7

Rectifiable Sets in Metric Spaces

7.1 Definition and Norm

Let (X, d) be a metric space. There is no problem with the definition:

Definition 7.1 A set $E \subset X$ is *m-rectifiable* if there are Lipschitz maps $f_i : A_i \to X, A_i \subset \mathbb{R}^m, i = 1, 2, \ldots$ such that

$$\mathcal{H}^m\left(E \setminus \bigcup_{i=1}^{\infty} f_i(A_i)\right) = 0.$$

A set $E \subset X$ is *purely m-unrectifiable* if $\mathcal{H}^m(E \cap F) = 0$ for every m-rectifiable set $F \subset X$.

But everything else is problematic. What properties can we prove? There are no linear subspaces, so can we talk about tangent planes or projections? Any metric space X can be embedded isometrically into a Banach space, and if X is separable, which rectifiable sets are, into l^∞. Thus we may consider X as a metric subspace of a Banach space Y, that is, we can assume that the metric d is given by a norm $\|\cdot\|$. This is often convenient and it gives us linear subspaces. But it does not solve everything. Anyway, Lipschitz maps from subsets of $\mathbb{R}^m$ to Banach spaces can be extended to all of $\mathbb{R}^m$, so in the definition we could consider $f_i : \mathbb{R}^m \to Y$. But not necessarily $f_i : \mathbb{R}^m \to X$.

Let us begin with densities, which anyway are defined as before.

7.2 Densities when $m = 1$

Using [16, Example 6.4], we first observe that there is no hope, even in Hilbert spaces, to get the Besicovitch–Preiss theorem's 'existence of density implies rectifiability': let $X = (0, 1)$ with the metric $d(x, y) = \sqrt{|x - y|}$. Then $\mathcal{H}^2_d = (\pi/4)\mathcal{L}^1$, so $\Theta^2(X, x) = 1/2$ for all $x \in X$. However, X is purely 2-unrectifiable by Theorem 7.7. To see the same in Hilbert spaces, consider $E = \{\chi_{[0,t]} : 0 < t < 1\} \subset L^2([0, 1])$. Nevertheless, maybe the weaker 'density 1 implies rectifiability' could be true?

There is no counter-example to Besicovitch's 1/2-conjecture 3.11 even in general metric spaces. The best result known is the following theorem of Preiss and Tiser [384], which improves and extends Besicovitch's 3/4 Theorem 3.10:

Theorem 7.2 *If $E \subset X$ is $\mathcal{H}^1$ measurable, $\mathcal{H}^1(E) < \infty$ and*

$$\Theta^1_*(E, x) > \frac{2 + \sqrt{46}}{12}$$

for $\mathcal{H}^1$ almost all $x \in E$, then E is 1-rectifiable.

Notice that $\frac{2+\sqrt{46}}{12}$ is between $\frac{58}{80}$ and $\frac{59}{80}$, so it is less than but close to $\frac{3}{4}$.

Corollary 7.3 *If $E \subset X$ is $\mathcal{H}^1$ measurable and $\mathcal{H}^1(E) < \infty$, then E is 1-rectifiable if and only if $\Theta^1(E, x) = 1$ for $\mathcal{H}^1$ almost all $x \in E$.*

That rectifiable sets have density 1 also in metric spaces follows from Kirchheim's theorem which we shall soon discuss.

The proof of Theorem 7.2 has similar basic ingredients as that of Theorem 3.10; Besicovitch circle pairs, in a generalized form, are used to find a continuum C with finite measure which intersects E in a set of positive measure. By Theorem 2.1 such a C is rectifiable even in metric spaces. This result of Eilenberg and Harrold is perhaps the first result on rectifiability in metric spaces.

To say a bit more, recall from the discussion on the proof of Theorem 3.10 that for any $\alpha > 0$ we had for some compact subset F of E with $\mathcal{H}^1(F) > 0$: if $\sigma = \frac{3}{4} + \alpha$,

$$\mathcal{H}^1(E \cap U(x, r)) > \sigma 2r \text{ for } x \in F, 0 < r < r_0, \tag{7.1}$$

and

$$\mathcal{H}^1(E \cap B) \leq (1 + \alpha)d(B) \text{ whenever } E \cap B \neq \emptyset \text{ and } d(B) < r_0, \tag{7.2}$$

then

$$\mathcal{H}^1(E \cap U(x, y)) \geq \alpha|x - y| \text{ for } x, y \in F \text{ with } d(x, y) < r_0/3, \tag{7.3}$$

where $U(x, y) = U(x, r) \cap U(y, r), r = |x - y|$. For this one cannot push σ below 3/4. But Preiss and Tiser showed that it is possible to use other sets in place of $U(x, y)$ to reduce σ to $\frac{2+\sqrt{46}}{12}$. More precisely (but not quite precisely), they showed that the following condition holds with any $\sigma > \frac{2+\sqrt{46}}{12}$ and some $\tau > 0$: whenever (7.2) holds with small $\alpha > 0$ and E_1, E_2 are Borel subsets of E with $d(E_1, E_2) > 0$ small satisfying (7.1) in place of F, then there is $U \subset X$ meeting both E_1 and E_2 such that $\mathcal{H}^1(E \cap U \setminus (E_1 \cup E_2)) > \tau d(U)$. Then they showed that this condition together with $\Theta^1_*(E, x) > \sigma$ (with any $\sigma > 0$) implies that some continuum with finite measure intersects E in a set of positive measure. To relate the Preiss–Tiser condition to (7.3), observe that $E \cap U(x_1, x_2) \setminus (E_1 \cup E_2) = E \cap U(x_1, x_2)$ if $d(E_1, E_2) = |x_1 - x_2|, x_i \in E_1$.

Recently Bate [59] proved the analogue of Theorem 4.11 for one-dimensional measures in a large class of metric spaces. In the following, doubling means that there is N such that every ball of radius $2r$ can be covered with N balls of radius r and geodesic means that the distance between any two points is the minimal length of the curves connecting them.

Theorem 7.4 *Let (X, d) be a doubling geodesic metric space and let $\mu \in \mathcal{M}(X)$ be such that the positive and finite limit $\lim_{r\to 0} r^{-1}\mu(B(x, r))$ exists for μ almost all $x \in X$. Then μ is 1-rectifiable.*

This result follows from a complete classification of 1-uniform metric measure spaces, of which there are three up to scaled isometry. The conditions on X imply that two of these spaces cannot be tangent spaces to (X, d, μ, x) for positively many x and so the rectifiability of μ follows from Theorem 7.13.

Bate's method applies to Heisenberg groups, which extends Antonelli's and Merlo's result Theorem 8.12.

7.3 Densities and Area Formula for General *m*

We still consider X as a metric subspace of a Banach space Y and we would like to use Rademacher's theorem. However, Lipschitz maps from $\mathbb{R}^m$ to Y need not be differentiable in the standard sense. Kirchheim found a useful substitute in [276]. This paper has been very influential in analysis and geometric measure theory in metric spaces.

Kirchheim's idea was to introduce *metric differentials* $MD(f, x)$, which are seminorms on $\mathbb{R}^m$, by

$$MD(f, x)(v) = \lim_{r\to 0} \|f(x + rv) - f(x)\|/r, \ \ x, v \in \mathbb{R}^m,$$

whenever the limit exists. He showed that if f is Lipschitz, it does exist for almost all $x \in \mathbb{R}^m$, and then for all $y, z \in \mathbb{R}^m$,

$$\|f(z) - f(y)\| - MD(f, x)(z - y) = o(|z - x| + |y - x|). \tag{7.4}$$

So we have something like (4.1) and a substitute for Rademacher's theorem.

Then Kirchheim proceeded to prove an area formula. For this define for any seminorm s on $\mathbb{R}^m$ the 'Jacobian'

$$J(s) = \alpha(m)m\left(\int_{S^{m-1}} s(x)^{-m} d\mathcal{H}^{m-1}x\right)^{-1}.$$

Then

Theorem 7.5 *If $f\colon \mathbb{R}^m \to X$ is Lipschitz and $A \subset \mathbb{R}^m$ Lebesgue measurable, then*

$$\int \operatorname{card} A \cap f^{-1}\{y\}\, d\mathcal{H}^m y = \int_A J(MD(f,x))\, d\mathcal{L}^m x.$$

The proof is based on the following lemma, [276, Lemma 4], going back to Federer's proof of the Euclidean area theorem and [203, Lemma 3.2.2]:

Lemma 7.6 *Let $f\colon \mathbb{R}^m \to X$ be Lipschitz and let B be the set of $x \in \mathbb{R}^m$ for which $MD(f,x)$ exists and is a norm. Then for any $\lambda > 1$ there are norms $\|\cdot\|_i$ on $\mathbb{R}^m$ and a Borel partition (B_i) of B such that*

$$\|x-y\|_i/\lambda \le d(f(x), f(y)) \le \lambda\|x-y\|_i \text{ for } x, y \in B_i, i = 1, 2, \ldots.$$

After this one gets

Theorem 7.7 *If $E \subset X$ is $\mathcal{H}^m$ measurable, m-rectifiable and $\mathcal{H}^m(E) < \infty$, then $\Theta^m(E,x) = 1$ for $\mathcal{H}^m$ almost all $x \in E$.*

As we saw, for $m = 1$ this is a characterization of rectifiability. Whether it is a characterization when $m > 1$ is open; perhaps surprisingly no counterexample is known. To emphasize this problem I state it as a conjecture, although I am rather suspicious about its validity (but for no good reason):

Conjecture 7.8 *An $\mathcal{H}^m$ measurable set $E \subset X$ with $\mathcal{H}^m(E) < \infty$ is m-rectifiable if and only if $\Theta^m(E,x) = 1$ for $\mathcal{H}^m$ almost all $x \in E$.*

So this is true in the Euclidean case and when $m = 1$. Examples of non-Euclidean homogeneous groups where it is true are due to Julia and Merlo [267]. If we replace the Hausdorff measure $\mathcal{H}^m$ by the spherical Hausdorff measure $\mathcal{S}^m$, then Heisenberg groups give us purely m-unrectifiable metric spaces X with $\mathcal{S}^m(X) < \infty$ and $\Theta^m(\mathcal{S}^m \llcorner X, x) = 1$ for $\mathcal{S}^m$ almost all $x \in X$. We shall come back to these facts at the end of Section 8.5.

7.4 Tangent Planes

Since we are considering X as a subspace of a Banach space Y, we have linear subspaces and we can hope for a tangent plane characterization of rectifiability. Now Y is l^∞, or more generally the dual of a separable Banach space. In addition to Kirchheim's metric differentiability result, Ambrosio and Kirchheim [16] proved the weak star differentiability for a Lipschitz map $f\colon \mathbb{R}^m \to Y$: for almost all $x \in \mathbb{R}^m$ there exists a w^*-differential of f at x, that is, a linear map $wdf_x\colon \mathbb{R}^m \to Y$ such that

$$w^* - \lim_{y\to x}(f(y) - f(x) - wdf_x(y-x))/|y-x|) = 0. \tag{7.5}$$

If $E \subset X$ is $\mathcal{H}^m$ measurable and m-rectifiable, then by Lemma 7.6 we can decompose almost all of it into sets $f_i(B_i)$, where $B_i \subset \mathbb{R}^m$ is a Borel set and f_i is bi-Lipschitz on B_i. Then for $\mathcal{H}^m$ almost all $a \in E$, we can define the *weak approximate tangent plane* of E at $a = f_i(x) \in f_i(B_i)$ as

$$\mathrm{Tan}^{(m)}(E, a) = wdf_{ix}(\mathbb{R}^m).$$

It further follows that if $\pi_a\colon Y \to \mathrm{Tan}^{(m)}(E, a)$ is a weak star continuous projection ($\pi_a(x) = x$ for $x \in \mathrm{Tan}^{(m)}(E, a)$), then $\|\pi_a(x) - a\|/\|x - a\| \to 1$ as $x \to a, x \in E_a$, where $\Theta^m(E \setminus E_a, a) = 0$.

Ambrosio and Kirchheim also had a converse in [16, Theorem 6.3]:

Theorem 7.9 *Let $E \subset Y$ be $\mathcal{H}^m$ measurable with $\mathcal{H}^m(E) < \infty$. Then E is m-rectifiable if and only if for $\mathcal{H}^m$ almost all $a \in E$ there is a weak star continuous linear map $\pi_a\colon Y \to Y, \dim \pi_a(Y) = m$ such that for some $s > 0$,*

$$\lim_{r\to 0} r^{-m}\mathcal{H}^m(\{x \in E \cap B(a,r)\colon \|\pi_a(x-a)\| < s\|x-a\|\}) = 0.$$

Ambrosio and Kirchheim had this with positive lower density condition, but using the argument from the Euclidean case one easily sees that this is not needed, see [321, Lemma 15.14].

Ambrosio and Kirchheim also proved in [16] an area formula for Lipschitz maps between rectifiable sets and a coarea formula for $\mathbb{R}^k$ valued Lipschitz maps on rectifiable subsets of metric spaces.

7.5 Cheeger's Differentiability Spaces and Alberti Representations

As discussed above, Kirchheim showed that Lipschitz maps from $\mathbb{R}^m$ to X are differentiable in a sense. In [90], Cheeger introduced conditions under which the differentiability of Lipschitz maps from X to $\mathbb{R}^m$ also makes sense and is true. This was further developed and generalized by Keith [273]. Suppose that X is equipped with a Borel measure μ. We say that (X, d, μ) is an m-dimensional *Lipschitz differentiability space*, LDS, if there is a countable collection of local Lipschitz charts $\phi_i\colon U_i \to \mathbb{R}^m, U_i \subset X, X = \bigcup_i U_i$, with respect to which every Lipschitz function $f\colon X \to \mathbb{R}$ is differentiable at μ almost all $x \in U_i$ in the sense that there is a unique linear function $Df(x)\colon \mathbb{R}^m \to \mathbb{R}$ such that

$$f(y) - f(x) = Df(x)(\phi_i(y) - \phi_i(x)) + o(d(x, y)), y \in U_i.$$

Clearly, Euclidean spaces and Riemannian manifolds with the Lebesgue measure are LDS. Not all, even compact, metric measure spaces are LDS, but there are a lot of non-Euclidean ones that are, for example, Heisenberg groups.

Recall the role of Alberti representations in Euclidean spaces from Section 4.8. *Alberti representation* of a measure μ on X is a Fubini-type decomposition of μ into 1-rectifiable measures μ_γ:

$$\mu(B) = \int \mu_\gamma(B)\, dP\gamma, \ B \subset X \text{ Borel}. \tag{7.6}$$

Here P is a probability measure on the space of curve fragments, that is, bi-Lipschitz mappings $\gamma\colon C_\gamma \to X, C_\gamma \subset \mathbb{R}$ compact, and the measures μ_γ are absolutely continuous with respect to $\mathcal{H}^1 \llcorner \gamma(C_\gamma)$.

For $\phi\colon X \to \mathbb{R}^m$, the Alberti representations $\gamma_i, i = 1, \ldots, m$ are said to be ϕ-independent if the derivatives of $\phi \circ \gamma_i, i = 1, \ldots, m$ are linearly independent and belong to disjoint cones.

Bate characterized in [56] the Lipschitz differentiability spaces via Alberti representations. The following theorem tells us a great deal of relations between Lipschitz differentiability spaces, Alberti representations and rectifiability:

Theorem 7.10 *Suppose that $\mathcal{H}^m(X) < \infty$ and $\Theta^m_*(X, x) > 0$ for $\mathcal{H}^m$ almost all $x \in X$. Then the following conditions are equivalent.*

(1) *X is m-rectifiable.*
(2) *There are Borel sets $U_i \subset X, i = 1, 2, \ldots,$ with $\mathcal{H}^m(X \setminus \bigcup_i U_i) = 0$ such that each $(U_i, d, \mathcal{H}^m \llcorner U_i)$ is an m-dimensional LDS.*
(3) *There are Borel sets $U_i \subset X, i = 1, 2, \ldots,$ with $\mathcal{H}^m(X \setminus \bigcup_i U_i) = 0$ and Lipschitz functions $\phi_i\colon U_i \to \mathbb{R}^m$ such that each $\mathcal{H}^m \llcorner U_i$ has m ϕ_i-independent Alberti representations.*

That (1) implies (2) follows from Kirchheim's work [276]. He showed that in the definition of rectifiable sets, Lipschitz maps can be replaced by bi-Lipschitz maps, recall Lemma 7.6. Then these can be used to find the required charts. The implication from (2) to (3) follows from [56]. Rectifiability from independent Alberti representations was proved by Bate and Li in [61]. They used some of the ideas of David from [152], who proved related weaker results. A key feature in this technically very complicated argument is to show that the maps ϕ_i are bi-Lipschitz on large subsets.

Notice that m in the assumptions and in (2) is the same. This leaves out many interesting LDSs. For example, the Hausdorff dimension of the Heisenberg group $\mathbb{H}^1$ is 4 but it is a 2-dimensional LDS.

David and Kleiner [153] proved a general result for a measure μ in a metric space without any density conditions: if at μ almost all points, μ is pointwise doubling, it has two independent Alberti representations, and its pointed Gromov–Hausdorff limits (see Section 7.7) are homeomorphic to $\mathbb{R}^2$, then μ is 2-rectifiable.

7.6 Projections as Lipschitz Images

First some bad news. The Besicovitch–Federer projection theorem fails in every infinite-dimensional separable Banach space Y: Bate, Csörnyei and Wilson [60] constructed a purely unrectifiable set $E \subset Y$ with $\mathcal{H}^1(E) < \infty$ for which $\mathcal{L}^1(L(E)) > 0$ for every non-zero continuous linear function $L\colon Y \to \mathbb{R}$. An earlier weaker result was given by De Pauw in [172].

However, instead of linear maps, Bate [57] considered Lipschitz maps from the metric space X to $\mathbb{R}^n$ and obtained a rectifiability characterization in the spirit of the Besicovitch–Federer projection theorem. There is no natural measure on the space of Lipschitz maps (at least for this purpose), but the Baire category gives a notion of typical maps. Related results were proven by Pugh [385] and Galeski [218].

For the rest of this section we shall assume that X is complete. To state Bate's result, let us equip the space of bounded Lipschitz functions $f\colon X \to \mathbb{R}^n$ such that $\mathrm{Lip}(f) \le 1$ with the supremum norm to have the complete metric space $\mathrm{Lip}_1(X, n)$. A subset of $\mathrm{Lip}_1(X, n)$ is *residual* if it contains a countable intersection of dense open sets. These are the complements of the 'small' first category sets, countable unions of nowhere dense sets. Bate proved the following:

Theorem 7.11 *Let $0 < m \le n$. If $E \subset X$ is purely m-unrectifiable with $\mathcal{H}^m(E) < \infty$ and $\Theta^m_*(E, x) > 0$ for $\mathcal{H}^m$ almost all $x \in E$, then the set of all $f \in Lip_1(X, n)$ with $\mathcal{H}^m(f(E)) = 0$ is residual.*

Conversely, if $F \subset X$ is m-rectifiable with $\mathcal{H}^m(F) > 0$, then the set of all $f \in Lip_1(X, n)$ with $\mathcal{H}^m(f(F)) > 0$ is residual.

Bate also showed that the positive lower density assumption is not needed if $X = \mathbb{R}^n$ or $m = 1$. It may be that it is never needed, but this depends on an announced but unpublished result of Csörnyei and Jones.

The first statement of the theorem is the more essential one. To prove it one needs to approximate well any $f \in \mathrm{Lip}_1(X, n)$ with $g \in \mathrm{Lip}_1(X, n)$ for which $\mathcal{L}^m(g(E)) = 0$, or at least arbitrarily small. There are several very interesting ingredients in this argument.

The main tools consist of Alberti representations and weak tangent fields. Recall from Section 4.8 that both play essential roles in the investigation of differentiability of Lipschitz maps in [3] and [4]. According to Definition 4.23, a set $E \subset \mathbb{R}^n$ has a weak $(m-1)$-dimensional tangent field if, generically, the tangents of its 1-rectifiable subsets span at most $(m-1)$-dimensional subspaces. In the metric space X this spanning condition is interpreted via Lipschitz maps $f\colon E \to \mathbb{R}^n, 0 < m \leq n$. A little more precisely, a Borel function $\tau\colon E \to G(n, m-1)$ is a weak tangent field with respect to f if for 1-rectifiable sets $\gamma \subset E$ at almost all points $x \in \gamma$ the tangent of $f(\gamma)$ at $f(x)$ lies in $\tau(x)$.

If E is purely m-unrectifiable, then by Theorem 7.10 E can have at most $m-1$ independent Alberti representations, which roughly means that one can move along E from the points of E at most to $m-1$ directions. This leads to the existence of a weak tangent field. The next step in the proof of Theorem 7.11 is to perturb f slightly to a Lipschitz function g with $\mathcal{H}^m(g(E))$ small. This can be done contracting along directions orthogonal to the planes $\tau(x)$. As these planes are $(m-1)$-dimensional, the small measure is achieved.

In connection with their study of Lipschitz analysis of metric spaces, Aliaga, Gartland, Petitjean and Prochzka [6] used Bate's work [57] to verify that a compact metric space is purely 1-unrectifiable if and only if locally flat (local Lipschitz constants tend to 0) Lipschitz functions separate its points uniformly, a result conjectured by Weaver. They also gave an independent proof.

As an application of Bate's result, David and Le Donne proved in [154] that if a subset of a compact metric space has finite $\mathcal{H}^m$ measure, positive lower density and topological dimension m, then it is not purely m-unrectifiable. In Euclidean spaces this follows, without positive lower density assumption, from the Besicovitch–Federer projection theorem, see [200].

There exist easy examples of sets in Euclidean spaces with positive Hausdorff m-measure which project to measure zero in *all* m-planes. This is true in the Lipschitz setting too. Vitushkin, Ivanov and Melnikov [432] constructed a purely 1-unrectifiable subset E of the plane with $\mathcal{H}^1(E) = 1$ such that $\mathcal{L}^1(f(E)) = 0$ for every Lipschitz function $f\colon E \to \mathbb{R}$, see [274] for a simplification. Modifying an idea of Konyagin, Ambrosio and Kirchheim [16] showed that for any $m > 0$, not necessarily an integer, there exists a compact metric space X such that $\mathcal{H}^m(X) = 1$ and $\mathcal{H}^m(f(X)) = 0$ for every Lipschitz map f from X into a Euclidean space.

Kun, Maleva and Máthé [283] proved that an analytic subset $A \subset \mathbb{R}^n$ is purely 1-unrectifiable if and only if for any compact subset F of A, $\mathcal{L}^1(f(F)) = 0$ for every local Lipschitz quotient map $f\colon F \to \mathbb{R}$. A Lipschitz function $f\colon F \to \mathbb{R}$ is a local Lipschitz quotient if there is $c > 0$ such that $f(F) \cap B(f(x), cr) \subset f(B(x,r))$ for all $x \in F, r > 0$.

7.7 Metric Tangents

In Theorem 7.9 we had a characterization of rectifiability in terms of approximate tangent planes, which are linear subspaces of the bigger Banach space. Bate [58] proved corresponding results approximating in the Gromov–Hausdorff distance by m-dimensional Banach spaces. Even more, he proved an analogue of Theorem 4.9 in metric spaces.

First we have to define the relevant concepts. The *Gromov–Hausdorff distance* $d_{GH}(X_1, X_2)$ between metric spaces $(X_i, d_i), i = 1, 2$ is the infimum of those $\varepsilon > 0$ for which there exists a metric space Z and isometric embedddings Z_1 and Z_2 of X_1 and X_2 into Z such that the Hausdorff distance $d_H(Z_1, Z_2) < \varepsilon$. Then $d_{GH}(X_1, X_2) = 0$ if and only if the completions of (X_1, d_1) and (X_2, d_2) are isometric. If $(X_i, d_i), i = 0, 1, 2, \dots$ are metric spaces and $x_i \in X_i$, we say that the sequence (X_i, d_i, x_i) of pointed metric spaces converges to (X_0, d_0, x_0) if for any $r > 0$ there is a sequence $r_i \to r$ such that $d_{GH}(B_{d_i}(x_i, r_i), B_{d_0}(x_0, r)) \to 0$ as $i \to \infty$. A *pointed metric measure space* (X, d, μ, x) consists of a metric space (X, d), a locally finite (bounded sets have finite measure) Borel measure μ on X and a distinguished point $x \in \operatorname{spt} \mu$.

Let (X, d) be a complete metric space. For $K \geq 1$ let $\operatorname{biLip}(K)$ be the set of metric spaces $Y = (\mathbb{R}^n, \rho)$ such that ρ is K-bi-Lipschitz equivalent to the Euclidean norm. Denote by $\operatorname{biLip}(K)^*$ the set of pointed metric measure spaces (X, d, μ, x) with $(X, d) \in \operatorname{biLip}(K)$. First a geometric version from [58]:

Theorem 7.12 *Let $E \subset X$ be $\mathcal{H}^m$ measurable with $\mathcal{H}^m(E) < \infty$. Suppose that for $\mathcal{H}^m$ almost all $x \in E$, $\Theta^m_*(E, x) > 0$ and there exists $K_x \geq 1$ such that for each $r > 0$ there exist $Y_r \in \operatorname{biLip}(K_x)$ and $E_r \subset E \cap B(x, r)$ for which*

$$\lim_{r \to 0} r^{-m} \mathcal{H}^m(E \cap B(x, r) \setminus E_r) = 0$$

and

$$\lim_{r \to 0} r^{-1} d_{GH}(Y_r \cap B(0, r), E_r) = 0.$$

Then E is m-rectifiable.

This result is new also in Euclidean spaces because to conclude rectifiability, it allows much more general approximating sets than planes.

Next we define the tangent measure spaces. For a pointed metric measure space (X, d, μ, x) let $\mathrm{Tan}(X, d, \mu, x)$ be the set of all pointed metric measure spaces (Y, ρ, ν, y) such that there exist $r_i > 0$ for which $r_i \to 0$ and

$$\left(X, r_i^{-1} d, \mu(B(x, r_i))^{-1}\mu, x\right) \to (Y, \rho, \nu, y). \tag{7.7}$$

Finding a metric that is suitable to define this convergence is a bit of a delicate matter, since for the standard metric measure space convergence there may not be any tangent measure spaces. Standard meaning that one uses the Hausdorff distance for the spaces Z, as in the definition of d_{GH}, and a distance metrizing the weak convergence for the push-forwards of the measures. Instead, Bate defined a metric d_* that only considers the distance between the measures and disregards the Hausdorff distance between the embedded metric spaces.

Theorem 7.13 *Let $E \subset X$ be $\mathcal{H}^m$ measurable with $\mathcal{H}^m(E) < \infty$ and with $\Theta^m_*(E, x) > 0$ for $\mathcal{H}^m$ almost all $x \in E$. Then the following are equivalent:*

(1) *E is m-rectifiable.*
(2) *For $\mathcal{H}^m$ almost all $x \in E$ there exists an m-dimensional Banach space $(\mathbb{R}^m, \|\cdot\|_x)$ such that*

$$\mathrm{Tan}(X, d, \mathcal{H}^m \llcorner E, x) = \{(\mathbb{R}^m, \|\cdot\|_x, \mathcal{L}^m, 0)\}.$$

(3) *For $\mathcal{H}^m$ almost all $x \in E$ there exists $K_x \geq 1$ such that*

$$\mathrm{Tan}(X, d, \mathcal{H}^m \llcorner E, x) \subset \mathrm{biLip}(K_x)^*.$$

That (1) implies (2) uses Kirchheim's results in [276] discussed above. The main and hardest arguments involve the proofs of Theorem 7.12 and the implication (3) ⇒ (1) in Theorem 7.13. They use Bate's Lipschitz projection Theorem 7.11. Recall that the proof of the Euclidean Theorem 4.9 also used projections. Another basic tool needed consists of approximation of E with continuous images of cubes in $\mathbb{R}^m$. Its formulation is rather complicated, so I only give Bate's own informal description from his introduction. Roughly speaking, it shows that, under the hypotheses of Theorem 7.12, for $\mathcal{H}^m$ almost all $x \in E$, the following is true: for any $\varepsilon > 0$ and any sufficiently small $r > 0$, there exists a metric space $\tilde{E}$ containing E and a continuous (in fact Hölder) map $\iota\colon [0, r]^m \to \tilde{E}$ such that $\iota|\partial[0, r]^m$ is close to having Lipschitz inverse and $\mathcal{H}^m_\infty(\iota([0,r]^m) \setminus E) < \varepsilon r^m$. Here $\mathcal{H}^m_\infty$ denotes the m-dimensional Hausdorff content.

To finish the proofs Bate showed that as a consequence of Theorem 7.11 E is rectifiable if all of its subsets possess this approximation property. To get some (not quite correct) idea how Theorem 7.11 is applied, imagine that for some $x \in E, r > 0$, with $\mathcal{H}^m(E \cap B(x,r)) \sim r^m$ we would have $\iota([0,r]^m) = E \cap B(x,r)$ for ι as above. Extending, if possible, $(\iota|\partial[0,r]^m)^{-1}$ to a Lipschitz map f from $E \cap B(x,r)$ onto $[0,r]^m$ we would then have $\mathcal{L}^m(f(E \cap B(x,r))) \sim r^m$. By some topology the same would be true for small perturbations of f. If E were purely unrectifiable this would contradict Theorem 7.11.

In addition, for the proof of Theorem 7.13 Bate extensively developed the properties of the metric d_* and the metric tangent measure spaces.

7.8 Menger Curvature

The curvature of a smooth plane curve C at a point p can be defined as the limit of the reciprocals of the largest radii of the circles approaching and only touching C at p. The obstruction for not being able to make the radii bigger comes from triples of points on the curve near p. So the curvature at p is also the limit of $c(x,y,z)$, where $x,y,z \in C$ go to p and $c(x,y,z) = 1/R$, with R equal to the radius of the circle passing through x,y and z. If we take three points in a metric space X, there is an isometric triple in the plane. Thus we can define the *Menger curvature* $c(x,y,z)$ for any $x,y,z \in X$. It is not necessary to pass through the plane because there is a formula that gives $c(x,y,z)$ in terms of the three distances, see [229], [230] or [336].

Menger introduced his curvature in [336] as a part of extensive studies of geometry of metric spaces. In particular, he characterized in terms of this curvature the metric arcs which are isometric to Euclidean line segments. An interested reader might also wish to have a look at the book [68].

We already discussed this topic in Section 3.4. In [229], Hahlomaa proved a variant of Jones's Theorem 3.16 in general metric spaces. Now $\beta(F)$ is defined in terms of the Menger curvature. In [230], he generalized the David–Léger Theorem 3.18 to metric spaces:

Theorem 7.14 *If $\mathcal{H}^1(X) < \infty$ and*

$$\int_X \int_X \int_X c(x,y,z)^2 \, d\mathcal{H}^1 x \, d\mathcal{H}^1 y \, d\mathcal{H}^1 z < \infty,$$

then X is 1-rectifiable.

The proof follows lines similar to David's original proof in the plane, but much also has to be changed.