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PARTIAL REGULARITY AND EVERYWHERE CONTINUITY FOR A MODEL PROBLEM FROM NON-LINEAR ELASTICITY

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Abstract

We prove a new energy or Caccioppoli type estimate for minimisers of the model functional $\int_\Omega |Du|^2 + (\det Du)^2$, where $\Omega \subset \mathbb{R}^2$ and $u : \Omega \to \mathbb{R}^2$. We apply this to establish C^∞ regularity for minimisers except on a closed set of measure zero. We also prove a maximum principle and use this to establish everywhere continuity of minimisers.

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1. Introduction

Since the first paper by Evans [4] on the subject appeared, partial regularity of minimisers (that is, smoothness except on a closed set of measure zero) of functionals of the form

$$I[u] = \int_\Omega F(Du)dx, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $u : \Omega \to \mathbb{R}^N$ and F is quasi-convex, has been extensively studied (see Acerbi and Fusco [1], Evans and Gariepy [5], Fusco and Hutchinson [6], Giaquinta and Modica [9]).

Recently in [7] the authors have studied the partial regularity of minimisers of certain polyconvex functionals (that is functionals which are convex in the various minors of the matrix $[Du]$). Polyconvex functionals are quasiconvex and arise in non-linear elasticity (Ball [2, 3]), and all known 'natural' examples of quasiconvex functionals are polyconvex.

A model problem covered by the results in Fusco and Hutchinson [7] is given by $N = n = 2$ and

$$I[u] = \int_\Omega |Du|^2 + (\det Du)^2. \tag{1.2}$$

The first term in the integrand is the usual Dirichlet energy (for line elements); the second term is the Dirichlet energy for area elements (compare Fusco and Hutchinson [7]). There is clearly no loss of generality in having the same coefficient in front of each term, as follows by a scaling argument.

We show in particular that mininimisers of this functional are smooth except on a closed set of measure zero, and are everywhere continuous.

Although simple, this example already contains all the difficulties which appear in handling the general case to which the main theorem in Fusco and Hutchinson [7] applies. Firstly, the set of competing functions, those for which the integral is finite, is not a linear subspace of $W^{1,2}$. Secondly, the term $(\det Du)^2$ can grow, in certain directions, like a fourth power of Du, which one has no way of controlling by using the leading term $|Du|^2$ appearing in $I[u]$.

These two difficulties were overcome in Fusco and Hutchinson [7]. There we used the standard technique of blowing up a minimiser u of I around a convergent sequence of points in order to obtain a sequence v_m of functions which converges weakly in $W^{1,2}$ to the solution v of a linear elliptic system with constant coefficients. Then the crucial fact that we proved in order to obtain the decay Lemma 3.1 and hence the partial regularity result Theorem 3.2 was that the functions v_m converge strongly in $W^{1,2}$ to v.

The first main result in this paper is an energy estimate of Caccioppoli type which leads to an alternative proof of the decay estimate Lemma 3.1. Namely, we prove that if $u \in W^{1,2}$ is a minimizer of I then for any ball $B_R \subset\subset \Omega$

$$\int_{B_{R/2}} |Du|^2 + (\det Du)^2 \le \frac{c}{R^2} \int_{B_R} |u - u_R|^2 + \frac{c}{R^2} \left(\int_{B_R} |Du|^2 \right)^2. \tag{1.3}$$

This estimate is proved using the comparison function (2.9) introduced in Fusco and Hutchinson [7]. Once one has (1.3), the proof of the Lemma 3.1 becomes simpler than in [7].

In order to make clear the ideas and techniques used in the proof of partial regularity, we restrict ourselves to the model case $I[u]$. To prove Theorem 3.2 in the general case one should use the appropriate version of the estimate (1.3) containing all the higher order minors of the matrix $[Du]$. This requires only technical complications, but no new ideas.

Finally, we remark that the blow-up argument used here and in [7] works because the leading term in $I[u]$ is quadratic and there is no degeneracy in ellipticity. It would not work if we had something like

$$I[u] = \int_\Omega |Du|^p + (\det Du)^p, \tag{1.4}$$

with $p > 2$, or its n-dimensional version. However, the extension of Theorem 3.2 to functionals having a degenerate elliptic leading term will be treated in a forthcoming paper by the two authors.

The second main result in this paper is a proof of everywhere continuity of minimisers of the functional $I[u]$. This uses an earlier unpublished maximum principle result of the authors, and of Leonetti [10, 11]. We give here a simple geometric proof. Everywhere continuity follows from an application of the Courant-Lebesgue lemma. We remark that this result does not extend to the general class of problems considered in [7]. The second author would like to thank Michael Grüter and Ulrich Dierkes for pointing out the relevance of the Courant-Lebesgue lemma in this setting.

2. The Energy Estimate

If $\Omega \subset \mathbb{R}^2$ is a bounded open set and λ is any real number, we consider for $v \in W^{1,2}(\Omega; \mathbb{R}^2)$ the functional

$$I_\lambda[v, \Omega] = \int_\Omega |Dv|^2 + \lambda^2 (\det Dv)^2. \tag{2.1}$$

If $Q \geq 1$ is a real number we say that $u \in W^{1,2}(\Omega; \mathbb{R}^2)$ is a *Q-minimiser* of I_λ if, for any $\phi \in W^{1,2}(\Omega; \mathbb{R}^2)$ whose support is a compact subset of Ω,

$$I_\lambda(u) \leq Q I_\lambda(u + \phi). \tag{2.2}$$

If $Q = 1$ then u is a *minimiser*.

In the following we write

$$⨍_E g = \frac{1}{|E|} \int_E g = g_E \tag{2.3}$$

for any measurable set E of positive measure and any $g \in L^1(E; \mathbb{R}^k)$. If E is the open ball $B_R(x_0)$ we also write

$$⨍_{B_R(x_0)} g = (g)_{x_0,R} = g_R. \tag{2.4}$$

The key ingredient in obtaining the decay Lemma 3.1 is the following energy estimate for a Q-minimiser of I_λ.

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LEMMA 2.1. *If $u \in W^{1,2}(\Omega;\mathbb{R}^2)$ is a Q-minimiser of I_λ, then there exists a constant c depending only on Q such that for any $B_R \subset\subset \Omega$*

$$\int_{B_{R/2}} |Du|^2 + \lambda^2(\det Du)^2 \leq \frac{c}{R^2}\int_{B_R} |u-u_R|^2 + \frac{c\lambda^2}{R^2}\left(\int_{B_R}|Du|^2\right)^2. \tag{2.5}$$

PROOF. Fix $B_R \subset\subset \Omega$ and $R/2 < s < t < R$. Define

$$E_{s,t} = \left\{\rho \in (s,t) : \int_{\partial B_\rho} |Du|^2 \leq \frac{2}{t-s}\int_{B_t\setminus B_s} |Du|^2\right\}. \tag{2.6}$$

Then

$$|E_{s,t}| \geq \frac{t-s}{2}. \tag{2.7}$$

Let $\omega = x/|x|$. For a.e. $\rho \in (s,t)$ the function $\omega \mapsto u(\rho\omega)$ belongs to $W^{1,2}(\partial B_1)$ and

$$|u(\rho\omega) - u_{\partial B_\rho}| \leq c\sqrt{\rho}\left(\int_{\partial B_\rho}|Du|^2\right)^{1/2} \tag{2.8}$$

for all ω (taking the continuous representative of $\omega \mapsto u(\rho\omega)$), where $u_{\partial B_\rho} = 1/(2\pi\rho)\int_{\partial B_\rho} u$.

For each ρ we introduce the comparison function

$$\phi(r\omega) = \begin{cases} u_{\partial B_\rho} & \text{if } r \leq s;\\ \dfrac{\rho - r}{\rho - s}u_{\partial B_\rho} + \dfrac{r-s}{\rho-s}u(\rho\omega) & \text{if } s \leq r \leq \rho;\\ u(r\omega) & \text{if } \rho \leq r \leq R.\end{cases} \tag{2.9}$$

It easily follows that

$$|D\phi(r\omega)| \leq c\left(\frac{|u(\rho\omega) - u_{\partial B_\rho}|}{\rho - s} + |Du(\rho\omega)|\right) \tag{2.10}$$

and

$$|\det D\phi(r\omega)| \leq c\frac{|u(\rho\omega) - u_{\partial B_\rho}|}{\rho - s}|Du(\rho\omega)| \tag{2.11}$$

for any $r \in [s,\rho]$ (recall $r/2 < s < \rho < R$).

From (2.8), (2.10), (2.11) and the Q-minimising property of u, it follows that for a.e. $\rho \in E_{s,t}$

$$\begin{aligned}\int_{B_\rho} |Du|^2 + \lambda^2(\det Du)^2 &\leq Q\int_{B_\rho\setminus B_s} |D\phi|^2 + \lambda^2(\det D\phi)^2\\ &\leq \frac{c}{\rho - s}\int_{\partial B_\rho}|u - u_{\partial B_\rho}|^2 + c(\rho - s)\int_{\partial B_\rho}|Du|^2 + \frac{c\lambda^2}{\rho - s}\int_{\partial B_\rho}|u-u_{\partial B_\rho}|^2|Du|^2\\ &\leq \frac{c}{\rho-s}\int_{\partial B_\rho}|u-u_R|^2 + c(\rho - s)\int_{\partial B_\rho}|Du|^2 + \frac{c\rho\lambda^2}{\rho - s}\left(\int_{\partial B_\rho}|Du|^2\right)^2. \end{aligned} \tag{2.12}$$

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From (2.7) it follows that

$$\left|\left\{\rho \in E_{s,t} : \rho - s \geq (t-s)/4\right\}\right| \geq (t-s)/4. \tag{2.13}$$

Multiplying (2.12) by $\rho - s$ and integrating with respect to ρ over $E_{s,t}$, and again using (2.7) and the definition of $E_{s,t}$, we see that

$$\begin{aligned}&\frac{(t-s)^2}{16}\int_{B_s}|Du|^2+\lambda^2(\det\, Du)^2\\ &\leq c\int_{B_t\setminus B_s}|u-u_R|^2+c(t-s)^2\int_{B_t\setminus B_s}|Du|^2+\frac{cR\lambda^2}{t-s}\left(\int_{B_t\setminus B_s}|Du|^2\right)^2.\end{aligned} \tag{2.14}$$

Dividing through by $(t-s)^2$ and using the 'hole-filling' trick, (that is adding $c\int_{B_s}|Du|^2$ $+c\int_{B_s}\lambda^2(\det\, Du)^2$ to both sides of the inequality) it follows there exists $\theta \in (0, 1)$ such that

$$\begin{aligned}I_\lambda(u;B_s) \leq \theta I_\lambda(u;B_t) &+ \frac{c}{(t-s)^2}\int_{B_R\setminus B_{R/2}}|u-u_R|^2\\ &+\frac{cR\lambda^2}{(t-s)^3}\left(\int_{B_R\setminus B_{R/2}}|Du|^2\right)^2.\end{aligned} \tag{2.15}$$

The Lemma now follows from a straightforward extension of Giaquinta [8, Chapter 5, Lemma 3.1].

3. Partial Regularity

The essential tool in obtaining the partial regularity result on Theorem 3.2 is the following decay estimate for the quantity

$$U(x,r) = \fint_{B_r(x)}|Du-(Du)_r|^2+(\det(Du-(Du)_r))^2. \tag{3.1}$$

LEMMA 3.1. *Let $u \in W^{1,2}(\Omega;\mathbb{R}^2)$ be a minimiser of the functional $I[\cdot]$. Suppose $M > 0$. Then for some constant $c(M)$ and any $\tau \in (0, 1/2)$ there exists $\epsilon(\tau, M)$ such that if*

$$|(Du)_{x,r}| \leq M \quad \text{and} \quad U(x,r) < \epsilon \tag{3.2}$$

then

$$U(x,\tau r) \leq c(M)\tau^2 U(x,r). \tag{3.3}$$

PROOF. Fix M. Arguing by way of contradiction we assume that there exists a sequence of balls $B_{r_m} \subset\subset \Omega$ for which

$$|(Du)_{x_m,r_m}| \leq M \quad \text{and} \quad \lambda_m^2 = U(x_m, r_m) \to 0 \tag{3.4}$$

but

$$\frac{U(x_m, \tau r_m)}{\lambda_m^2} > c(M)\tau^2, \tag{3.5}$$

where $c(M)$ will be chosen later.

We set $A_m = (Du)_{x_m,r_m}$ and

$$v_m(y) = \frac{u(x_m + r_m y) - (u)_{x_m,r_m} - r_m A_m y}{\lambda_m r_m} \tag{3.6}$$

for all $y \in B_1(0)$.

Then

$$\int_{B_1} |Dv_m|^2 + \lambda_m^2 (\det Dv_m)^2 = 1 \tag{3.7}$$

and $(v_m)_{0,1} = 0$. So we may suppose on passing to a subsequence that

$$\begin{array}{rcll} Dv_m & \to & Dv & \text{weakly in } L^2(B_1), \\ v_m & \to & v & \text{strongly in } L^2(B_1), \\ \lambda_m \det Dv_m & \to & 0 & \text{weakly in } L^2(B_1), \\ A_m & \to & A. & \end{array} \tag{3.8}$$

The third claim in (3.8) comes from observing that $\lambda_m \det Dv_m$ converges weakly in $L^2(B_1)$, but also for any $\phi \in C_0^1(B_1)$

$$\int \det Dv_m \phi \to \int \det Dv\phi, \tag{3.9}$$

as one can check by writing $\det Dv_m$ in divergence form.

It is also immediate to check that v_m minimises the functional

$$w \mapsto \int_{B_1} \lambda_m^2 |Dw|^2 + (\det(\lambda_m Dw + A_m))^2. \tag{3.10}$$

We introduce the bilinear form $\odot$ defined by

$$\det(A + B) = \det A + A \odot B + \det B \tag{3.11}$$

where $A, B \in \operatorname{Hom}(\mathbb{R}^2, \mathbb{R}^2)$.

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Expanding out the second term on the right side of (3.10) and expressing $\det(\lambda_m Dw + A_m)$ in divergence form, it follows that v_m minimises the functional

$$w \mapsto \int_{B_1} |Dw|^2 + (A_m \odot Dw + \lambda_m \det Dw)^2 . \tag{3.12}$$

Hence v_m satisfies the Euler-Lagrange system

$$0 = \int_{B_1} Dv_m D\phi + (A_m \odot Dv_m + \lambda_m \det Dv_m)(A_m \odot D\phi + \lambda_m Dv_m \odot D\phi) \tag{3.13}$$

for all $\phi \in W_0^{1,2}(B_1; \mathbb{R}^2)$. Letting $m \to \infty$ and using (3.8) we obtain

$$0 = \int DvD\phi + (A \odot Dv)(A \odot D\phi). \tag{3.14}$$

Thus v satisfies a linear elliptic system with constant coefficients.

By standard regularity results (see for example Giaquinta [8, Chapter III]) we have for any $\tau \in (0, 1)$ that

$$⨍_{B_\tau} |Dv - (Dv)_\tau|^2 \le c\tau^2 ⨍_{B_1} |Dv - (Dv)_\tau|^2 \le c\tau^2 \tag{3.15}$$

and

$$|(Dv)_{2\tau} - (Dv)_\tau|^2 \le c\tau^2, \tag{3.16}$$

where c depends only on the ellipticity constants of the system (3.14) and hence only on M. Notice that the last inequality in each case follows from (3.7) and (3.8).

On rescaling (3.1) we have for any $\tau \in (0, 1/2)$ that

$$\frac{U(x_m, \tau r_m)}{\lambda_m^2} = ⨍_{B_\tau} |Dv_m - (Dv_m)_\tau|^2 + \lambda_m^2 (\det(Dv_m - (Dv_m)_\tau))^2 \tag{3.17}$$

Setting

$$w_m(y) = v_m(y) - (Dv_m)_\tau y - (v_m)_{2\tau} \tag{3.18}$$

and using both the fact that v_m minimises the functional (3.12) and the divergence structure of 'det', we see that w_m minimises the functional

$$w \mapsto \int_{B_1} |Dw|^2 + ((A_m + \lambda_m (Dv_m)_\tau) \odot Dw + \lambda_m \det Dw)^2 . \tag{3.19}$$

Hence w_m is a Q-minimiser of $I\lambda_m[w; B_1]$ for some $Q = Q(M)$.

Then from Lemma 2.1 we obtain for $\tau \in (0, 1/2)$ that

$$\begin{aligned} \frac{U(x_m, \tau r_m)}{\lambda_m^2} &= \fint_{B_\tau} |Dw_m|^2 + \lambda_m^2 (\det Dw_m)^2 \\ &\le \frac{c}{\tau^2} \fint_{B_{2\tau}} |w_m - (w_m)_{2\tau}|^2 + c\lambda_m^2 \left(\fint_{B_{2\tau}} |Dw_m|^2 \right)^2 . \end{aligned} \tag{3.20}$$

Passing to the limit as $m \to \infty$ and using (3.8), the Poincaré inequality, (3.15) and (3.16), we obtain for $\tau \in (0, 1/2)$ that

$$\begin{aligned} \limsup_{m \to \infty} \frac{U(x_m, \tau r_m)}{\lambda_m^2} &\le \frac{c}{\tau^2} \fint_{B_{2\tau}} |v - (v)_{2\tau} - (Dv)_\tau y|^2 \\ &\le \frac{c}{\tau^2} \fint_{B_{2\tau}} |v - (v)_{2\tau} - (Dv)_{2\tau} y|^2 + |(Dv)_{2\tau} - (Dv)_\tau|^2 \\ &\le c \left(\fint_{B_{2\tau}} |Dv - (Dv)_{2\tau}|^2 \right) + c\tau^2 \\ &\le c_1(M)\tau^2 . \end{aligned} \tag{3.21}$$

This contradicts (3.5) if $c(M)$ is chosen larger than $c_1(M)$.

A standard iteration and bootstrapping argument ([7, Lemma 6.1] or [8, Chapter VI]) implies:

THEOREM 3.2. *Let $u \in W^{1,2}_{loc}(\Omega; \mathbb{R}^2)$ be a local minimiser of the functional $I[\cdot]$. Then $u \in C^{1,\infty}(\Omega_0)$, where Ω_0 is an open set such that $|\Omega \backslash \Omega_0| = 0$. Moreover*

$$\Omega_0 = \left\{ x \in \Omega : \limsup_{r \to 0} |(Du)_{x,r}| < \infty, \quad \lim_{r \to 0} \fint_{B_r(x)} |Du - (Du)_{x,r}|^2 = 0, \right. \\ \left. \text{and} \quad \lim_{r \to 0} \fint_{B_r(x)} \left(\det(Du - (Du)_{x,r}) \right)^2 = 0 \right\} .$$

4. Everywhere Continuity

We first establish a maximum principle using a geometric-type argument. See Leonetti [10, 11] for another proof.

Suppose that

$$u : \Omega (\subset \mathbb{R}^n) \to \mathbb{R}^N \tag{4.1}$$

is a minimiser of a functional of the form

$$I[u] = \int_\Omega F(|Du|, |\Lambda_2 Du|, \dots, |\Lambda_n Du|). \tag{4.2}$$

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Here $|\Lambda_k Du(x)|$ is the Euclidean norm of the map

$$\Lambda_k Du(x) : \Lambda_k \mathbb{R}^n \to \Lambda_k \mathbb{R}^N, \tag{4.3}$$

given by

$$\Lambda_k Du(x)(w_1 \wedge \cdots \wedge w_k) = Du(x)(w_1) \wedge \cdots \wedge Du(x)(w_k), \tag{4.4}$$

and

$$|\Lambda_k Du(x)|^2 = \sum_{i_1 < \cdots < i_k} |\Lambda_k Du(x)(\tau_{i_1} \wedge \cdots \wedge \tau_{i_k})|^2 \tag{4.5}$$

for any orthonormal basis $\tau_1, \dots, \tau_k$ of $\mathbb{R}^n$.

Note that in case $n = N = 2$, then $\det Du(x)$ can be regarded as the map $\Lambda_2 Du(x)$ operating on 2-vectors, and $|\Lambda_2 Du(x)|^2 = (\det Du(x))^2$.

Assume moreover that $F = F(p_1, p_2, \dots, p_n)$ is *convex* in each of the arguments p_k separately, and that F is *monotone* in the sense that

$$F(q_1, q_2, \dots, q_n) < F(p_1, p_2, \dots, p_n) \tag{4.6}$$

provided $q_1 < p_1$ and $q_k \le p_k$ for $k = 2, \dots, n$. Notice that the model problem (1.1) is of this type.

We then have the following result.

THEOREM 4.1. *Let $u \in W^{1,N}(\Omega; \mathbb{R}^2)$ be a minimiser of the functional $I[\cdot]$, where I is as in* (4.2) *and is convex and monotone. Let $E \subset\subset \Omega$ be an open set with Lipschitz boundary ∂E. Then $u(E) \subset \mathcal{C}(u(\partial E))$, where $\mathcal{C}(u(\partial E))$ is the closed convex hull of $u(\partial E)$.*

PROOF. For any closed ball $B_R = B_R(y_0) \subset \mathbb{R}^N$ let $\psi : \mathbb{R}^N \to B_R$ be the radial projection map defined by

$$\psi(y) = \begin{cases} y, & if\ y \in B_R, \\ y_0 + R\dfrac{(y - y_0)}{|y - y_0|}, & y \in \mathbb{R}^N \setminus B_R. \end{cases} \tag{4.7}$$

Fix $y \in \mathbb{R}^N$, choose $\tau_1 = (y - y_0)/|y - y_0|$ and extend τ_1 to an orthonormal basis $\tau_1, \tau_2, \dots, \tau_N$ for $\mathbb{R}^N$. Then for any $y \in \mathbb{R}^N \setminus B_R$

$$D\psi(\tau_i) = \gamma_i \tau_i \tag{4.8}$$

where $|\gamma_i| < 1$. In particular, it follows that

$$|D\psi(y)(w)| < |w| \tag{4.9}$$

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for any $0 \neq w \in \mathbb{R}^N$.

More generally, it follows from (4.4) and (4.8) that if $y \in \mathbb{R}^N \backslash B_R$ then

$$\Lambda_k D\psi(y)(\tau_{i_1} \wedge \cdots \wedge \tau_{i_k}) = \gamma_{i_1} \ldots \gamma_{i_k}(\tau_{i_1} \wedge \cdots \wedge \tau_{i_k}), \tag{4.10}$$

and hence using the orthonormal basis $\tau_{i_1} \wedge \cdots \wedge \tau_{i_k}$ where $i_1 < \cdots < i_k$, that

$$|\Lambda_k D\psi(y)(w_1 \wedge \cdots \wedge w_k)| \leq |w_1 \wedge \cdots \wedge w_k| \tag{4.11}$$

for any k-vector $w_1 \wedge \cdots \wedge w_k \in \Lambda_k \mathbb{R}^N$.

Now suppose $\mathcal{C}(u(\partial E)) \subset B_R$ and define

$$v(x) = \begin{cases} u(x), & \text{if } x \notin E, \\ \psi(u(x)), & \text{if } x \in E. \end{cases} \tag{4.12}$$

Then $I[u] \leq I[v]$ by the minimising property of u.

On the other hand, $|\Lambda_k Dv(x)| = |\Lambda_k D\psi(u(x)) \circ \Lambda_k Du(x)|$ and so from (4.5) and (4.11),

$$|\Lambda_k Dv(x)| \leq |\Lambda_k Du(x)| \tag{4.13}$$

for any $x \in \Omega$ and $k = 1, \ldots, n$. Moreover, using (4.9), *strict* inequality holds in case $k = 1$ if $u(x) \in \mathbb{R}^N \backslash B_R$ and $Du(x) \neq 0$. If $E \subset\subset \Omega$ is open and $|u(E) \backslash \mathcal{C}(u(\partial E))| \neq 0$, then by the area formula there exists $E' \subset E$ with $|E'| \neq 0$, $|u(E') \backslash \mathcal{C}(u(\partial E))| \neq 0$ and $|Du(x)| \neq 0$ for $x \in E'$. By appropriate choice of B_R it follows from (4.6) that $I[v] < I[u]$, contradicting the minimising property of u.

We next recall an application of the Courant-Lebesgue Lemma.

PROPOSITION 4.2. *Suppose $u \in W^{1,2}(B_R; \mathbb{R}^2)$. Then there exists a sequence $\rho_n \to 0$ such that $\ell(u(\partial B_{\rho_n})) \to 0$, where $\ell(u(\partial B_{\rho_n}))$ is the length of $u(\partial B_{\rho_n})$.*

PROOF. First note that $\int_{\partial B_\rho} |Du|^2 < \infty$ for a.e. $0 < \rho < R$, and so $u|_{\partial B_\rho}$ is continuous for a.e. $0 < \rho < R$.

Let $M = \int_{B_R} |Du|^2$ and assume that $0 < \delta < \sqrt{\delta} < R$. We claim that

$$\int_{\partial B_\rho} \left| \frac{\partial u}{\partial \theta} \right|^2 d\theta \leq \frac{2M}{\log \frac{1}{\delta}} \tag{4.14}$$

for some $\rho \in (\delta, \sqrt{\delta})$.

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Suppose not. Then

$$
\begin{aligned}
M &\geq \int_{B_R} \frac{1}{\rho^2}\left|\frac{\partial u}{\partial \theta}\right|^2 \rho\, d\rho\, d\theta \\
&\geq \int_{\delta}^{\sqrt{\delta}} \frac{1}{\rho}\left(\int_{\partial B_\rho}\left|\frac{\partial u}{\partial \theta}\right|^2 d\theta\right) d\rho \\
&> \frac{2M}{\log\frac{1}{\delta}} \int_{\delta}^{\sqrt{\delta}} \frac{1}{\rho} d\rho \geq M. \qquad (4.15)
\end{aligned}
$$

This contradiction establishes (4.14).

It follows that

$$
\begin{aligned}
\ell(u(\partial B_\rho)) &= \int_{\partial B_\rho}\left|\frac{\partial u}{\partial \theta}\right| d\theta \\
&\leq \sqrt{2\pi}\left(\int_{\partial B_\rho}\left|\frac{\partial u}{\partial \theta}\right|^2 d\theta\right)^{1/2} \\
&\leq \sqrt{\frac{4\pi M}{\log\frac{1}{\delta}}}. \qquad (4.16)
\end{aligned}
$$

This establishes the Proposition.

Everywhere continuity of minimisers of (1.1) now follows from the previous two results.

THEOREM 4.3. *Suppose* $u \in W^{1,2}(\Omega; \mathbb{R}^2)$ *is a minimiser of* (1.1). *Then* u *has a continuous representative.*

PROOF. For each $x \in \Omega$ there exists by Proposition 4.2 a sequence $\rho_n \to 0$ such that $\ell(u(\partial B_{\rho_n}(x))) \to 0$. By the maximum principle Theorem 4.1, $u(B_{\rho_n}(x)) \subset C(u(\partial B_{\rho_n}(x)))$, and so $\operatorname{diam} u(B_{\rho_n}(x)) \to 0$.

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