BULL. AUSTRAL. MATH. SOC. 10F40, 10K05
VOL. 31 (1985), 329-338.

ON THE DISCREPANCY OF THE SEQUENCE FORMED FROM MULTIPLES OF AN IRRATIONAL NUMBER

TONY VAN RAVENSTEIN

This paper demonstrates a connection between two measures of discrepancy of sequences which arise in the theory of uniform distribution modulo one. The sequence formed from the non-negative integer multiples of an irrational number ξ is investigated and, by an application of the "Steinhaus Conjecture", some values of the two discrepancies are obtained using continued fractions.

1. Introduction

Let $\nu = (x_1, x_2, x_3, \ldots)$ be an infinite sequence of real numbers located in the unit interval. We define the *standard* discrepancy of the first N terms of ν as

$$(1.1) \qquad D_N(\nu) = \sup_{0\le\alpha\le\beta<1} \left|\frac{A([\alpha,\beta];\nu;N)}{N} - (\beta-\alpha)\right| ,$$

where $A([\alpha, \beta]; \nu; N)$ counts the number of the first N elements of ν which belong to $[\alpha, \beta]$.

The discrepancy is a measure of how closely the first N elements of ν approximate a uniform distribution. A related measure (the *extreme*

Received 30 October 1984. The author is most grateful to Dr Keith Tognetti of the University of Wollongong for helpful discussions.

discrepancy where $\alpha = 0$) is given by

$$D_N^*(\nu) = \sup_{0 \le \beta < 1} \left| \frac{A([0,\beta];\nu;N)}{N} - \beta \right| . \tag{1.2}$$

In this paper we first demonstrate an interesting relationship between these two measures. In particular, we consider the sequence ω formed by the fractional parts of non-negative integer multiples of ξ , where ξ is an irrational number. $D_N(\omega)$ and $D_N^*(\omega)$ are then evaluated for some particular values of N based on the continued fraction expansion of ξ . The formula largely derives from a result related to the "Steinhaus Conjecture" (see Section 5). An upper bound of $\liminf_{N\to\infty} ND_N(\omega)$ is offered as a corollary.

2. Representation of ν on a circle

Suppose that the sequence ν is represented on the circle C of unit circumference, rather than on the unit interval. Let $[\alpha : \beta]$ denote the arc from α to β $(0 \le \alpha, \beta < 1)$ on the circle in the direction of increasing co-ordinate. That is,

$$[\alpha : \beta] = \begin{cases} [\alpha, \beta] , & 0 \le \alpha \le \beta < 1 , \\ C - (\beta, \alpha) , & 0 \le \beta < \alpha < 1 . \end{cases} \tag{2.1}$$

Note that the length of such an arc is equal to $\{\beta-\alpha\}$, the fractional part of $\beta - \alpha$.

With respect to this representation, the discrepancy of ν may be given by

$$\hat{D}_N(\nu) = \sup_{0 \le \alpha, \beta < 1} \left| \frac{A([\alpha:\beta];\nu;N)}{N} - \{\beta-\alpha\} \right| . \tag{2.2}$$

LEMMA. $D_N(\nu) = \hat{D}_N(\nu)$.

Proof. Let

$$R(\alpha, \beta) = \frac{A([\alpha:\beta];\nu;N)}{N} - \{\beta-\alpha\} . \tag{2.3}$$

Clearly

https://doi.org/10.1017/S000497270000928X Published online by Cambridge University Press

$$\hat{D}_N(\nu) = \sup\left(\sup_{0\le\alpha,\beta<1} R(\alpha, \beta), \sup_{0\le\alpha,\beta<1} -R(\alpha, \beta)\right) .$$

But $-R(\alpha, \beta) \le R(\beta, \alpha)$ since $A([\alpha : \beta]; \nu; N) + A([\beta : \alpha]; \nu; N) \ge N$ and $\{-x\} = 1 - \{x\}$ for real x. Hence

$$\hat{D}_N(\nu) = \sup_{0\le\alpha,\beta<1} \left|\frac{A([\alpha:\beta];\nu;N)}{N} - \{\beta-\alpha\}\right| , \tag{2.4}$$

or

$$\hat{D}_N(\nu) = \sup\left(\sup_{0\le\alpha\le\beta<1} R(\alpha, \beta), \sup_{0\le\beta<\alpha<1} R(\alpha, \beta)\right) . \tag{2.5}$$

From (2.1),

$$R(\alpha, \beta) = \begin{cases} \dfrac{A([\alpha,\beta];\nu;N)}{N} - (\beta-\alpha) , & 0 \le \alpha \le \beta \le 1 , \\ \alpha - \beta - \dfrac{A([\beta,\alpha];\nu;N)}{N} , & 0 \le \beta < \alpha \le 1 . \end{cases}$$

Replacing $R(\alpha, \beta)$ in (2.5) by this expression completes the proof. □

Thus if the sequence is represented on the unit interval or the circle of unit circumference, the measure of discrepancy is the same.

3. A relation between $D_N^*(\nu)$ and $D_N(\nu)$

The following proposition relates the two functions $D_N^*(\nu)$ and $D_N(\nu)$.

PROPOSITION. $D_N(\nu) = D_N^*(\nu) + \inf\left(D_N^+(\nu), D_N^-(\nu)\right)$, *where*

$$D_N^+(\nu) = \sup_{0\le\beta<1} \left|\frac{A([0,\beta];\nu;N)}{N} - \beta\right| ,$$

$$D_N^-(\nu) = \sup_{0\le\beta<1} \left|\beta - \frac{A([0,\beta];\nu;N)}{N}\right| ,$$

$$D_N^*(\nu) = \sup\left(D_N^+(\nu), D_N^-(\nu)\right) .$$

Proof. We need only show that $D_N(\nu) = D_N^+(\nu) + D_N^-(\nu)$. Without loss of generality assume that the elements x_j , $1 \le j \le N$, are arranged in ascending order of magnitude. For notational convenience, let $x_0 = 0$ and

$x_{N+1} = 1$. Then the numbers $x_0, x_1, \ldots, x_{N+1}$ partition the unit interval so that

$$D_N^+(\nu) = \sup_{\substack{x_j \le \beta < x_{j+1} \\ j=0,1,2,\ldots,N}} \left(\frac{A([0,\beta];\nu;N)}{N} - \beta\right) .$$

Evaluating the supremum over each sub-interval $[x_j, x_{j+1})$ gives

$$D_N^+(\nu) = \sup_{j=1,2,\ldots,N} \left(\frac{j}{N} - x_j\right) . \tag{3.1}$$

Similarly

$$D_N^-(\nu) = \sup_{j=1,2,\ldots,N} \left(x_j - \frac{j-1}{N}\right) . \tag{3.2}$$

From (2.4) and the fact that $R(\alpha, \beta) \le R(x_i, x_j)$ where $x_i \le \alpha < x_{i+1}$, $x_j \le \beta < x_{j+1}$ (for suitable i and j) yields

$$D_N(\nu) = \sup_{0 \le i, j \le N} \left(\frac{A([x_i : x_j];\nu;N)}{N} - \{x_j - x_i\}\right) .$$

Alternatively

$$\begin{aligned} D_N(\nu) &= \sup_{0 \le i, j \le N} \left(\frac{j-i+1}{N} - x_j + x_i\right) \\ &= \sup_{0 \le j \le N} \left(\frac{j}{N} - x_j\right) + \sup_{0 \le i \le N} \left(x_i - \frac{i-1}{N}\right) \\ &= D_N^+(\nu) + D_N^-(\nu) . \end{aligned}$$

This follows from (3.1) and (3.2). □

4. The sequence formed from multiples of ξ

We now turn to evaluating the discrepancies $D_N^*(\omega)$ and $D_N(\omega)$ for some particular values of N related to the simple continued fraction expansion of ξ , where ω is the sequence of fractional parts of consecutive non-negative integer multiples of the irrational number ξ . The following notation is used. Write $t_0 = \xi$ and express (for $n = 0, 1, 2, \ldots$),

https://doi.org/10.1017/S000497270000928X Published online by Cambridge University Press

$$a_n = [t_n] ,$$

$$t_{n+1} = \frac{1}{\{t_n\}} ,$$

where [ ] is the truncation operator.

In this way the continued fraction expansion of ξ is identified by the expression

$$\xi = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

$$= \{a_0;\ a_1,\ a_2,\ a_3,\ \ldots\} .$$

The partial convergents to ξ are defined as

$$\frac{p_{n+1,i}}{q_{n+1,i}} = \{a_0;\ a_1,\ a_2,\ \ldots,\ a_n,\ i\} , \quad i = 1, 2, \ldots, a_{n+1} .$$

Note that

$$\frac{p_{n+1,i}}{q_{n+1,i}} = \frac{p_{n-1}+ip_n}{q_{n-1}+iq_n} , \quad p_{-2} = q_{-1} = 0 , \quad q_{-2} = p_{-1} = 1 ,$$

$$\frac{p_{n+1,k}}{q_{n+1,k}} = \frac{p_{n+1}}{q_{n+1}} ,$$

where $k = a_{n+1}$.

The total convergents p_n/q_n , $n = 0, 1, 2, \ldots$, are important in Diophantine approximation theory since they provide the unique sequence of best rational approximations to ξ in the sense that

$$\|q_n\xi\| < \|q\xi\| , \quad 0 < q < q_{n+1} , \quad q \neq q_n ,$$

where

$$\|q\xi\| = |q\xi-p| , \quad p = [q\xi+\tfrac{1}{2}] .$$

(That is, $\|q\xi\|$ is equal to the absolute difference between $q\xi$ and its nearest integer. Note that $p_n = [q_n\xi+\frac{1}{2}]$.) We quote some results from the theory of continued fractions which we will need later:

(4.1) $$q_n p_{n+1,i} - p_n q_{n+1,i} = (-1)^n ;$$

(4.2) $$q_{n+1,i}\|q_n\xi\| + q_n\|q_{n+1,i}\xi\| = 1 ;$$

(4.3) $$p_{n+1,i}\|q_n\xi\| + p_n\|q_{n+1,i}\xi\| = \xi .$$

5. The three gap theorem (the Steinhaus conjecture)

This theorem, originally conjectured by H. Steinhaus states that any N consecutive elements of ω partition C (or the unit interval) into sub-intervals or gaps of at most three different lengths and at least two if ξ is irrational. Various proofs have appeared in the literature (see, for example, references [1], [5]-[9]). The theorem is also related to the ordering of the first N elements of ω. Let $(\{u_j\xi\})$, $j = 1, 2, \ldots, N$, be that ordered sequence. That is, $\{u_1, u_2, \ldots, u_N\} = \{0, 1, \ldots, N-1\}$ where $\{u_j\xi\} < \{u_{j+1}\xi\}$. It may be found from references [5] and [6] that the elements u_j are obtained by the following relation,

(5.1) $$u_{j+1} = \begin{cases} u_j + u_2 , & 0 \le u_j \le N-u_2 , \\ u_j + u_2 - u_N , & N-u_2 < u_j < u_N , \\ u_j - u_N , & u_N \le u_j \le N , \end{cases}$$

for $j = 1, 2, \ldots, N$, $u_1 = 0$, where

(5.2) $$\begin{cases} u_2 = \begin{cases} q_n , & n \text{ even} \\ q_{n+1,i-1} , & n \text{ odd} , \end{cases} \\ u_N = \begin{cases} q_{n+1,i-1} , & n \text{ even,} \\ q_n , & n \text{ odd} , \end{cases} \end{cases}$$

which holds for $q_{n+1,i-1} < N \le q_{n+1,i}$, $2 \le i \le a_{n+1}$ $(n \ge 1)$.

For $q_n < N \le q_{n+1,i}$ $(n \ge 1)$,

https://doi.org/10.1017/S000497270000928X Published online by Cambridge University Press

$$(5.3)\qquad \begin{cases} u_2 = \begin{cases} q_n\ , & n \text{ even}, \\ q_{n-1}\ , & n \text{ odd}\ , \end{cases} \\ u_N = \begin{cases} q_{n-1}\ , & n \text{ even}, \\ q_n\ , & n \text{ odd}\ . \end{cases} \end{cases}$$

THEOREM. *The first* $u_2 + u_N = q_{n+1,i}$ $(i = 1, 2, \ldots, a_{n+1}\ ,\ n \geq 1)$ *elements of* ω *partition the circle into gaps of two different lengths. In this case*

$$D^*_{q_{n+1,i}}(\omega) = \begin{cases} \dfrac{1}{q_{n+1,i}} + \left(q_{n+1,i}-1\right)\left(\dfrac{p_{n+1,i}}{q_{n+1,i}} - \xi\right)\ , & n \ \textit{even}, \\ \sup\left(\dfrac{1}{q_{n+1,i}},\ \left(q_{n+1,i}-1\right)\left(\xi - \dfrac{p_{n+1,i}}{q_{n+1,i}}\right)\right)\ , & n \ \textit{odd}, \end{cases}$$

$$D_{q_{n+1,i}}(\omega) = \frac{1}{q_{n+1,i}} + \left(q_{n+1,i}-1\right)\left|\frac{p_{n+1,i}}{q_{n+1,i}} - \xi\right|\ .$$

Proof. With $N = u_2 + u_N = q_{n+1,i}$, (5.1) becomes

$$(5.4)\qquad u_{j+1} = \begin{cases} u_j + u_2\ , & 0 \leq u_j < u_N\ , \\ u_j - u_N\ , & u_N \leq u_j \leq N\ , \end{cases}$$

for $j = 1, 2, \ldots, N$ $(u_1 = 0)$. Hence, for this value of N , there are only two gap lengths $\|u_2\xi\|$ and $\|u_N\xi\|$. From (5.1) note that if $N \neq u_1 + u_2$, the circle is partitioned into gaps of three different lengths.

(5.4) is equivalent to

$$(5.5)\qquad u_j = \left((j-1)u_2\right) \bmod N\ ,\quad j = 1, 2, \ldots, N\ .$$

Substituting (5.2) and (5.3) into (5.5) yields

$$(5.6)\qquad u_j = \left((-1)^n(j-1)q_n\right) \bmod q_{n+1,i}\ ,\quad j = 1, 2, \ldots, q_{n+1,i}\ .$$

It is seen that there exists a non-negative integer k so that

https://doi.org/10.1017/S000497270000928X Published online by Cambridge University Press

(5.7) $$u_j = (-1)^n(j-1)q_n - (-1)^n kq_{n+1,i} .$$

Solving this linear Diophantine equation by use of (4.1) yields

(5.8) $$j - 1 = q_{n+1,i}\left\{\frac{u_j p_{n+1,i}}{q_{n+1,i}}\right\} ,$$

(5.9) $$k = u_j p_n - \left[\frac{u_j p_{n+1,i}}{q_{n+1,i}}\right]q_n .$$

From (5.7) it follows that

(5.10) $$\{u_j\xi\} = (j-1)\|q_n\xi\| + k\|q_{n+1,i}\xi\| .$$

Substituting expressions for j and k from (5.8), (5.9) and using (4.2) and (4.3) yields

(5.11) $$\{u_j\xi\} = u_j\xi - \left[\frac{u_j p_{n+1,i}}{q_{n+1,i}}\right] .$$

That is

(5.12) $$[u_j\xi] = \left[\frac{u_j p_{n+1,i}}{q_{n+1,i}}\right] .$$

Substitution of (5.8) into (3.2) with the inclusion of (5.11) yields

$$\begin{aligned} D^-_{q_{n+1,i}}(\omega) &= \sup_{j=1,2,\ldots,q_{n+1,i}} u_j\left(\xi - \frac{p_{n+1,i}}{q_{n+1,i}}\right) \\ &= \sup_{u_j=0,1,\ldots,q_{n+1,i}-1} u_j\left(\xi - \frac{p_{n+1,i}}{q_{n+1,i}}\right) \\ &= \begin{cases} 0 , & n \text{ even}, \\ \left(q_{n+1,i}-1\right)\left(\xi - \dfrac{p_{n+1,i}}{q_{n+1,i}}\right) , & n \text{ odd}. \end{cases} \end{aligned}$$

This follows from the fact that $\xi - p_{n+1,i}/q_{n+1,i}$ is negative for even n and positive otherwise.

To determine $D^+_{q_{n+1,i}}(\omega)$ note that

https://doi.org/10.1017/S000497270000928X Published online by Cambridge University Press

$$D^+_{q_{n+1,i}}(\omega) = \frac{1}{q_{n+1,i}} - \inf_{j=1,2,\ldots,q_{n+1,i}} \left(\{u_j\xi\} - \frac{j-1}{q_{n+1,i}}\right) .$$

Following the same procedure as above, it is found that

$$D^+_{q_{n+1,i}}(\omega) = \begin{cases} \dfrac{1}{q_{n+1,i}} + \left(q_{n+1,i}-1\right)\left(\dfrac{p_{n+1,i}}{q_{n+1,i}} - \xi\right) , & n \text{ even}, \\ \\ \dfrac{1}{q_{n+1,i}} , & n \text{ odd}. \end{cases}$$

The theorem now follows from the proposition. □

COROLLARY. $1 \leq \liminf_{N\to\infty} ND_N(\omega) \leq 1 + \frac{1}{\sqrt{5}}$.

Proof. The lower bound is easily found. (See Kuipers and Neiderreiter [4], page 90.) For the upper bound, first note that

$$\liminf_{N\to\infty} ND_N(\omega) \leq \liminf_{n\to\infty} q_n D_{q_n}(\omega) .$$

From the theorem,

$$\begin{aligned} \liminf_{n\to\infty} q_n D_{q_n}(\omega) &= \liminf_{n\to\infty} 1 + \left(q_n-1\right)\|q_n\xi\| \\ &= 1 + \liminf_{n\to\infty} q_n\|q_n\xi\| . \end{aligned}$$

From a theorem of Hurwitz (see, for example, Hardy and Wright [2], Theorem 194),

$$\begin{aligned} \sup_{\xi} \liminf_{q\to\infty} q\|q\xi\| &= \sup_{\xi} \liminf_{n\to\infty} q_n\|q_n\xi\| \\ &= \frac{1}{\sqrt{5}} . \end{aligned}$$

The supremum occurs at all values of ξ which have $t_j = (1+\sqrt{5})/2$ for some non-negative integer j .

Thus the corollary follows. □

https://doi.org/10.1017/S000497270000928X Published online by Cambridge University Press

References

[1] J.H. Halton, "The distribution of the sequence $\{n\xi\}$ $(n = 0, 1, 2, \ldots)$ ", *Proc. Cambridge Philos. Soc.* **61** (1965), 665-670.

[2] G.H. Hardy and E.M. Wright, *An introduction to the theory of numbers*, 4th ed. (Clarendon Press, Oxford, 1960; reprinted 1971).

[3] A.Y. Khintchine, *Continued fractions* (translated by P. Wynn. Noordhoff, Groningen, 1963).

[4] L. Kuipers and H. Neiderreiter, *Uniform distribution of sequences* (John Wiley and Sons, New York, London, 1974).

[5] A. Révuz, "Sur la repartition des points $e^{\nu i\theta}$ ", *C.R. Acad. Sci.* **228** (1949), 1966-1967.

[6] N.B. Slater, "Gaps and steps for the sequence $n\theta$ mod 1 ", *Proc. Cambridge Philos. Soc.* **63** (1967), 1115-1122.

[7] V.T. Sós, "On the theory of Diophantine approximations, I", *Acta Math. Acad. Sci. Hungar.* **8** (1957), 461-472.

[8] V.T. Sós, "On the distribution mod 1 of the sequence $n\alpha$ ", *Ann. Univ. Sci. Pudapest. Eötvös Sect. Math.* **1** (1958), 127-134.

[9] S. Świerckowski, "On successive settings of an arc on the circumference of a circle", *Fund. Math.* **46** (1958), 187-189.

Department of Mathematics,
University of Wollongong,
Wollongong,
New South Wales 2500,
Australia.