*Glasgow Math. J.* **51** (2009) 257–261. © 2008 Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089508004679. Printed in the United Kingdom

# COMPACT TOEPLITZ OPERATORS WITH CONTINUOUS SYMBOLS

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(Received 11 May 2008; Accepted 13 Sept 2008)

**Abstract.** For any rotation-invariant positive regular Borel measure $\nu$ on the closed unit ball $\overline{\mathbb{B}}_n$ whose support contains the unit sphere $\mathbb{S}_n$, let $L^2_a$ be the closure in $L^2 = L^2(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$ of all analytic polynomials. For a bounded Borel function $f$ on $\overline{\mathbb{B}}_n$, the Toeplitz operator $T_f$ is defined by $T_f(\varphi) = P(f\varphi)$ for $\varphi \in L^2_a$, where $P$ is the orthogonal projection from $L^2$ onto $L^2_a$. We show that if $f$ is continuous on $\overline{\mathbb{B}}_n$, then $T_f$ is compact if and only if $f(z) = 0$ for all $z$ on the unit sphere. This is well known when $L^2_a$ is replaced by the classical Bergman or Hardy space.

2002 *Mathematics Subject Classification.* Primary 47B35

**1. Introduction.** As usual, for any integer $n \geq 1$, let $\mathbb{B}_n$ denote the open unit ball and $\mathbb{S}_n$ the unit sphere in $\mathbb{C}^n$. The closure of $\mathbb{B}_n$ in the Euclidean metric on $\mathbb{C}^n$ is denoted by $\overline{\mathbb{B}}_n$. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $|z|$ denotes the Euclidean norm of $z$. For any multi-index $m = (m_1, \ldots, m_n)$ in $\mathbb{N}^n$ (where $\mathbb{N}$ denotes the set of all non-negative integers), $z^m = z_1^{m_1} \cdots z_n^{m_n}$ and $\overline{z}^m = \overline{z}_1^{m_1} \cdots \overline{z}_n^{m_n}$. We also write $|m| = m_1 + \cdots + m_n$ and $m! = m_1! \cdots m_n!$. Let $\sigma$ denote the rotation-invariant positive Borel measure on $\mathbb{S}_n$, which is normalized so that $\sigma(\mathbb{S}_n) = 1$. Let $\mu$ be a positive regular Borel measure on the closed interval $[0, 1]$ with $\mu([0, 1]) = 1$, and 1 is in the support of $\mu$. Let $\nu$ be the product measure of $\mu$ and $\sigma$. So $\nu$ is a regular Borel measure on $\overline{\mathbb{B}}_n$ with unit total mass, such that for any $f \in L^1(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$, we have the integration in polar coordinate formula:

$$\int_{\overline{\mathbb{B}}_n} f(z)\,\mathrm{d}\nu(z) = \int_{[0,1]} \left( \int_{\mathbb{S}_n} f(r\zeta)\mathrm{d}\sigma(\zeta) \right) \mathrm{d}r. \tag{1}$$

Let $L^2_a(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$ be the closure of the space of all holomorphic polynomials in $L^2(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$, and let $P$ denote the orthogonal projection from $L^2(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$ onto $L^2_a(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$.

If $\mathrm{d}\mu(r) = \frac{2\,\Gamma(n+\alpha+1)}{\Gamma(n)\,\Gamma(\alpha+1)} r^{2n-1}(1 - r^2)^{\alpha}\mathrm{d}r$ for some $\alpha > -1$, then $\nu$ is a weighted Lebesgue measure on $\mathbb{B}_n$, and $L^2_a(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$ is the familiar weighted Bergman space. If $\mu$ is the point mass measure at 1, then $L^2_a(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$ can be identified with the Hardy space $H^2$ on $\mathbb{B}_n$. See **[4]** for more detail about Bergman and Hardy spaces.

For any bounded Borel function $f$ defined on $\overline{\mathbb{B}}_n$, the Toeplitz operator $T_f$ is the operator on $L^2_a(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$, defined by $T_f\varphi = P(f\varphi)$ for $\varphi \in L^2_a$. The function $f$ is called the symbol of $T_f$. It is clear that $T_f$ is a bounded operator with $\|T_f\| \leq \|f\|_{L^\infty(\overline{\mathbb{B}}_n, \mathrm{d}\nu)}$. It follows from the density in $C(\overline{\mathbb{B}}_n)$ of polynomials (in $z$ and $\overline{z}$) that if $T_f = 0$, then

$f(z) = 0$ for $\nu$-almost all $z$ in $\overline{\mathbb{B}}_n$. So the map $f \mapsto T_f$ from $L^\infty(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$ into the $C^*$-algebra $\mathfrak{B}(L^2_a(\overline{\mathbb{B}}_n, \mathrm{d}\nu))$ of all bounded linear operators on $L^2_a(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$ is an injective contraction. This map is not an isometry in general. Note that if $\mu(\{1\}) = 0$, then the values of $f$ on the unit sphere do not affect the operator $T_f$. On the other hand, the values of $f$ on the unit sphere play an important role when $\mu(\{1\}) > 0$.

In this paper we are interested in Toeplitz operators whose symbols behave well near the boundary of $\mathbb{B}_n$. Toeplitz operators (on the classical Hardy and Bergman spaces) whose symbols are continuous functions on $\overline{\mathbb{B}}_n$ and the $C^*$-algebras generated by them were studied by L. Coburn [**1**] back in the 1970s. One of many results on this subject is the theorem given next.

THEOREM 1.1. *Suppose $f$ is in $C(\overline{\mathbb{B}}_n)$. Then $T_f$ is a compact operator if and only if $f(\zeta) = 0$ for all $\zeta \in \mathbb{S}_n$.*

In this paper we will show that Theorem 1.1 still holds true for Toeplitz operators acting on any $L^2_a(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$. That $f|_{\mathbb{S}_n} \equiv 0$ implies the $T_f$ is compact is not new. The proof is similar to that of the classical case. On the other hand, the proof of the converse requires a different argument. The usual approach which involves reproducing kernels does not seem to work for general $\nu$. The reason is that for such a $\nu$, even though reproducing kernels exist, there is no useful formula for them. Theorem 1.1 for a general rotation-invariant positive Borel measure $\nu$ on the unit disk was shown by T. Nakazi and R. Yoneda [**2**]. This paper was in fact inspired by theirs.

**2. Toeplitz operators with compactly supported symbols.** In this section we show that if $f$ is a bounded Borel function whose support is contained in a compact subset of $\mathbb{B}_n$, then $T_f$ is a Hilbert–Schmidt operator.

For multi-indexes $m, k \in \mathbb{N}^n$, from formula (1) and Propositions 1.4.8 and 1.4.9 in [**3**], we have

$$\begin{aligned}\int_{\overline{\mathbb{B}}_n} z^m \bar{z}^k \,\mathrm{d}\nu(z) &= \int_{[0,1]} \left( \int_{\mathbb{S}_n} \zeta^m \zeta^k \mathrm{d}\sigma(\zeta) \right) r^{2|m|}\,\mathrm{d}\mu(r) \\ &= \begin{cases} 0 & \text{if } m \neq k, \\ \dfrac{(n-1)!\, m!}{(n-1+|m|)!} \displaystyle\int_{[0,1]} r^{2|m|}\mathrm{d}\mu(r) & \text{if } m = k. \end{cases}\end{aligned}$$

For $s \in \mathbb{N}$, let $\alpha_s = \int_{[0,1]} r^{2s}\mathrm{d}\mu(r)$. For $m \in \mathbb{N}^n$ and $z \in \mathbb{C}^n$, put

$$e_m(z) = \left( \frac{(n-1+|m|)!}{(n-1)!\, m!\, \alpha_{|m|}} \right)^{1/2} z^m.$$

Then from the above computation and the definition of $L^2_a(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$, it follows that the set $\{e_m : m \in \mathbb{N}^n\}$ is an orthonormal basis for $L^2_a(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$.

PROPOSITION 2.1. *Let $f$ be a bounded Borel function on $\overline{\mathbb{B}}_n$, such that for some $0 < \delta < 1$, $f(z) = 0$ whenever $|z| > \delta$. Then $T_f$ is a Hilbert–Schmidt operator.*

https://doi.org/10.1017/S0017089508004679 Published online by Cambridge University Press

*Proof.* For $z \in \mathbb{B}_n$ with $|z| \le \delta$, we have

$$\begin{aligned}\sum_{m\in\mathbb{N}^n} |e_m(z)|^2 &= \sum_{m\in\mathbb{N}^n} \frac{(n-1+|m|)!}{(n-1)!\, m_1!\cdots m_n!}\,\frac{|z_1|^{2m_1}\cdots|z_n|^{2m_n}}{\alpha_{|m|}}\\ &= \sum_{M=0}^{\infty} \frac{(n-1+M)!}{(n-1)!\, M!\,\alpha_M} \sum_{|m|=M} \frac{M!}{m_1!\cdots m_n!}|z_1|^{2m_1}\cdots|z_n|^{2m_n}\\ &= \sum_{M=0}^{\infty} \frac{(n-1+M)!}{(n-1)!\, M!\,\alpha_M}(|z_1|^2+\cdots+|z_n|^2)^M\\ &\le \sum_{M=0}^{\infty} \frac{(n-1+M)!}{(n-1)!\, M!\,\alpha_M}\delta^{2M}. \end{aligned} \tag{2}$$

Now $\lim_{M\to\infty}(\alpha_M)^{1/M} = \lim_{M\to\infty}(\int_{[0,1]} r^{2M}\mathrm{d}\mu(r))^{1/M} = \|r^2\|_{L^\infty([0,1],\mathrm{d}\mu)} = 1$, where the last identity follows from the fact that 1 is in the support of $\mu$. Thus the infinite sum in (2) is convergent. So for each $0 < \delta < 1$, there is a constant $C(\delta) < \infty$, such that $\sum_{m\in\mathbb{N}^n} |e_m(z)|^2 \le C(\delta)$ for all $|z| \le \delta$.

Now suppose $f$ satisfies the hypothesis of the proposition. Then

$$\begin{aligned}\sum_{m,k\in\mathbb{N}^n} |\langle T_f e_m, e_k\rangle|^2 &\le \sum_{m,k\in\mathbb{N}^n}\left(\int_{\mathbb{B}_n} |f(z)e_m(z)e_k(z)|\mathrm{d}\nu(z)\right)^2\\ &\le \sum_{m,k\in\mathbb{N}^n}\int_{\mathbb{B}_n} |f(z)|^2|e_m(z)|^2|e_k(z)|^2\mathrm{d}\nu(z)\\ &\quad \text{(by Holder's inequality)}\\ &= \int_{|z|\le\delta} |f(z)|^2\left(\sum_{m\in\mathbb{N}^n}|e_m(z)|^2\right)\left(\sum_{k\in\mathbb{N}^n}|e_k(z)|^2\right)\mathrm{d}\nu(z)\\ &\le (C(\delta))^2\int_{|z|\le\delta} |f(z)|^2\mathrm{d}\nu(z) < \infty.\end{aligned}$$

This shows that $T_f$ is a Hilbert–Schmidt operator. □

The corollary given below proves the 'if' part of Theorem 1.1. The 'only if' part will follow from a more general result which will be presented in Section 3.

COROLLARY 2.2. *If $f \in C(\overline{\mathbb{B}}_n)$ such that $f(\zeta) = 0$ for all $|\zeta| = 1$, then $T_f$ is compact.*

*Proof.* Since $f$ can be uniformly approximated on $\overline{\mathbb{B}}_n$ by continuous functions with compact supports in $\mathbb{B}_n$, Proposition 2.1 shows that $T_f$ can be approximated in the operator norm by Hilbert–Schmidt operators. Hence $T_f$ is a compact operator. □

**3. Compact Toeplitz operators with continuous symbols.** We begin this section with a proposition that relates the boundary values of $f$ with $\langle T_f e_m, e_m\rangle$ as $|m| \to \infty$.

PROPOSITION 3.1. *Let $f$ be a bounded Borel function on $\overline{\mathbb{B}}_n$, such that for $\sigma$-almost all $\zeta \in \mathbb{S}_n$, we have $f(\zeta) = \lim_{r\uparrow 1} f(r\zeta)$. If $\lim_{|m|\to\infty}\langle T_f e_m, e_m\rangle = \alpha$, then $\int_{\mathbb{S}_n} f(\zeta)\mathrm{d}\sigma(\zeta) = \alpha$.*

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*Proof.* Without loss of generality, we may assume that $\alpha = 0$. For any function $g$ in $L^1(\mathbb{B}_n, \mathrm{d}\nu)$ and any positive integer $M$ we have

$$\begin{aligned}
&\sum_{|m|=M} \langle T_g e_m, e_m\rangle \\
&= \sum_{|m|=M} \frac{(n-1+|m|)!}{(n-1)!\, m!\, \alpha_{|m|}} \int_{\overline{\mathbb{B}}_n} g(z) z^m \bar{z}^m \mathrm{d}\nu(z) \\
&= \frac{(n-1+M)!}{(n-1)!\, M!\, \alpha_M} \int_{\overline{\mathbb{B}}_n} g(z) \left\{ \sum_{|m|=M} \frac{M!}{m_1! \cdots m_n!} |z_1|^{2m_1} \cdots |z_n|^{2m_n} \right\} \mathrm{d}\nu(z) \qquad (3)\\
&= \frac{(n-1+M)!}{(n-1)!\, M!\, \alpha_M} \int_{\overline{\mathbb{B}}_n} g(z)(|z_1|^2 + \cdots + |z_n|^2)^M \mathrm{d}\nu(z) \\
&= \frac{(n-1+M)!}{(n-1)!\, M!\, \alpha_M} \int_{[0,1]} \left( \int_{\mathbb{S}_n} g(r\zeta)\mathrm{d}\sigma(\zeta) \right) r^{2M} \mathrm{d}\mu(r).
\end{aligned}$$

In particular, if $g(z) = 1$ for all $z \in \overline{\mathbb{B}}_n$, then $\frac{(n-1+M)!}{(n-1)!\, M!} = \sum_{|m|=M} 1$. This shows that the set $\{m = (m_1, \ldots, m_n) \in \mathbb{N}^n : m_1 + \cdots + m_n = M\}$ has $\frac{(n-1+M)!}{(n-1)!\, M!}$ elements. This formula can, of course, be shown directly by an elementary combinatoric argument.

Let $\epsilon > 0$ be given. There is an integer $M_\epsilon$ such that for all $m \in \mathbb{N}^n$ with $|m| > M_\epsilon$ we have $|\langle T_f e_m, e_m\rangle| < \epsilon$. Thus for any $M > M_\epsilon$, (3) with $f$ in place of $g$ gives

$$\begin{aligned}
\left| \frac{1}{\alpha_M} \int_{[0,1]} \left( \int_{\mathbb{S}_n} f(r\zeta)\mathrm{d}\sigma(\zeta) \right) r^{2M} \mathrm{d}\mu(r) \right| &\leq \frac{(n-1)!\, M!}{(n-1+M)!} \sum_{|m|=M} |\langle T_f e_m, e_m\rangle| \\
&\leq \frac{(n-1)!\, M!}{(n-1+M)!} \sum_{|m|=M} \epsilon \\
&= \epsilon.
\end{aligned}$$

This shows that

$$\lim_{M\to\infty} \frac{1}{\alpha_M} \int_{[0,1]} \left( \int_{\mathbb{S}_n} f(r\zeta)\mathrm{d}\sigma(\zeta) \right) r^{2M} \mathrm{d}\mu(r) = 0. \qquad (4)$$

For each $0 \leq r \leq 1$, let us put $\varphi(r) = \int_{\mathbb{S}_n} f(r\zeta)\mathrm{d}\sigma(\zeta)$. Since $f$ is bounded on $\overline{\mathbb{B}}_n$ and $f(r\zeta) \to f(\zeta)$ as $r \uparrow 1$ for $\sigma$-almost all $\zeta \in \mathbb{S}_n$, Lebesgue's dominated convergence theorem implies that $\varphi(r) \to \varphi(1)$ as $r \uparrow 1$. We now show that $\lim_{M\to\infty} \frac{1}{\alpha_M} \int_{[0,1]} \varphi(r) r^{2M} \mathrm{d}\mu(r) = \varphi(1)$. Let $\epsilon > 0$ be given. There is a $\delta$ in [0, 1) such that $|\varphi(r) - \varphi(1)| < \epsilon$ for all $a \leq r \leq 1$. Therefore,

$$\begin{aligned}
\left| \left( \frac{1}{\alpha_M} \int_{[0,1]} \varphi(r) r^{2M} \mathrm{d}\mu(r) \right) - \varphi(1) \right| &= \left| \frac{1}{\alpha_M} \int_{[0,1]} (\varphi(r) - \varphi(1)) r^{2M} \mathrm{d}\mu(r) \right| \\
&\leq \frac{1}{\alpha_M} \int_{[0,a)} |\varphi(r) - \varphi(1)| r^{2M} \mathrm{d}\mu(r) \\
&\quad + \frac{1}{\alpha_M} \int_{[a,1]} |\varphi(r) - \varphi(1)| r^{2M} \mathrm{d}\mu(r) \\
&\leq 2\|\varphi\|_\infty \frac{1}{\alpha_M} \int_{[0,a)} r^{2M} \mathrm{d}\mu(r) + \epsilon.
\end{aligned}$$

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Now since 1 is in the support of $\mu$, an elementary argument shows that $\lim_{M\to\infty} \frac{1}{\alpha_M}\int_{[0,a)} r^{2M} \mathrm{d}\mu(r) = 0$. (See [**2**, Lemma 2] for a detailed proof.) By taking $M \to \infty$ in the above inequalities, we conclude that

$$\limsup_{M\to\infty} \left| \left( \frac{1}{\alpha_M} \int_{[0,1]} \varphi(r) r^{2M} \mathrm{d}\mu(r) \right) - \varphi(1) \right| \le \epsilon.$$

Since $\epsilon$ was arbitrary, we get

$$\lim_{M\to\infty} \frac{1}{\alpha_M} \int_{[0,1]} \varphi(r) r^{2M} \mathrm{d}\mu(r) = \varphi(1). \tag{5}$$

Now (4) and (5) imply that $\varphi(1) = 0$, which means $\int_{\mathbb{S}_n} f(\zeta) \mathrm{d}\sigma(\zeta) = 0$. □

COROLLARY 3.2. *Suppose $f$ is a bounded Borel function on $\overline{\mathbb{B}}_n$, such that for $\sigma$-almost all $\zeta \in \mathbb{S}_n$, $f(\zeta) = \lim_{r\uparrow 1} f(r\zeta)$ and that $T_f$ is a compact operator on $L^2_a(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$. Then $f(\zeta) = 0$ for $\sigma$-almost all $\zeta$ in $\mathbb{S}_n$. From this, the 'only if' part of Theorem 1.1 follows.*

*Proof.* For all multi-indexes $l_1, l_2 \in \mathbb{N}^n$, the operator $T_{fe_{l_1}\bar{e}_{l_2}} = T_{\bar{e}_{l_2}} T_f T_{e_{l_1}}$ is compact. Thus we have $\lim_{|m|\to\infty} \langle T_{fe_{l_1}\bar{e}_{l_2}} e_m, e_m \rangle = 0$. By Proposition 3.1 and the fact that for $\sigma$-almost all $\zeta \in \mathbb{S}_n$, $\lim_{r\uparrow 1} f(r\zeta) e_{l_1}(r\zeta) \bar{e}_{l_2}(r\zeta) = f(\zeta) e_{l_1}(\zeta) \bar{e}_{l_2}(\zeta)$, which is a positive multiple of $f(\zeta)\zeta^{l_1}\bar{\zeta}^{l_2}$, we conclude that $\int_{\mathbb{S}_n} f(\zeta)\zeta^{l_1}\bar{\zeta}^{l_2} \mathrm{d}\sigma(\zeta) = 0$. Since this is true for all multi-indexes $l_1$ and $l_2$, we have $f(\zeta) = 0$ for $\sigma$-almost all $\zeta \in \mathbb{S}_n$. □

ACKNOWLEDGEMENTS. This work was completed when the author was a postdoctoral fellow at the Department of Mathematics and the Fields Institute for Research in Mathematical Sciences, University of Toronto. The author is grateful for their support.

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