LINEAR TRANSFORMATIONS ON MATRICES: THE INVARIANCE OF A CLASS OF GENERAL MATRIX FUNCTIONS

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1. Introduction. Let $M_m(F)$ be the vector space of m-square matrices

$$X = (x_{ij}), \qquad x_{ij} \in F; i, j = 1, \ldots, m,$$

where F is a field; let f be a function on $M_m(F)$ to some set R. It is of interest to determine the linear maps T: $M_m(F) \to M_m(F)$ which preserve the values of the function f; i.e., $f(T(X)) = f(X)$ for all X. For example, if we take $f(X)$ to be the rank of X, we are asking for a determination of the types of linear operations on matrices that preserve rank. Other classical invariants that may be taken for f are the determinant, the set of eigenvalues, and the rth elementary symmetric function of the eigenvalues. Dieudonné **(1)**, Hua **(2)**, Jacobs **(3)**, Marcus **(4, 6, 8)**, Morita **(9)**, and Moyls **(6)** have conducted extensive research in this area. A class of matrix functions that have recently aroused considerable interest **(4; 7)** is the generalized matrix functions in the sense of I. Schur **(10)**. These are defined as follows: let S_m be the full symmetric group of degree m and let λ be a function on S_m with values in F. The matrix function associated with λ is defined by

$$d_\lambda(X) = \sum_{\sigma \in S_m} \lambda(\sigma) \prod_{i=1}^{m} x_{i\sigma(i)}.$$

These functions clearly include the classical determinant, permanent **(5)**, and imminent functions **(11)**.

Let C be a transitive cyclic subgroup of S_m and suppose $\lambda: S_m \to F$ is such that $\lambda(\sigma) = 0$ if $\sigma \notin C$. Our main result is a characterization of all linear maps $T: M_m(F) \to M_m(F)$ that satisfy

$$(1) \qquad d_\lambda(T(X)) = d_\lambda(X) \qquad \text{for all } X \in M_m(F).$$

The results are first established for the case when C is the group generated by the cycle π given by

$$\pi(i) \equiv i + 1, \qquad \pi^k(i) \equiv i + k \pmod m, \quad k \text{ integer},$$

and the function λ is identically equal to 1 on C. We then extend the results to other transitive cyclic subgroups and other functions by showing that it is possible to convert one matrix function uniformly into another by a linear

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transformation. We assume throughout that the field F contains more than m elements, where m is the size of the matrices under consideration, and that $m \geqslant 3$.

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2. Definitions and main results. Let π be the cycle defined before, let C be the cyclic subgroup of S_m generated by π, and let $\lambda: S_m \to F$ be defined by

$$\lambda(\sigma) = 1 \quad \text{if } \sigma \in C, \qquad \lambda(\sigma) = 0 \quad \text{if } \sigma \notin C.$$

We denote the generalized matrix function associated with λ by d. Clearly we may write

$$d(X) = \sum_{k=1}^{m} \prod_{i=1}^{m} x_{i\pi^k(i)}.$$

Definition. If M is a subspace of $M_m(F)$, then

(a) M is a 0-subspace if $\dim M = m^2 - m$ and $d(X) = 0$ for all $X \in M$.

(b) M is of type α, where $\alpha = (\alpha_1, \ldots, \alpha_m)$ is an ordered m-tuple of integers $1 \leqslant \alpha_i \leqslant m$, if $\dim M = m^2 - m$ and the $(\alpha_k, \pi^k(\alpha_k))$ entry of every $X \in M$ is zero for $k = 1, \ldots, m$.

The following characterization of the 0-subspaces of $M_m(F)$ turns out to be very useful in the determination of the structure of the set of linear maps of $M_m(F)$ into itself satisfying (1).

THEOREM 1. *Any 0-subspace M of $M_m(F)$ is of type α for some unique sequence α.*

If $P = (p_{ij})$ is the permutation matrix corresponding to the cycle π (i.e. $p_{ij} = \delta_{i\pi(j)}$), then $X \in M_m(F)$ can be uniquely written

$$X = \sum_{i=1}^{m} X_i P^i$$

where the X_i are diagonal matrices. We use this representation in defining the following linear maps of $M_m(F)$ into itself. Set $X_i = \operatorname{diag}(x_{i1}, \ldots, x_{im})$, and define three classes of maps by:

(a) If $\sigma \in S_m$, then

$$T(\sigma)(X) = \sum_{i=1}^{m} X_i P^{\sigma(i)}.$$

(b) If $\tau \in S_m$ and $1 \leqslant k \leqslant m$, then

$$S_k(\tau)(X) = \sum_{i=1}^{m} X'_i P^i$$

where $X'_i = X_i$ for $i \neq k$ and $X'_k = \operatorname{diag}(x_{k\tau(1)}, \ldots, x_{k\tau(m)})$.

(c) If $1 \leqslant k \leqslant m$ and $a_i \in F$ are such that

$$\prod_{i=1}^{m} a_i = 1,$$

then

$$M_k(a_1, \ldots, a_m)(X) = \sum_{i=1}^{m} X''_i P^i$$

where $X''_i = X_i$ for $i \neq k$ and $X''_k = \operatorname{diag}(a_1 x_{k1}, \ldots, a_m x_{km})$.

It is clear that each of the above types of linear transformations $T(\sigma)$, $S_k(\sigma)$, and $M_k(a_1, \ldots, a_m)$ is non-singular. We let G be the (multiplicative) subgroup of $\mathrm{GL}(m^2)$ (the group of non-singular linear maps of $M_m(F)$ into itself) generated by the above three types.

If C is a transitive cyclic subgroup of S_m and $\lambda: S_m \to F$ is such that $\lambda(\sigma) = 0$ for $\sigma \notin C$, let $N = \{\sigma \in C: \lambda(\sigma) = 0\}$. We define a linear map A_N of $M_m(F)$ into itself by

$$A_N(E_{ij}) = \begin{cases} 0 & \text{if } j = \sigma(i) \text{ for some } \sigma \in N, \\ E_{ij} & \text{otherwise,} \end{cases}$$

and extend A_N linearly. Here E_{ij} is the m-square matrix with a 1 in the (i, j) position and zeros elsewhere.

Let I be the identity map of $M_m(F)$ into itself. We can now state our main result.

THEOREM 2. *Let C be a transitive cyclic subgroup of S_m and $\lambda: S_m \to F$ be such that $\lambda(\sigma) = 0$ if $\sigma \notin C$. Let d_λ be the generalized matrix function associated with λ. There exists a non-singular linear transformation R of $M_m(F)$ onto itself such that a linear map T of $M_m(F)$ into itself satisfies*

$$d_\lambda(T(X)) = d_\lambda(X) \qquad \textit{for all } X$$

if and only if $A_N(T - I) + I \in R^{-1}GR$.

A specific formula for R will be given in the next section. It should be noted that R is independent of the map T but is not unique.

3. Proofs. Recall that d is the generalized matrix function associated with the function $\lambda: S_m \to F$ defined by $\lambda(\sigma) = 1$ if σ belongs to the cyclic group generated by π and $\lambda(\sigma) = 0$ otherwise. Here π is defined by

$$\pi(i) \equiv i + 1 \pmod{m}.$$

We let P be this permutation matrix corresponding to π and say a matrix K is a k-diagonal matrix if $K = DP^k$ for some diagonal matrix D. We now prove some lemmas needed to prove Theorem 1.

LEMMA 1. *If M is a 0-subspace, then M contains a non-zero k-diagonal matrix for each $k = 1, \ldots, m$.*

Proof. It is enough to show that M contains a diagonal matrix, for then we can apply this result to the 0-subspace MP^k. Suppose M does not contain a non-zero diagonal matrix. Let D be the subspace of diagonal matrices and M_k the subspace formed by adjoining P^k to M. It is easy to check that $d(P^k) = 1$; therefore $P^k \notin M$ and $\dim M_k = m^2 - m + 1$. Then $D \cap M_k \neq 0$, for $\dim D = m$ and a simple dimension argument yields the result. Let D_k be a non-zero element of $D \cap M_k$. We may assume (by multiplying by a suitable constant) that $D_k = M_k - P^k$ where $M_k \in M$. Let $D_k = \text{diag}(d_{k1}, \ldots, d_{km})$. Then

$$d(M_k) = d(D_k + P^k) = \prod_{i=1}^{m} d_{ki} + 1 \qquad \text{for } k \neq m.$$

Recall that M is a 0-subspace, so we must have

$$\prod_{i=1}^{m} d_{ki} = -1.$$

Hence no d_{ki} is equal to zero for $k \neq m$. Let T be the subspace generated by adjoining the matrix E_{n1} to D. Then $\dim T = m + 1$ and a dimension argument again shows that $T \cap M \neq 0$. Let

$$B = \sum_{i=1}^{m} b_i E_{ii} + cE_{n1}$$

be a non-zero matrix in $T \cap M$. Then $M_1 + zB \in M$ for all $z \in F$ since M_1 and B belong to the subspace M. Now

$$d(M_1 + zB) = \prod_{i=1}^{m} (d_{1i} + zb_i) + 1$$

and it is evident that this is equal to zero if and only if $b_i = 0$ for all i. Hence $B = cE_{n1}$. Since B and M_{m-1} belong to M, $M_{m-1} + zB$ belongs to M for all $z \in F$. Now

$$d(M_{m-1} + zB) = \prod_{i=1}^{m} d_{m-1,i} + (1 + cz) = cz.$$

This, however, implies that $c = 0$, for $M_{m-1} + zBM$. Hence $B = 0$, a contradiction.

LEMMA 2. *If M is a 0-subspace and $X \in M$, then for each $k = 1, \ldots, m$ there exists a unique integer j_k, independent of X, such that the $(j_k, \pi^k(j_k))$ entry of X is zero.*

Proof. We first show that for any $X \in M$ the ordered set

$$D(X, k) = \{x_{1,\sigma(1)}, \ldots, x_{m,\sigma(m)} : \sigma = \pi^k\}$$

(i.e., the kth diagonal of X) contains at least one zero. For some $X \in M$ and some k assume that $0 \notin D(X, k)$. By Lemma 1, let $K \in M$ be a non-zero

k-diagonal matrix. Let $K = DP^k$ where $D = \text{diag}(d_1, \ldots, d_m)$ and suppose that $d_t \neq 0$. Then $Z = X - x_{t,\pi^k(t)}/d_t K$ belongs to M and since $K_{ij} = d_i\,\delta_{i,\pi^k(j)}$,

$$d(Z) = \sum_{\substack{j=1\\ j\neq t}}^{m} \prod_{i=1}^{m} x_{i,\pi^j(i)} = -\prod_{i=1}^{m} x_{i,\pi^t(i)}.$$

Hence $d(Z) \neq 0$ since $0 \notin D(X, k)$, a contradiction.

We now show that the position of the zero in the set $D(X, k)$ is independent of X. Suppose that this is not the case. Then for some integer k, $1 \leqslant k \leqslant m$, there exist m matrices $X^{(1)}, \ldots, X^{(m)}$ belonging to M such that

$$x_{i,\pi}{}^{(i)}{}_{k(i)} \neq 0 \qquad \text{for } i = 1, \ldots, m.$$

A standard argument shows that we can choose $c_i \in F$ $(i = 1, \ldots, m)$ such that

$$d_i = \sum_{t=1}^{m} c_t\, x_{i,\pi}{}^{(t)}{}_{k(i)} \neq 0, \qquad i = 1, \ldots, m.$$

Define

$$Y = \sum_{t=1}^{m} c_t X^{(t)}.$$

Then $Y \in M$ and $0 \notin D(Y, k) = \{d_1, \ldots, d_m\}$. This, however, contradicts the fact that $0 \in D(Y, k)$ if $Y \in M$.

To see that the integer j_k is unique, note that the above shows that the subspace M consists of matrices that have zeros in at least m fixed positions, with at least one in each diagonal. It is not hard to see that if there was more than one zero in some diagonal, then dim M would be less than $m^2 - m$, a contradiction.

We now prove Theorem 1 by simply taking the sequence α to be $(j_1, \ldots, j_m)$.

Let T be a linear transformation of $M_m(F)$ into itself satisfying (1) where C is the group generated by the permutation defined by $\pi(i) \equiv i + 1 \pmod m$ and the function λ is identically equal to 1 on C and 0 off C. Let d be the generalized matrix function associated with C and λ.

LEMMA 3. *The linear transformation T is non-singular.*

Proof. Suppose T is singular. Then $T(A) = 0$ for some $A \neq 0$. Therefore

$$d(X - A) = d(T(X - A)) = d(T(X) - T(A)) = d(T(X)) = d(X)$$

for all X. If $A = (a_{ij})$, then $a_{ij} \neq 0$ for some i, j. The group C is transitive, so there exists an integer k such that $\pi^k(i) = j$. Set

$$B = \sum_{\substack{r=1\\ r\neq k}}^{m} \prod_{t=1}^{m} a_{t\pi^r(t)}.$$

Then

$$d(A) = \prod_{t=1}^{m} a_{t\pi^k(t)} + B = 0,$$

since $0 = d(0) = d(T(A)) = d(A)$.

We consider two cases:

(1) $$\prod_{t=1}^{m} a_{t\pi^k(t)} = -B = 0.$$

Let $X = a_{ij}P^k$; then $d(X) = a_{ij}^m \neq 0$. On the other hand,

$$d(X - A) = \prod_{t=1}^{m} (a_{t\pi^k(t)} - a_{ij}) + B = 0$$

since $a_{i\pi^k(i)} = a_{ij}$.

(2) $$\prod_{t=1}^{m} a_{t\pi^k(t)} = -B \neq 0.$$

Let $X = a_{ij}E_{ij}$; then $d(X) = 0$. However, we also have

$$d(X - A) = \prod_{t=1}^{m} (\delta_{j\pi^k(t)} a_{ij} - a_{t\pi^k(t)}) + B = 0 + B \neq 0$$

since $\delta_{j\pi^k(i)} a_{ij} - a_{i\pi^k(i)} = 0$. Hence we have $d(X) \neq d(X - A)$, a contradiction.

Let $M_i(M^j)$ be the subspace of $M_m(F)$ consisting of all matrices with row i (column j) zero. Clearly M_i and M^j are 0-subspaces. Let $R_i = T(M_i)$ and $R^j = T(M^j)$. Then R_i and R^j are 0-subspaces; for, by Lemma 3, T is non-singular and so preserves dimension and T preserves the values of the matrix function d by assumption. Applying Theorem 1, we may conclude that R_i is of type $\beta_{(i)} = (\beta_{i1}, \ldots, \beta_{im})$ and R^j is of type $\beta^{(j)} = (\beta_1{}^j, \ldots, \beta_m{}^j)$ for some unique sequences $\beta_{(i)}$ and $\beta^{(j)}$.

In order to determine the structure of T, it is convenient to let $X = (x_{ij})$ be a matrix of m^2 indeterminates. If we do this, we can consider $T(X)$ as a matrix of m^2 linear forms, $L(i, j)$, where

$$L(i,j) = \sum_{r=1}^{m} \sum_{s=1}^{m} t(i, j, r, s)x_{rs}, \qquad t(i, j, r, s) \in F.$$

We now use the fact that $R_i(R^j)$ is of type $\beta_{(i)}(\beta^{(j)})$ to determine the coefficients $t(i, j, r, s)$ in each linear form $L(i, j)$. Clearly once we have done this, the structure of the linear map T will be known.

Lemma 4. *Each linear form $L(i, j)$ involves only one indeterminate* (i.e. $L(i, j) = c_{rs} x_{rs}$ for some r, s) *and different linear forms involve different indeterminates.*

Proof. Consider $L(\beta_{ik}, \pi^k(\beta_{ik}))$. If $x_{i1} = \ldots = x_{im} = 0$, then $L(\beta_{ik}, \pi^k(\beta_{ik})) = 0$ because R_i is of type $(\beta_{i1}, \ldots, \beta_{im})$. Hence $t(\beta_{ik}, \pi^k(\beta_{ik}), r, s) = 0$ if $r \neq i$. A similar argument shows that $L(\beta_j{}^k, \pi^k(\beta_j{}^k)) = 0$ if $x_{1j} = \ldots = x_{mj} = 0$.

Now notice that if $i \neq j$, then $\beta_{it} \neq \beta_{jt}$ for any t. To see this, suppose that $\beta_{it} = \beta_{jt}$ for some t and some $i \neq j$. The argument above shows that $L(\beta_{it},$

$\pi^k(\beta_{it}))$ involves only the indeterminates $x_{i1}, \ldots, x_{im}$. But we have assumed that $L(\beta_{it}, \pi^t(\beta_{it})) = L(\beta_{jt}, \pi^t(\beta_{jt}))$; hence $L(\beta_{it}, \pi^t(\beta_{it}))$ involves only the indeterminates $x_{j1}, \ldots, x_{jm}$. Hence $L(\beta_{it}, \pi^t(\beta_{it})) = 0$ since

$$\{x_{i1}, \ldots, x_{im}\} \cap \{x_{j1}, \ldots, x_{jm}\} = \emptyset.$$

This, however, implies that T is singular, contradicting Lemma 3. We may now conclude that for each $i, t = 1, \ldots, m$ there exists an integer r such that $\beta_{rt} = i$; for we have shown that $\beta_{rt} \neq \beta_{st}$ for $r \neq s$ and $1 \leqslant \beta_{uv} \leqslant m$ by definition. The group C is transitive; hence, we can choose an integer t such that $\pi^t(i) = j$. Then the above arguments show that $L(i, j) = L(\beta_{rt}, \pi^t(\beta_{rt}))$ involves only the indeterminates $x_{r1}, \ldots, x_{rm}$. Similarly $L(i, j)$ involves only the indeterminates $x_{1s}, \ldots, x_{ms}$ for some integer s. Now

$$\{x_{r1}, \ldots, x_{rm}\} \cap \{x_{1s}, \ldots, x_{ms}\} = x_{rs},$$

so $L(i, j)$ involves only the indeterminate x_{rs}.

If two different linear forms involved the same indeterminate, then, since there are m^2 linear forms and m^2 indeterminates, some indeterminate, say x_{uv}, would not appear in any linear form. Then T is singular for $T(E_{uv}) = 0$, a contradiction.

Let G be the subgroup of $\mathrm{GL}(m^2)$ defined in §2. We now prove a special case of Theorem 2.

LEMMA 5. *A linear transformation T of $M_m(F)$ into itself satisfies $d(T(X)) = d(X)$ for all X if and only if $T \in G$.*

Proof. First note that if

$$X = (x_{ij}) \in M_m(F) \quad \text{and} \quad X_i = \operatorname{diag}(x_{1,\pi^i(1)}, \ldots, x_{m,\pi^i(m)})$$

then

$$X = \sum_{i=1}^{m} X_i P^i$$

and this representation is unique.

If $x_1, \ldots, x_m$ are indeterminates and $X = \operatorname{diag}(x_1, \ldots, x_m)P^k$, then, by Lemma 4, $T(X)$ has precisely m non-zero entries. Further,

$$d(X) = \prod_{i=1}^{m} x_i = d(T(X));$$

hence the non-zero entries in $T(X)$ must lie in a k-diagonal for some k so $T(X)$ is a k-diagonal matrix. Let $T(X) = \operatorname{diag}(L_1, \ldots, L_m)P^k$ where the L_i are linear forms in the indeterminates $x_1, \ldots, x_m$. By Lemma 4, $L_i = a_i x_{\sigma(i)}$ for some permutation $\sigma \in S_m$. Hence

$$d(T(X)) = \prod_{i=1}^{m} a_i x_{\sigma(i)} = \prod_{i=1}^{m} a_i \prod_{i=1}^{m} x_i = \prod_{i=1}^{m} x_i = d(X)$$

and we must have

$$\prod_{i=1}^{m} a_i = 1.$$

If D_1 and D_2 are diagonal matrices and $i \neq j$, then we have shown that $T(D_1 P^i) = D'_i P^r$ and $T(D_2 P^j) = D'_2 P^s$ for diagonal matrices D'_1 and D'_2. In addition, we can conclude from the non-singularity of T that $r \neq s$. Therefore, using the linearity of T, if

$$X = \sum_{i=1}^{m} X_i P^i, \qquad \text{then } T(X) = \sum_{i=1}^{m} X'_i P^{\sigma(i)}$$

where $\sigma \in S_m$; and if $X_i = \operatorname{diag}(x_{i1}, \ldots, x_{im})$, then

$$X'_1 = \operatorname{diag}(a_{i1} x_{i\tau(1)}, \ldots, a_{im} x_{i\tau(m)})$$

for a permutation $\tau = \tau_i \in S_m$, and the a_{ij} satisfy

$$\prod_{j=1}^{m} a_{ij} = 1.$$

Now notice that if $M_k(a_1, \ldots, a_m)$ and $S_k(\tau)$ are the linear transformations defined in §2 and D is a diagonal matrix, then $S_k(\tau)(DP^i) = DP^i$ and $M_k(a_1, \ldots, a_m)(DP^i) = DP^i$ if $i \neq j$. This is an immediate consequence of the fact that these two transformations only affect the k-diagonal of the matrix on which they operate.

Finally, let

$$S = T(\sigma) \prod_{i=1}^{m} M_i(a_{i1}, \ldots, a_{im}) S_i(\tau_i).$$

Then $S \in G$ and a straightforward computation shows that $S(X) = T(X)$ for all X.

Now let C be any transitive cyclic subgroup of S_m. It is well known that C consists of all powers of a cycle σ of length m, and we call σ a generator of C. We use this fact in the following preliminary version of Theorem 2.

LEMMA 6. *Let C be any transitive cyclic subgroup of S_m, σ a generator of C, and λ a function on S_m to F such that $\lambda(\tau) = 0$ if $\tau \notin C$ and $\lambda(\tau) \neq 0$ if $\tau \in C$. Let d_λ be the generalized matrix function associated with λ. There exists a non-singular linear transformation R of $M_m(F)$ onto itself such that a linear transformation T satisfies*

(2) $$d_\lambda(T(X)) = d_\lambda(X) \qquad \textit{for all } X$$

if and only if $R^{-1}TR \in G$.

Proof. It is well known that if two permutations $\alpha, \beta \in S_m$ have the same cycle structure, then there exists a permutation $\mu \in S_m$ such that $\mu\alpha\mu^{-1} = \beta$. By the above remarks the permutations π and σ have the same cycle structure.

Let $\phi \in S_m$ be such that $\phi\sigma\phi^{-1} = \pi$. Define the map R by

$$R(X) = P(\phi^{-1})M(X)P(\phi)$$

where $P(\phi)$ and $P(\phi^{-1})$ are the permutation matrices corresponding to ϕ and ϕ^{-1} and

$$M = \prod_{k=1}^{m} M_k((\lambda(\sigma^k))^{-1}, 1, \ldots, 1).$$

A straightforward computation shows that

$$d_\lambda(R(X)) = d(X) \qquad \text{for all } X$$

and

$$d(R^{-1}(X)) = d_\lambda(X) \qquad \text{for all } X.$$

The map R is clearly linear and non-singular.

If T is a linear transformation of $M_m(F)$ into itself satisfying (2), then

$$d(R^{-1}TR(X)) = d_\lambda(TR(X)) = d_\lambda(R(X)) = d(X).$$

Hence, by Lemma 5, $R^{-1}TR \in G$.

We now remove the restriction that the values of the function λ must be non-zero on the group C. Recall that $N = \{\sigma \in C: \lambda(\sigma) = 0\}$ and

$$A_N(E_{ij}) = \begin{cases} 0 & \text{if } j = \sigma(i) \text{ for some } \sigma \in N, \\ E_{ij} & \text{otherwise.} \end{cases}$$

It is easy to check that if $X = (x_{ij}) \in M_m(F)$, then as τ runs over C, the ordered sets (the diagonals),

$$D(X, \tau) = \{x_{1,\tau(1)}, \ldots, x_{m,\tau(m)}: \tau \in C\}$$

form a partition of the elements of the matrix X. We define $\bar{\lambda}: S \to F$ by $\bar{\lambda}(\tau) = 1$ if $\tau \in N$ and $\bar{\lambda}(\tau) = \lambda(\tau)$ otherwise. Let $\bar{d}_\lambda$ be the generalized matrix function associated with $\bar{\lambda}$.

LEMMA 7. *Let $T: M_m(F) \to M_m(F)$ be a linear map satisfying $d(T(X)) = d(X)$ for all X and let $S = A_N(T - I) + I$ where I is the identity transformation. Then $\bar{d}_\lambda(S(X)) = \bar{d}_\lambda(X)$ for all X.*

Proof. For any matrix $X \in M_m(F)$ we have

$$\bar{d}_\lambda(X) = \sum_{\sigma \in N} \bar{\lambda}(\sigma) \prod_{i=1}^{m} x_{i\sigma(1)} + \sum_{\sigma \notin N} \bar{\lambda}(\sigma) \prod_{i=1}^{m} x_{i\sigma(i)} = \sum_{\sigma \in N} \prod_{i=1}^{m} x_{i\sigma(i)} + d_\lambda(X).$$

$$(A_N T(X) - A_N(X) + X)_{i\sigma(i)} = \begin{cases} x_{i\sigma(i)} & \text{if } \sigma \in N, \\ T(X)_{i\sigma(i)} & \text{if } \sigma \notin N. \end{cases}$$

Hence

$$\bar{d}_\lambda(S(X)) = \sum_{\sigma \in N} \prod_{i=1}^{m} x_{i\sigma(i)} + d_\lambda(T(X)) = \bar{d}_\lambda(X).$$

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It is clear that the function $\bar{d}_\lambda$ and the map $S = A_N(T - I) + I$ satisfy the hypotheses of Lemma 6, so there exists a non-singular linear transformation R of $M_m(F)$ into itself, independent of T, such that $A_N(T - I) + I \in R^{-1}GR$. This completes the proof of Theorem 2.

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