Bull. Aust. Math. Soc. **101** (2020), 294–298
doi:10.1017/S0004972719000777

A q-ANALOGUE OF A HYPERGEOMETRIC CONGRUENCE

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(Received 4 June 2019; accepted 9 June 2019; first published online 18 July 2019)

Abstract

We give a q-analogue of the following congruence: for any odd prime p,

$$\sum_{k=0}^{(p-1)/2}(-1)^k(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 8^k}\sum_{j=1}^{k}\bigg(\frac{1}{(2j-1)^2}-\frac{1}{16j^2}\bigg)\equiv 0 \pmod p,$$

which was originally conjectured by Long and later proved by Swisher. This confirms a conjecture of the second author ['A q-analogue of the (L.2) supercongruence of Van Hamme', *J. Math. Anal. Appl.* **466** (2018), 749–761].

2010 *Mathematics subject classification*: primary 33D15; secondary 11A07, 11F33.

Keywords and phrases: basic hypergeometric series, q-congruence, congruence, cyclotomic polynomial.

1. Introduction

In 1914, Ramanujan [11] obtained a number of fast approximations of $1/\pi$. The following equation, which is a special case of a ${}_4F_3$ summation formula of Gosper [1], is not in the list of [11], but gives such an example:

$$\sum_{k=0}^{\infty}(-1)^k(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 8^k}=\frac{2\sqrt{2}}{\pi}. \tag{1.1}$$

Here $(a)_n = a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol. In 1997, Van Hamme [14] proposed 13 amazing p-adic analogues of Ramanujan-type formulas, such as

$$\sum_{k=0}^{(p-1)/2}(-1)^k(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 8^k}\equiv p\Big(\frac{-2}{p}\Big) \pmod{p^3}, \tag{1.2}$$

where p is an odd prime and $(\frac{\cdot}{p})$ denotes the Legendre symbol modulo p. Van Hamme's supercongruence (1.2) was first proved by Swisher [13]. We point out that the last supercongruence of Van Hamme was proved by Osburn and Zudilin [9]

The second author was partially supported by the National Natural Science Foundation of China (grant 11771175).

in 2016. There have been many studies of q-analogues of such supercongruences in recent years (see, for example, [2–5, 7, 12]).

Swisher [13] also deduced the following interesting congruence from (1.2): for any odd prime p,

$$\sum_{k=0}^{(p-1)/2}(-1)^k(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 8^k}\sum_{j=1}^{k}\Big(\frac{1}{(2j-1)^2}-\frac{1}{16j^2}\Big)\equiv 0 \pmod{p}. \tag{1.3}$$

This congruence was conjectured by Long [8]. In this note we shall prove the following q-analogue of (1.3), which was originally observed by the second author [2, Conjecture 4.4].

THEOREM 1.1. *Let n be a positive odd integer. Then*

$$\sum_{k=0}^{(n-1)/2}(-1)^k[6k+1]\frac{(q;q^2)_k^3}{(q^4;q^4)_k^3}\sum_{j=1}^{k}\Big(\frac{q^{2j-1}}{[2j-1]^2}-\frac{q^{4j}}{[4j]^2}\Big)\equiv 0 \pmod{\Phi_n(q)}. \tag{1.4}$$

Here and throughout the paper, we adopt the standard q-notation:

$$(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$$

is the *q-shifted factorial*, $[n]=1+q+\cdots+q^{n-1}$ is the *q-integer* and $\Phi_n(q)$ stands for the nth *cyclotomic polynomial* in q, which may be defined as

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(n,k)=1}}(q-\zeta^k),$$

where ζ is an nth primitive root of unity.

2. Proof of Theorem 1.1

We first give the following lemma, which is a q-analogue of [8, Lemma 4.4].

LEMMA 2.1. *Let n be a positive odd integer. Then*

$$\sum_{k=0}^{(n-1)/2}(-1)^k[6k+1]\frac{(q;q^2)_k(q^{1+n};q^2)_k(q^{1-n};q^2)_k}{(q^4;q^4)_k(q^{4+n};q^4)_k(q^{4-n};q^4)_k}=[n](-q)^{-(n-1)(n+5)/8}. \tag{2.1}$$

PROOF. The second author and Zudilin [6] gave a q-analogue of a formula for $1/\pi$ by using the following formula of Rahman [10, (4.6)]:

$$\sum_{k=0}^{\infty}\frac{(a;q)_k(1-aq^{3k})(d;q)_k(q/d;q)_k(b;q^2)_k}{(q^2;q^2)_k(1-a)(aq^2/d;q^2)_k(adq;q^2)_k(aq/b;q)_k}\frac{a^kq^{\binom{k+1}{2}}}{b^k}$$
$$=\frac{(aq;q^2)_\infty(aq^2;q^2)_\infty(adq/b;q^2)_\infty(aq^2/bd;q^2)_\infty}{(aq/b;q^2)_\infty(aq^2/b;q^2)_\infty(aq^2/d;q^2)_\infty(adq;q^2)_\infty}. \tag{2.2}$$

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Letting $q\to q^2$ and $b\to\infty$ in (2.2) and then taking $a=q$ and $d=aq$,

$$\sum_{k=0}^{\infty}(-1)^k[6k+1]\frac{(aq;q^2)_k(q/a;q^2)_k(q;q^2)_kq^{3k^2}}{(aq^4;q^4)_k(q^4/a;q^4)_k(q^4;q^4)_k}=\frac{(q^3;q^4)_\infty(q^5;q^4)_\infty}{(aq^4;q^4)_\infty(q^4/a;q^4)_\infty},$$

which for $a=q^n$ gives

$$\sum_{k=0}^{(n-1)/2}(-1)^k[6k+1]\frac{(q;q^2)_k(q^{1+n};q^2)_k(q^{1-n};q^2)_k}{(q^4;q^4)_k(q^{4+n};q^4)_k(q^{4-n};q^4)_k}q^{3k^2}=(-1)^{(n-1)(n+5)/8}[n]q^{(n-1)(n-3)/8}.$$

Replacing q by q^{-1}, we obtain (2.1). □

Proof of Theorem 1.1. Using (2.2), the second author and Zudilin (see [7, Theorem 4.4] with $a\to 1$) proved that

$$\sum_{k=0}^{(n-1)/2}(-1)^k[6k+1]\frac{(q;q^2)_k^3}{(q^4;q^4)_k^3}\equiv[n](-q)^{-(n-1)(n+5)/8}\pmod{[n]\Phi_n(q)^2},\tag{2.3}$$

which was first conjectured in [2, Conjecture 1.1]. Consider the difference of the left-hand side and the right-hand side of (2.3). By (2.1),

$$\begin{aligned}
&\sum_{k=0}^{(n-1)/2}(-1)^k[6k+1]\frac{(q;q^2)_k^3}{(q^4;q^4)_k^3}-[n](-q)^{-(n-1)(n+5)/8}\\
&=\sum_{k=0}^{(n-1)/2}(-1)^k[6k+1]\frac{(q;q^2)_k}{(q^4;q^4)_k}\bigg(\frac{(q;q^2)_k^2}{(q^4;q^4)_k^2}-\frac{(q^{1+n};q^2)_k(q^{1-n};q^2)_k}{(q^{4+n};q^4)_k(q^{4-n};q^4)_k}\bigg)\\
&=\sum_{k=0}^{(n-1)/2}(-1)^k[6k+1]\frac{(q;q^2)_k}{(q^4;q^4)_k}\\
&\quad\times\frac{(q;q^2)_k^2(q^{4+n};q^4)_k(q^{4-n};q^4)_k-(q^4;q^4)_k^2(q^{1+n};q^2)_k(q^{1-n};q^2)_k}{(q^4;q^4)_k^2(q^{4+n};q^4)_k(q^{4-n};q^4)_k}.
\end{aligned}$$

Noticing that

$$(1-q^{a+n+dj})(1-q^{a-n+dj})=(1-q^{a+dj})^2-(1-q^n)^2q^{a+dj-n}$$

and $1-q^n\equiv 0\pmod{\Phi_n(q)}$,

$$\begin{aligned}
(q^{4+n};q^4)_k(q^{4-n};q^4)_k&=\prod_{j=1}^{k}(1-q^{n+4j})(1-q^{-n+4j})\\
&=\prod_{j=1}^{k}((1-q^{4j})^2-(1-q^n)^2q^{4j-n})\\
&\equiv(q^4;q^4)_k^2-(q^4;q^4)_k^2\sum_{j=1}^{k}\frac{(1-q^n)^2}{(1-q^{4j})^2}q^{4j-n}\pmod{\Phi_n(q)^4},
\end{aligned}$$

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since the remaining terms are multiples of $(1-q^n)^4$. Similarly,

$$(q^{1+n};q^2)_k(q^{1-n};q^2)_k \equiv (q^2;q^2)_k^2 - (q^2;q^2)_k^2 \sum_{j=1}^{k} \frac{(1-q^n)^2}{(1-q^{2j-1})^2} q^{2j-n-1} \pmod{\Phi_n(q)^4}.$$

It follows that

$$\begin{aligned}&(q;q^2)_k^2(q^{4+n};q^4)_k(q^{4-n};q^4)_k - (q^4;q^4)_k^2(q^{1+n};q^2)_k(q^{1-n};q^2)_k\\ &\equiv (q;q^2)_k^2(q^4;q^4)_k^2[n]^2 \sum_{j=1}^{k}\Big(\frac{q^{2j-n-1}}{[2j-1]^2} - \frac{q^{4j-n}}{[4j]^2}\Big) \pmod{\Phi_n(q)^4}.\end{aligned}$$

From (2.3),

$$\sum_{k=0}^{(n-1)/2} \frac{(-1)^k[6k+1](q;q^2)_k^3}{(q^4;q^4)_k(q^{4+n};q^4)_k(q^{4-n};q^4)_k} \sum_{j=1}^{k}\Big(\frac{q^{2j-n-1}}{[2j-1]^2} - \frac{q^{4j-n}}{[4j]^2}\Big) \equiv 0 \pmod{\Phi_n(q)},$$

which is equivalent to the desired congruence (1.4) by observing that $q^n \equiv 1 \pmod{\Phi_n(q)}$.

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