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EMBEDDING THE HOPF AUTOMORPHISM GROUP INTO THE BRAUER GROUP

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ABSTRACT. Let H be a faithfully projective Hopf algebra over a commutative ring k. In [8, 9] we defined the Brauer group $\mathrm{BQ}(k,H)$ of H and an homomorphism π from Hopf automorphism group $\mathrm{Aut}_{\mathrm{Hopf}}(H)$ to $\mathrm{BQ}(k,H)$. In this paper, we show that the morphism π can be embedded into an exact sequence.

Introduction. In this paper k is a commutative ring with unit. Let H be a Hopf k-algebra having a bijective antipode. It is possible to introduce a "Quantum-Brauer" group of H which can be obtained by taking isomorphism classes of H-Azumaya algebras in the category of Yetter-Drinfel'd H-modules modulo H-Morita equivalence and use the braided product to define multiplication. This group, denoted by $\mathrm{BQ}(k,H)$, was introduced in [8, 9]. Since then we dream of calculating it, or specific parts of it, for some popular quantum groups. When H is commutative and cocommutative and a faithfully projective Hopf algebra, the group $\mathrm{BQ}(k,H)$ turns out to be the Brauer-Long group $\mathrm{BD}(k,H)$ introduced by F. W. Long in [12, 13]. In fact, even for the Brauer-Long group no good (cohomological) calculative methods were known before the more recent results of [5, 6, 7]. Various subgroups of the Brauer-Long group could more easily be studied *cf.* [1, 2, 3, 5, 6, 7, 10]. For example, Deegan's subgroup introduced in [10] which in fact turns out to be isomorphic to the Hopf algebra automorphism group $\mathrm{Aut}(H)$ (*cf.* [10, 6]). The connection between $\mathrm{Aut}(H)$ and $\mathrm{BD}(k,H)$ for special commutative and cocommutative H was probably first studied by M. Beattie in [1] where she established the existence of an exact sequence $(*)$:

$$1 \longrightarrow BC(k,G)/\mathrm{Br}(k) \times \mathrm{BM}(k,G)/\mathrm{Br}(k) \longrightarrow B(k,G)/\mathrm{Br}(k) \xrightarrow{\beta} \mathrm{Aut}(G) \longrightarrow 1$$

where G is a finite abelian group and k is a connected ring. Based on Beattie's construction of the map β, Deegan constructed his subgroup $\mathrm{BT}(k,G)$ which is then isomorphic to $\mathrm{Aut}(G)$; the resulting embedding of $\mathrm{Aut}(G)$ in the Brauer-Long group (group case) is known as Deegan's embedding theorem. In [6], S. Caenepeel looked at β by means of the Picard group of a Hopf algebra, and so extended Deegan's embedding theorem from abelian groups to commutative and cocommutative Hopf algebras. But if H is a quantum group (*i.e.* (co-)quasi-triangular Hopf algebra) or just any non-commutative non-cocommutative Hopf algebra then it seems that the map β can not be extended to a map from some subgroup of $\mathrm{BQ}(k,H)$ to the automorphism group $\mathrm{Aut}(H)$. In fact,

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Aut(H) can no longer be embedded in BQ(k, H). On the other hand, we do have the map $\pi\colon \mathrm{Aut}(H) \longrightarrow \mathrm{BQ}(k, H)$ constructed in [9]. This map essentially deals with the actions and coactions of H on itself and this inspired us to pass to the action of the Drinfel'd double $D(H)$. In this way the kernel of π may be related to the group-like elements of $D(H)$ and $D(H)^*$. More precisely, we obtain an exact sequence:

$$(**) \qquad 1 \longrightarrow G\big(D(H)^*\big) \longrightarrow G\big(D(H)\big) \longrightarrow \mathrm{Aut}(H) \xrightarrow{\ \pi\ } \mathrm{BQ}(k, H)$$

The group $G\big(D(H)^*\big)$ is an abelian group *cf.* [16]. In case $D(H)$ is commutative, then π is injective and this is then the variant of Deegan-Caenepeel's embedding result for the Brauer-Long group. The meaning of the exact sequence $(**)$ goes beyond this because in our opinion it indicates a crucial difference between the Brauer-Long group and the quantum Brauer group. The reader may find this more obvious after looking at Examples 7, 8. Of course when k is a field of characteristic 0 and H is a finite dimensional commutative and cocommutative Hopf algebra, then Aut(H) is a finite group. This indicates that the Brauer-Long group of a finite dimensional Hopf algebra might be a torsion group as for usual Brauer groups. However, in the quantum group case, even when H is a small Hopf algebra, Aut(H) may be an infinite non-torsion group! Example 8 learns that, while Br($\mathbb{C}$) is trivial BQ($\mathbb{C}, H$) is not even torsion for Radford's Hopf algebra H as $\mathrm{GL}_n(\mathbb{C})$ modulo some finite subgroup embeds in it ! Now with Aut(H) out of the way we hope to find nice properties for Coker(π).

1**. Preliminaries.** Throughout k is a commutative ring with unit and H is a faithfully projective Hopf algebra over k. A Yetter-Drinfel'd H-module (simply, YD H-module) M is a crossed H-bimodule [18]. That is, M is a k-module which is at once a left H-module and a right H-comodule satisfying the following equivalent compatibility conditions [11, 5.1.1]:

(i) $\sum h_{(1)} \cdot m_{(0)} \otimes h_{(2)} m_{(1)} = \sum (h_{(2)} \cdot m)_{(0)} \otimes (h_{(2)} \cdot m)_{(1)} h_{(1)}$

(ii) $\chi(h \cdot m) = \sum (h_{(2)} \cdot m_{(0)}) \otimes h_{(3)} m_{(1)} S^{-1}(h_{(1)})$.

A Yetter-Drinfel'd H-module algebra is a YD H-module A such that A is a left H-module algebra and a right H^{op}-comodule algebra. For detail on H-(co)module and H-(co)algebras we refer to the standard book [17].

In [8] we defined the Brauer group of a Hopf algebra H by considering isomorphism classes of H-Azumaya algebras. A YD H-module algebra A is called *H-Azumaya* if it is faithfully projective as a k-module and if the following YD H-module algebra maps are isomorphisms:

$$F\colon A \# \overline{A} \longrightarrow \mathrm{End}(A), \quad F(a \# \overline{b})(c) = \sum a c_{(0)} (c_{(1)} \cdot b),$$
$$G\colon \overline{A} \# A \longrightarrow \mathrm{End}(A)^{\mathrm{op}}, \quad G(\overline{a} \# b)(c) = \sum a_{(0)} (a_{(1)} \cdot c) b,$$

where $\overline{A}$ is the H-opposite YD H-module algebra of A *cf.* [8]. For a faithfully projective YD H-module M the endomorphism ring $\mathrm{End}_k(M)$ a is a YD H-module algebra with

H-structures given by

$$(h\cdot f)(m)=\sum h_{(1)}\cdot f(S(h_{(2)})\cdot m), \tag{1}$$

$$\chi(f)(m)=\sum f(m_{(0)})_{(0)}\otimes S^{-1}(m_{(1)})f(m_{(0)})_{(1)}. \tag{2}$$

Two H-Azumaya algebras A and B are Brauer equivalent (denoted by $A\sim B$) if there exist two faithfully projective YD H-modules M and N such that $A\,\#\,\mathrm{End}(M)\cong B\,\#\,\mathrm{End}(N)$. $A\sim B$ if and only if A is H-Morita equivalent to B *cf.* [9, Theorem 2.10]. The Brauer group of the Hopf algebra H is denoted by $\mathrm{BQ}(k,H)$. The trivial element 1 in $\mathrm{BQ}(k,H)$ is represented by the endomorphism algebra of a faithfully projective YD H-module.

Let H be a faithfully projective Hopf algebra. The Drinfel'd double $\mathrm{D}(H)=(H^{\mathrm{op}})^*\bowtie H$ ($(H^{\mathrm{op}})^*=H^{*\mathrm{cop}}$) is a quasitriangular-Hopf algebra. If $\{h_i,h_i^*\}$ is the dual basis of H, then $R=\sum_i(h_i^*\bowtie 1)\otimes(1\bowtie h_i)$ is the canonical quasi-triangular structure of $D(H)$ *cf.* [14, 15]. It is well-known that a k-module M is a $D(H)$-module if and only if M is a YD-H-module *cf.* [14]. It follows from this fact that an algebra A is a left $\mathrm{D}(H)$-module algebras if and only if it is an Yetter-Drinfel'd H-module algebras. Since the Brauer equivalence is exactly the H-Morita equivalence, we obtain $\mathrm{BQ}(k,H)=\mathrm{BM}\big(k,\mathrm{D}(H)\big)$, where $\mathrm{BM}\big(k,\mathrm{D}(H)\big)$ is a subgroup of $\mathrm{BQ}\big(k,D(H)\big)$ represented by those $D(H)$-Azumaya algebras whose comodule structures stemming from the module structures by means of the quasi-triangular structure on $D(H)$. *cf.* [9].

2. **The exact sequence.** Let H be a faithfully projective Hopf algebra over k. In [9] we constructed an anti-homomorphism from $\mathrm{Aut}(H)$ to $\mathrm{BQ}(k,H)$, whose image in $\mathrm{BQ}(k,H)$ determines the action of $\mathrm{Aut}(H)$ on $\mathrm{BQ}(k,H)$ *cf.* [9, Theorem 4.11] We know that if M is a faithfully projective Yetter-Drinfel'd H-module, then $\mathrm{End}_k(M)$ is an H-Azumaya YD H-module algebra which represents the trivial element in $\mathrm{BQ}(k,H)$. However, if M is a a left H-module and a right H-comodule, but not a YD H-module, it may still happen that $\mathrm{End}_k(M)$ is a YD H-module algebra.

Take a non-trivial Hopf algebra isomorphism $\alpha\in\mathrm{Aut}(H)$. We define a left H-module and right H-comodule H_α as follows. As a k-module $H_\alpha=H$; we give H_α the obvious H-comodule structure Δ, and an H-module structure as follows:

$$h\cdot x=\sum\alpha(h_{(2)})xS^{-1}(h_{(1)}) \tag{3}$$

for $h\in H,\ x\in H_\alpha$. Since α is nontrivial H_α is not a YD H-module. Let $A_\alpha=\mathrm{End}(H_\alpha)$ with the induced H-structures given by (1) and (2).

LEMMA 1 [9, 4.6, 4.7]. *If H is a faithfully projective Hopf algebra and α is a Hopf algebra automorphism of H, then A_α is an Azumaya* YD-*module algebra and the following map defines a group homomorphism:*

$$\pi\colon \mathrm{Aut}(H)\longrightarrow \mathrm{BQ}(k,H),\qquad \alpha\longmapsto[A_{\alpha^{-1}}].$$

In the sequel, we will compute the kernel of the map π. Let $D(H)$ denote the Drinfel'd double of Hopf algebra H. Let A be an H-module algebra. Recall from [4] that the H-action on A is said to be *strongly inner* if there is an algebra map $f\colon H\longrightarrow A$ such that

$$h\cdot a=\sum f(h_{(1)})af(S(h_{(2)})),\qquad a\in A,\ h\in H.$$

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LEMMA 2. *Let M be a faithfully projective k-module. Suppose that* $\mathrm{End}(M)$ *is a* $D(H)$*-Azumaya algebra. Then* $[\mathrm{End}(M)] = 1$ *in* $\mathrm{BM}\big(k, D(H)\big)$ *if and only if the* $D(H)$*-action on* A *is strongly inner.*

PROOF. Suppose that the $D(H)$-action on A is strongly inner. There is algebra map $f: D(H) \longrightarrow A$ such that $t \cdot a = \sum f(t_{(1)})af\big(S(t_{(2)})\big)$, $t \in D(H), a \in A$. This inner action yields a $D(H)$-module structure on M given by

$$t \rightharpoonup m = f(t)(m), \quad t \in D(H),\ m \in M.$$

Since f is an algebra map the above action does define a module structure. Now it is a straightforward check that the $D(H)$-module structure on A is exactly induced by the $D(H)$-module structure on M defined above. By definition $[\mathrm{End}(M)] = 1$ in $\mathrm{BM}\big(k, D(H)\big)$.

Conversely, if $[A = \mathrm{End}(M)] = 1$, then there exists a faithfully projective $D(H)$-module N such that $A \cong \mathrm{End}(N)$ as $D(H)$-module algebras by [9, 2.11]. Now $D(H)$ acts strongly innerly on $\mathrm{End}(N)$. Let $u: D(H) \longrightarrow \mathrm{End}(N)$ be the algebra map. Now one may easily verify that the strongly inner action induced by the composite algebra map:

$$\mu: D(H) \xrightarrow{\ u\ } \mathrm{End}(N) \cong A$$

exactly defines the $D(H)$-module structure on A. ∎

LEMMA 3. *For a faithfully projective k-module M, let $u, \omega: H \longrightarrow \mathrm{End}(M)$ define H-module structures on M, call them M_u and M_ω. If* $\mathrm{End}(M_u) = \mathrm{End}(M_\omega)$ *as left H-modules via (1), then $(\omega \circ S) * u$ is an algebra map from H to k, i.e. a grouplike element in H^*. Similarly, if M admits two H-comodule structures ρ, χ such that the induced H-comodule structures on* $\mathrm{End}(M)$ *via (2) are same, then there is a grouplike element $g \in G(H)$ such that $\chi = (1 \otimes g)\rho$.*

PROOF. For any $m \in M$, $h \in H$, $\phi \in \mathrm{End}(M_u) = \mathrm{End}(M_\omega)$,

$$\sum u(h_{(1)})\Big[\phi\big[u\big(S(h_{(2)})\big)(m)\big]\Big] = \sum \omega(h_{(1)})\Big[\phi\big[\omega\big(S(h_{(2)})\big)(m)\big]\Big],$$

or equivalently,

$$\sum \omega\big(S(h_{(1)})\big)\Big[u(h_{(2)})\Big(\phi\big[u\big(S(h_{(3)})\big)(m)\big]\Big)\Big] = \phi\Big(\omega\big(S(h)\big)(m)\Big).$$

Let $\lambda = (\omega \circ S) * u: H \longrightarrow \mathrm{End}(M)$ with convolution inverse $(u \circ S) * \omega$. Letting $m = u(h_{(4)})(x)$ for any $x \in H$ in the equation above, we obtain $\lambda(h) \in Z\big(\mathrm{End}(M)\big) = k$ for all $h \in H$. Since u, ω are algebra maps, it is easy to see that λ is an algebra map from H to k. ∎

Given a group-like element $g \in G(H)$, g induces an inner Hopf automorphism of H denoted by $\bar{g}$, *i.e.*, $\bar{g}(h) = g^{-1}hg$, $h \in H$. Similarly, if λ is a group-like element of H^*, then λ induces a Hopf automorphism of H, denoted by $\bar{\lambda}$ where $\bar{\lambda}(h) = \sum \lambda(h_{(1)})h_{(2)}\lambda^{-1}(h_{(3)})$, $h \in H$. Since $G\big(D(H)\big) = G(H^*) \times G(H)$ (*cf.* [15, Proposition 9]) and $\bar{g}$ commutes with $\bar{\lambda}$ in $\mathrm{Aut}(H)$, we have a homomorphism θ:

$$G\big(D(H)\big) \longrightarrow \mathrm{Aut}(H), \quad \lambda \times g \longmapsto \overline{g\lambda}.$$

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Let $K(H)$ denote the subgroup of $G\big(D(H)\big)$ consisting of elements

$$\{\lambda \times g \mid \overline{g^{-1}}(h) = \overline{\lambda}(h), \forall h \in H\}.$$

LEMMA 4. *Let H be a faithfully projective Hopf algebra. Then $K(H) \cong G\big(D(H)^*\big)$.*

PROOF. By [15, Proposition 10], an element $g \otimes \lambda$ is in $G\big(D(H)^*\big)$ if and only if $g \in G(H), \lambda \in G(H^*)$ and g, λ satisfy the identity:

$$g(\lambda \rightharpoonup h) = (h \leftharpoonup \lambda)g, \quad \forall h \in H,$$

where, $\lambda \rightharpoonup h = \sum h_{(1)}\lambda(h_{(2)})$ and $h \leftharpoonup \lambda = \sum h_{(2)}\lambda(h_{(1)})$. Let $g \in G(H)$, $\lambda \in G(H^*)$, for any $h \in H$, we have

$$\begin{aligned}\sum gh_{(1)}\lambda(h_{(2)}) &= \sum \lambda(h_{(1)})h_{(2)}g \iff \sum h_{(1)}\lambda(h_{(2)})\\ &= \sum \lambda(h_{(1)})g^{-1}h_{(2)}g \iff \sum \lambda^{-1}(h_{(1)})h_{(2)}\lambda(h_{(3)}) = \sum g^{-1}hg.\end{aligned}$$

This means $g \otimes \lambda$ is in $G\big(D(H)^*\big)$ iff $g \times \lambda \in K(H)$. Therefore $K(H) = G\big(D(H)^*\big)$. ∎

Now we are able to prove the main theorem.

THEOREM 5. *Let H be a faithfully projective Hopf algebra over k. The following sequence is exact:*

$$1 \longrightarrow G\big(D(H)^*\big) \longrightarrow G\big(D(H)\big) \xrightarrow{\ \theta\ } \mathrm{Aut}(H) \xrightarrow{\ \pi\ } \mathrm{BQ}(k, H), \tag{4}$$

where $\pi(\alpha) = A_{\alpha^{-1}} = \mathrm{End}(H_{\alpha^{-1}})$.

PROOF. It is a routine verification that $\mathrm{Ker}(\theta) = K(H)$. Suppose that $[A_\alpha] = 1$. By Lemma 2, the $D(H)$-action on A_α induced by the H-structures of H_α is strongly inner. Denote by μ the algebra map from $D(H)$ into A_α which gives the strongly inner action on A_α. Taking into account the restriction to H of μ, we have an algebra map $\mu_H: H \longrightarrow A_\alpha$. We may use μ_H to define an H-module structure on H_α given by

$$h \rightharpoondown x = \mu_H(h)(x), \quad x \in H_\alpha, h \in H.$$

It is obvious that the induced (strongly inner) H-actions on A_α by $\rightharpoondown$ and $\cdot$ (see (3)) coincide. By Lemma 3, there exists a group-like element $\lambda \in H^*$ such that

$$h \rightharpoondown x = \sum \lambda(h_{(2)})h_{(1)} \cdot x, \quad h \in H,\ x \in H_\alpha. \tag{5}$$

Similarly, let $\mu_{H^{*\mathrm{cop}}}$ be the algebra map μ restricted to $H^{*\mathrm{cop}}$ and denote the $H^{*\mathrm{cop}}$-action on H_α by

$$p \rightharpoondown x = \mu_{H^{*\mathrm{cop}}}(p)(x), \quad p \in H^{*\mathrm{cop}},\ x \in H_\alpha.$$

Then there exists a group-like element $g \in H$ such that

$$p \rightharpoondown x = \sum p_{(1)}(g)p_{(2)} \cdot x = \sum p(gx_{(2)})x_{(1)}. \tag{6}$$

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Since A_α is a $D(H)$-module algebra and the $D(H)$-action is strongly inner, by Lemma 2 the algebra map μ gives H_α a $D(H)$-module structure. Let $p \in H^*$, $h \in H$. We have

$$p \bowtie h = \sum (\epsilon \bowtie h_{(2)})(p_{(2)} \bowtie 1)\langle p_{(1)}, h_{(3)}\rangle\langle p_{(3)}, S^{-1}(h_{(1)})\rangle.$$

Let both sides of the above equality act on element $x \in H_\alpha$, then we obtain the identity:

$$p \rightharpoonup (h \rightharpoonup x) = \sum h_{(2)} \rightharpoonup (p_{(2)} \rightharpoonup x)\langle p_{(1)}, h_{(3)}\rangle\langle p_{(3)}, S^{-1}(h_{(1)}\rangle.$$

Now applying relations (5), (6), we obtain

$$\begin{aligned} &(7) & p \rightharpoonup (h \rightharpoonup x) &= \sum \langle p_{(1)}, g\rangle p_{(2)} \cdot (\lambda(h_{(2)})h_{(1)} \cdot x) \\ &(8) & &= \sum \lambda(h_{(3)})\langle p_{(1)}, g\rangle p_{(2)} \cdot \big(\alpha(h_{(2)})xS^{-1}(h_{(1)})\big). \end{aligned}$$

On the other hand,

$$\begin{aligned} &(9) & &\sum h_{(2)} \rightharpoonup (p_{(2)} \rightharpoonup x)\langle p_{(1)}, h_{(3)}\rangle\langle p_{(3)}, S^{-1}(h_{(1)}\rangle \\ &(10) & &= \sum \lambda(h_{(4)})\langle p_{(2)}, g\rangle\langle p_{(1)}, h_{(5)}\rangle\langle p_{(4)}, S^{-1}(h_{(1)})\rangle\langle p_{(3)}, x_{(2)}\rangle \\ &(11) & &\quad \alpha(h_{(3)})x_{(1)}S^{-1}(h_{(2)}) \\ &(12) & &= \sum \lambda(h_{(3)})\langle p, h_{(4)}gx_{(2)}S^{-1}(h_{(1)})\rangle h_{(2)} \cdot x_{(1)}. \end{aligned}$$

Now let ϵ act on (8) and (12), then we obtain:

$$\sum \langle p, g\alpha(h_{(2)})\lambda(h_{(3)})xS^{-1}(h_{(1)})\rangle = \sum \langle p, \lambda(h_{(2)})h_{(3)}gxS^{-1}(h_{(1)})\rangle,$$

for $h \in H, p \in H^*$. Since p is arbitrary, we get:

$$(13) \qquad \sum g\alpha(h_{(2)})\lambda(h_{(3)})xS^{-1}(h_{(1)}) = \sum \lambda(h_{(2)})h_{(3)}gxS^{-1}(h_{(1)}), \quad h \in H.$$

Let $x = 1$ in (13). We obtain

$$\begin{aligned} \sum g\alpha(h_{(1)})\lambda(h_{(2)}) &= \sum g\alpha(h_{(3)})\lambda(h_{(4)})S^{-1}(h_{(2)})h_{(1)} \\ &= \sum \lambda(h_{(3)})h_{(4)}gS^{-1}(h_{(2)})h_{(1)} \\ &= \sum \lambda(h_{(1)})h_{(2)}g \end{aligned}$$

Thus we have

$$\begin{aligned} \alpha(h) &= \sum g^{-1}\lambda^{-1}(h_{(3)})g\lambda(h_{(2)})\alpha(h_{(1)}) \\ &= \sum g^{-1}\lambda^{-1}(h_{(3)})\lambda(h_{(1)})h_{(2)}g \\ &= \overline{g}\overline{\lambda}(h) \end{aligned}$$

■

As a consequence of the theorem, we obtain the Deegan-Caenepeel's embedding theorem for a commutative and cocommutative Hopf algebra *cf.* [6, 10].

COROLLARY 6. *Let H be a faithfully projective Hopf algebra such that $G(H)$ and $G(H^*)$ are contained in the centers of H and H^* respectively. Then the map π in sequence (4) is a monomorphism. In particular, if H is a commutative and cocommutative faithfully projective Hopf algebra over k, then* $\mathrm{Aut}(H)$ *can be embedded into* $\mathrm{BQ}(k, H)$.

PROOF. In this case, $G\big(D(H)^*\big) = G\big(D(H)\big)$. It follows that the morphism θ is trivial, and hence the morphism π is a monomorphism. ∎

Now an interesting question arises: Is H commutative and cocommutative if $G(H)$ and $G(H^*)$ are in the centers of H and H^* respectively?

In the following, we present two examples of the exact sequence (4). It will follow that the Brauer group of a Hopf algebra need not be a torsion group like the classical Brauer group.

EXAMPLE 7. Let H be the Sweedler Hopf algebra over a field k. That is, $H = k\langle g, x\rangle / \langle g^2 = 1, x^2 = 0, gx = -xg\rangle$ with comultiplication given by

$$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes g + 1 \otimes x.$$

H is a self-dual Hopf algebra, *i.e.*, $H \cong H^*$ as Hopf algebras. It is straightforward to show that the Hopf automorphism group $\mathrm{Aut}(H)$ is isomorphic to $k^* = k \setminus 0$ via:

$$f \in \mathrm{Aut}(H), \quad f(g) = g, \quad f(x) = zx, \quad z \in k^*.$$

Considering the group $G\big(D(H)\big)$ of group-like elements, it is easy to see that

$$G\big(D(H)\big) = \{1 \times \epsilon, 1 \times \lambda, g \times \epsilon, g \times \lambda\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

where $\lambda = p_1 - p_g$, and p_1, p_g is the dual basis of $1, g$. One may calculate that the kernel of the map θ is given by:

$$K(H) = \{1 \times \epsilon, g \times \lambda\} \cong \mathbb{Z}_2$$

The image of θ is $\{\bar{1}, \bar{g}\}$ which corresponds to the subgroup $\{1, -1\}$ of k^*. Thus by Theorem 5 we have an exact sequence:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow k^* \longrightarrow \mathrm{BQ}(k, H),$$

It follows that $k^*/\mathbb{Z}_2$ can be embedded into the Brauer group $\mathrm{BQ}(H)$. In particular, if $k = R$, the real number field, then $\mathrm{Br}(R) = \mathbb{Z}_2 \subset \mathrm{BQ}(R, H)$, and $R^*/\mathbb{Z}_2$ is a non-torsion subgroup of $\mathrm{BQ}(R, H)$.

In the previous example, the subgroup $k^*/\mathbb{Z}_2$ of the Brauer group $\mathrm{BQ}(k, H)$ is still an abelian group. The next example shows the general linear group $\mathrm{GL}_n(k)$ for any positive number n may be embedded into the Brauer group $\mathrm{BQ}(k, H)$ of some finite dimensional Hopf algebra H by modulo some finite group of roots of unit.

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EXAMPLE 8. Let $m > 2, n$ be any positive numbers. Let H be the Radford's Hopf algebra of dimension $m2^{n+1}$ over $\mathbb{C}$ (complex field) generated by $g, x_i, 1 \le i \le n$ such that

$$g^{2m} = 1, \quad {x_i}^2 = 0, \quad gx_i = -x_i g, \quad x_i x_j = -x_j x_i.$$

The coalgebra structure Δ and the counit ϵ are given by

$$\Delta g = g \otimes g, \quad \Delta x_i = x_i \otimes g + 1 \otimes x_i, \quad \epsilon(g) = 1, \quad \epsilon(x_i) = 0, \quad 1 \le i \le n.$$

By [16, Proposition 11], the Hopf automorphism group of H is $\mathrm{GL}_n(\mathbb{C})$. Now we compute the group $G\big(D(H)\big)$ and $G\big(D(H)^*\big)$. It is easy to see that $G(H) = (g)$ (see also [16, p. 353]) is a cyclic group of order $2m$. Let $\omega_i, 1 \le i \le m$ be the m-th roots of 1, and let ζ_j be the m-th roots of -1. Define the algebra maps from H to $\mathbb{C}$ by

$$\eta_i(g) = \omega_i g, \ \eta_i(x_j) = 0, \quad 1 \le i, \ j \le m,$$

and

$$\lambda_i(g) = \zeta_i g, \ \lambda_i(x_j) = 0, \quad 1 \le i, \ j \le m.$$

One may check that $\{\eta_i, \lambda_i\}_{i=1}^n$ is the group $G(H^*)$. It follows that $G\big(D(H)\big) = G(H) \times G(H^*) \cong (g) \times U$, where U is the group of $2m$-th roots of 1. To compute $G\big(D(H)^*\big)$ it is enough to calculate $K(H)$. Since

$$\overline{g^i} = \begin{cases} \mathrm{id} & \text{if } i \text{ is even} \\ \phi & \text{if } i \text{ is odd} \end{cases}$$

where $\phi(g) = g$, $\phi(x_j) = -x_j, 1 \le j \le n$, and

$$\overline{\eta_i}(g) = g, \ \overline{\eta_i}(x_j) = \omega_i x_j, \quad 1 \le i, \ j \le n,$$
$$\overline{\lambda_i}(g) = g, \ \overline{\lambda_i}(x_j) = \zeta_i x_j, \quad 1 \le i, \ j \le n,$$

one may easily obtain that

$$K(H) = \{g^{2i} \times \epsilon, g^{2i-1} \times \psi, 1 \le i \le m\}$$

where ψ is is given by

$$\psi(g) = -g, \quad \psi(x_i) = 0.$$

It follows that $G\big(D(H)^*\big) \cong U$, Since the base field is $\mathbb{C}$, $(g) \cong U$, and we have an exact sequence

$$1 \longrightarrow U \longrightarrow U \times U \longrightarrow \mathrm{GL}_n(\mathbb{C}) \longrightarrow \mathrm{BQ}(\mathbb{C}, H).$$

The above two examples highlight the interest of the study of the Brauer group of a Hopf algebra. In Example 8, even though the classic Brauer group $\mathrm{Br}(\mathbb{C})$ is trivial, the Brauer group $\mathrm{BQ}(\mathbb{C}, H)$ is still large enough.

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