namely, R_M and R_N.*

Proof. (a) Lemma 3.3 guarantees us that we cannot have more than one local minimal overring, and Lemma 3.1 specifies that if one exists it must either be R_M or R_N. Assuming that

$$\{P \in \text{Spec }(R): P \subsetneq N\} \subsetneq \{Q \in \text{Spec }(R): Q \subsetneq M\}$$

and arguing as in Lemma 3.2, we can eliminate R_N as a candidate for a local minimal overring.

It remains to show that R_M is a minimal overring of R. Assume $R \subsetneq S \subseteq R_M$. Since R is Prüfer, S is flat over R, and since $S \not\subseteq R_N$, $NS = S$ [**10**, Theorem 1]. An application of [**5**, Theorem 22.1] gives $S = R_M$ to complete part (a).

(b) Let $R \subsetneq S \subseteq K$; thus $S \not\subseteq R_M \cap R_N$. Argue as in part (a) to see that R_M and R_N are local minimal overrings, and so the only possible ones.

COROLLARY 3.5. *Let R be a nonlocal Prüfer domain. Then, R has exactly two local minimal overrings, if and only if each overring T of R with $T \neq R$ is local.*

Proof. The "only if" part follows by combining Lemma 3.3 and the proof of Theorem 3.4 (b).

By Theorem 3.4 (b), it suffices to show that each nonmaximal prime ideal of R is contained in $J(R)$. If this is not the case, then we may assume there is a $P \in \text{Spec }(R)$ such that $P \subsetneq M$ and $P \not\subseteq N$ for $M, N \in \text{Spec }(R)$. But then $R_{R \setminus P \cup N}$ is a nonlocal overring of R different from R, which contradicts our assumption.

We end this section by analyzing the Noetherian case.

LEMMA 3.6. *Let R be Noetherian and not local. If* dim $(R) > 1$, *then there exists a nonmaximal height one prime ideal not contained in* $J(R)$.

Proof. Let M be a maximal ideal such that $\mathrm{ht}(M) > 1$ and let N be any other maximal ideal. Choose $x \in M \backslash N$ and let P be a prime ideal minimal over xR with $P \subseteq M$. Since $\mathrm{ht}(P) = 1$, $P \neq M$, and $P \nsubseteq J(R)$ because $J(R) \subsetneq N$, we are done.

PROPOSITION 3.7. *Let R be Noetherian and nonlocal. Then, R has exactly two local minimal overrings and is integrally closed if and only if R is a PID with exactly two nonzero prime ideals.*

Proof. The "if" part follows directly from Theorem 3.4 while the other direction is a consequence of Lemmas 3.3, 3.6 and [**7**, Theorem 96].

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REFERENCES

1. N. Bourbaki, *Commutative algebra* (Addison-Wesley, Reading, Massachusetts, 1972).
2. D. E. Dobbs and I. J. Papick, *When is $D + M$ coherent?*, to appear, Proc. Amer. Math. Soc.
3. D. Ferrand et J. P. Olivier, *Homomorphismes minimaux d'anneaux*, J. of Algebra *16* (1970), 461–471.
4. R. Gilmer and W. J. Heinzer, *Intersections of quotient rings of an integral domain*, J. Math. Kyoto University *7–2* (1967), 133–150.
5. R. Gilmer, *Multiplicative ideal theory*, Queen's Papers in Pure and Appl. Math., No. 12, Queen's University, Kingston, Ontario, 1968.
6. B. Greenberg, *Global dimension of Cartesian squares*, J. of Algebra *32* (1974), 31–43.
7. I. Kaplansky, *Commutative rings* (Allyn and Bacon, Boston, Massachusetts, 1970).
8. H. Matsumura, *Commutative algebra* (W. A. Benjamin, New York, 1970).
9. I. J. Papick, *Topologically defined classes of going-down domains*, to appear, Trans. Amer. Math. Soc.
10. F. Richman, *Generalized quotient rings*, Proc. Amer. Math. Soc. *16* (1965), 794–799.
11. W. V. Vasconcelos, *Divisor theory in module categories*, Mathematics Studies No. *14* (North-Holland, Amsterdam, 1974).

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