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Convex Functions on Discrete Time Domains

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Abstract. In this paper, we introduce the definition of a convex real valued function f defined on the set of integers, $\mathbb{Z}$. We prove that f is convex on $\mathbb{Z}$ if and only if $\Delta^2 f \geq 0$ on $\mathbb{Z}$. As a first application of this new concept, we state and prove discrete Hermite–Hadamard inequality using the basics of discrete calculus (*i.e.,* the calculus on $\mathbb{Z}$). Second, we state and prove the discrete fractional Hermite–Hadamard inequality using the basics of discrete fractional calculus. We close the paper by defining the convexity of a real valued function on any time scale.

1 Introduction

The convexity property of a given function plays an important role in obtaining integral inequalities. Proving inequalities for convex functions has a long and rich history in mathematics. We refer the reader to a monograph written by Dragomir and Pearce [8]. Recently, there have been some published results (especially, Hermite–Hadamard integral inequalities) using fractional integral operators [10–12]. The Hermite–Hadamard inequality states that if $f: I \to \mathbb{R}$ is a convex function, then the following inequality is satisfied:

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\Big(\int_a^b f(t)dt\Big) \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $a, b \in I$ and I is an interval in $\mathbb{R}$. To date, there has not been any published paper for discrete or discrete fractional version of the Hermite–Hadamard inequality.

Our goal in this paper is to state and prove the Hermite–Hadamard inequality for the real valued functions defined on the set of integers. In order to achieve our goal, we first define a convex function defined on the set of integers. To characterize the convexity for discrete functions, we introduce a midpoint condition, and then we prove that the discrete function is convex if and only if its second difference is positive on $\mathbb{Z}$.

The plan of the rest of the paper is as follows: In Section 3, with the use of the substitution method in the time scale calculus, we state and prove the discrete Hermite–Hadamard inequality. In Section 4, we first recall the definitions of the nabla fractional

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sum operator and the right delta fractional sum operator. We then state and prove the discrete fractional Hermite–Hadamard inequality. In Section 5, we briefly talk about convexity of a function defined on any time scale.

For further reading on the discrete calculus and the discrete fractional calculus, we refer the reader to a book by Kelley and Peterson [9], and the papers [1,3,7].

2 Preliminaries

Let $\mathbb{Z}$ be the set of integers and $a, b \in \mathbb{Z}$ with $a < b$. By $[a,b]_{\mathbb{Z}}$ we mean $[a,b] \cap \mathbb{Z}$. We define

$$\mathbb{T}_{[a,b]} = \left\{ u \mid u = \frac{t-b}{a-b} \text{ for } t \in [a,b]_{\mathbb{Z}} \right\}.$$

We note that $\mathbb{T}_{[a,b]}$ is a subset of the real interval $[0,1]$. Any nonempty closed subset of the set of real numbers, $\mathbb{R}$, is called a *time scale*. Hence one can consider $\mathbb{T}_{[a,b]}$ as an isolated time scale on which all the points are left and right scattered at the same time. For further reading on time scales, we refer the reader to an excellent book on the analysis of time scales [4].

For a function $f: \mathbb{Z} \to \mathbb{R}$, the Δ and ∇ operators are defined as

$$\Delta f(t) = f(t+1) - f(t) \quad \text{and} \quad \nabla f(t) = f(t) - f(t-1),$$

for every $t \in \mathbb{Z}$, respectively. We also recall that $\Delta^2 f(t) = \Delta(\Delta f(t))$.

Let Γ denote the usual special gamma function and recall the notation that is known as the falling factorial power

$$t^{(\mu)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\mu)}.$$

Throughout, we assume that if $t+1-\mu \in \{0,-1,\dots,-k,\dots\}$, then $t^{(\mu)} = 0$.

Now we are in a position to define the convexity for a real valued function defined on the set of integers. Next, we define the midpoint condition for a discrete function.

Definition 2.1 $f: \mathbb{Z} \to \mathbb{R}$ is called convex on $\mathbb{Z}$ if for every $x, y \in \mathbb{Z}$ with $x < y$ the inequality $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$ is satisfied for all $\lambda \in \mathbb{T}_{[x,y]}$.

Definition 2.2 $f: \mathbb{Z} \to \mathbb{R}$ satisfies the midpoint condition if

$$f\Big(\frac{a+b}{2}\Big) \le \frac{f(a)+f(b)}{2}$$

for every $a, b \in \mathbb{Z}$ with $a+b$ is an even number.

The discrete Taylor's theorem plays an important role in the proof of the first result of this paper. For the reader's convenience we state the theorem here.

Theorem 2.3 (Discrete Taylor's Theorem [2]) *Let $u(k)$ be defined on $\mathbb{N}_a$. Then, for all $k \in \mathbb{N}_a$ and $n \ge 1$,*

$$u(k) = \sum_{i=0}^{n-1} \frac{(k-a)^{(i)}}{i!} \Delta^i u(a) + \frac{1}{(n-1)!} \sum_{l=a}^{k-n} (k-(l+1))^{(n-1)} \Delta^n u(l),$$

where $\mathbb{N}_a = \{a, a+1, \dots\}$.

Next we characterize the convexity of a discrete function by considering the sign of its second difference.

Theorem 2.4 *Let* $f : \mathbb{Z} \to \mathbb{R}$ *be given. The following are equivalent:*

(i) f *is convex on* $\mathbb{Z}$.
(ii) f *satisfies the midpoint condition.*
(iii) $\Delta^2 f(t) \geq 0$ *for all* $t \in \mathbb{Z}$.

Proof We prove that (i) ⇒ (ii) ⇒ (iii) ⇒ (i).

(i) ⇒ (ii): Let $a, b \in \mathbb{Z}$ with $a < b$ and where $a + b$ is an even number. This implies that $\frac{1}{2} \in \mathbb{T}_{[a,b]}$. Hence, we choose $\lambda = \frac{1}{2}$ to obtain the midpoint condition.

Next we prove that (ii) implies (iii). Let $t \in \mathbb{Z}$. Since f has midpoint condition, we have

$$f(t+1) = f\big(\tfrac{1}{2}t + \tfrac{1}{2}(t+2)\big) \leq \tfrac{1}{2}f(t) + \tfrac{1}{2}f(t+2).$$

This implies that $f(t+2) - 2f(t+1) + f(t) \geq 0$. Hence $\Delta^2 f(t) \geq 0$.

Next we prove that (iii) implies (i). Let $x, y \in \mathbb{Z}$ with $x < y$. We need to prove that

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

for all $\lambda \in \mathbb{T}_{[x,y]}$. Fix $\lambda \in \mathbb{T}_{[x,y]} \backslash \{0,1\}$. Define $x_0 = \lambda x + (1-\lambda)y$. Using the discrete Taylor's Theorem (Theorem 2.3) at x_0, we have

$$f(y) = \sum_{i=0}^{1} \frac{(y-x_0)^{(i)}}{(i)!} \Delta^i f(x_0) + \frac{1}{1!} \sum_{l=x_0}^{y-2} (y-(l+1))^{(1)} \Delta^2 f(l).$$

Since $\Delta^2 f(t) \geq 0$ on $\mathbb{Z}$ and by the Mean Value Theorem, we have

$$f(x) \geq f(x_0) + (x-x_0)\Delta f(x_0),$$
$$f(y) \geq f(x_0) + (y-x_0)\Delta f(x_0).$$

If we multiply the first inequality by λ and the second inequality by $1-\lambda$ and adding inequalities side by side, we obtain

$$f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda)f(y) \text{ for all } \lambda \in \mathbb{T}_{[x,y]}.$$ ∎

3 Discrete Hermite–Hadamard Inequality

In this section, we prove the discrete Hermite–Hadamard inequality for convex functions defined on $\mathbb{Z}$. In the statement of the inequality, we use the notations of time scales calculus:

$$\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t) \quad \text{and} \quad \int_a^b f(t)\nabla t = \sum_{t=a+1}^{b} f(t).$$

The following substitution rule plays an important role in the proof.

Theorem 3.1 (Substitution rule on time scales [6]) *Assume $\nu:\mathbb{T}\to\mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}}:\nu(\mathbb{T})$ is a time scale. If $f:\mathbb{T}\to\mathbb{R}$ is a an rd-continuous function and ν is differentiable with rd-continuous derivative, then if $a,b\in\mathbb{T}$,*

$$\int_a^b f(t)\nu^{\Delta}(t)\Delta t = \int_{\nu(a)}^{\nu(b)} (f\circ\nu^{-1})(s)\widetilde{\Delta}s$$

or

$$\int_a^b f(t)\nu^{\nabla}(t)\nabla t = \int_{\nu(a)}^{\nu(b)} (f\circ\nu^{-1})(s)\widetilde{\nabla}s.$$

Theorem 3.2 *Suppose $f:\mathbb{Z}\to\mathbb{R}$ is a convex function on $[a,b]_{\mathbb{Z}}$ with $a,b\in\mathbb{Z}$, $a<b$, and $a+b$ an even number. Then*

$$f\Big(\frac{a+b}{2}\Big)\le\frac{1}{2(b-a)}\Big[\int_a^b f(t)\Delta t+\int_a^b f(t)\nabla t\Big]\le\frac{f(a)+f(b)}{2}. \tag{3.1}$$

Proof Fix $t\in\mathbb{T}_{[a,b]}\setminus\{0,1\}$. We define

$$x=ta+(1-t)b,\qquad y=(1-t)a+tb.$$

It is easy to see that $x,y\in[a,b]_{\mathbb{Z}}$ and $x+y$ is even. Hence, $\frac{1}{2}\in\mathbb{T}_{[x,y]}$ (or $\mathbb{T}_{[y,x]}$). Since f is convex on $[x,y]_{\mathbb{Z}}$ (or $[y,x]_{\mathbb{Z}}$), we have

$$f\Big(\frac{x+y}{2}\Big)\le\frac{f(x)+f(y)}{2}.$$

This implies that

$$f\Big(\frac{a+b}{2}\Big)\le\frac{1}{2}\big[f(ta+(1-t)b)+f((1-t)a+tb)\big].$$

Next we integrate each side of the inequality over $\mathbb{T}_{[a,b]}$ and obtain

$$\int_{\mathbb{T}_{[a,b]}} f\Big(\frac{a+b}{2}\Big)\widetilde{\Delta}t\le\frac{1}{2}\Big[\int_{\mathbb{T}_{[a,b]}} f(ta+(1-t)b)\widetilde{\Delta}t+\int_{\mathbb{T}_{[a,b]}} f((1-t)a+tb)\widetilde{\Delta}t\Big],$$

where $\widetilde{\Delta}$ represents the derivative operator on the time scale $\mathbb{T}_{[a,b]}$.

Let us first closely look at

$$\int_{\mathbb{T}_{[a,b]}} f(tb+(1-t)a)\widetilde{\Delta}t.$$

Choose $\nu(t)=\frac{t-a}{b-a}$. Then we have $f(tb+(1-t)a)=(f\circ\nu^{-1})(t)$. We observe that ν is strictly increasing and $\nu([a,b]_{\mathbb{Z}})=\mathbb{T}_{[a,b]}$. By using the substitution method on time scales (Theorem 3.1), we have

$$\int_{\mathbb{T}_{[a,b]}} f(tb+(1-t)a)\widetilde{\Delta}t=\frac{1}{b-a}\int_{[a,b]_{\mathbb{Z}}} f(t)\Delta t,$$

since $\nu^{\Delta}(t)=\frac{1}{b-a}$.

Next we claim that

$$\int_{\mathbb{T}_{[a,b]}} f\big(ta+(1-t)b\big)\widetilde{\Delta}t=\frac{1}{b-a}\int_{[a,b]_{\mathbb{Z}}} f(t)\nabla t. \tag{3.2}$$

We prove our claim using the basics of dual time scales given in [5].

Define $v(t) = \frac{t-b}{a-b}$. Hence we rewrite the left-hand side of equality (3.2) as

$$\begin{aligned}(3.3)\qquad \int_{\mathbb{T}_{[a,b]}} f(ta+(1-t)b)\widetilde{\Delta}t &= \int_{\mathbb{T}_{[a,b]}} (f\circ v^{-1})(t)\widetilde{\Delta}t\\ &= \int_0^1 (f\circ v^{-1})(t)\widetilde{\Delta}t = \int_{-1}^0 (f\circ v^{-1})^*(s)\widetilde{\widetilde{\nabla}}s\\ &= \int_{-1=(u^{-1}\circ v)(a)}^{0=(u^{-1}\circ v)(b)} f((u^{-1}\circ v)^{-1})(s)\widetilde{\widetilde{\nabla}}s,\end{aligned}$$

where

$$(f\circ v^{-1})^*(s) = f\big(v^{-1}(-s)\big) = f\big((v^{-1}\circ\ u)(s)\big) = f\big((u^{-1}\circ v)^{-1}\big)(s)$$

and $u(s) = -s$.

Here we also have

$$(u^{-1}\circ v)(s) = u^{-1}(v(s)) = u^{-1}\Big(\frac{s-b}{a-b}\Big) = \frac{b-s}{a-b},$$

$(u^{-1}\circ v)^{\nabla}(s) = \frac{1}{b-a} > 0 \Rightarrow u^{-1}\circ v$ is strictly increasing.

Using the substitution method (Theorem 3.1) for (3.3), we obtain

$$\int_{-1}^0 f(u^{-1}\circ v)^{-1}(t)\widetilde{\widetilde{\nabla}}t = \int_a^b f(t)\frac{1}{b-a}\nabla t,$$

as desired.

To prove the other half of the inequality in (3.1), we use the convexity of the function and have the following inequalities:

$$\begin{aligned} f\big(ta+(1-t)b\big) &\le tf(a)+(1-t)f(b)\\ f\big((1-t)a+tb\big) &\le (1-t)f(a)+tf(b).\end{aligned}$$

Adding these two inequalities side by side, we obtain

$$f\big(ta+(1-t)b\big) + f\big((1-t)a+tb\big) \le f(a)+f(b).$$

Integrating each side over $\mathbb{T}_{[a,b]}$, we have

$$\int_{\mathbb{T}_{[a,b]}} \big(f(ta+(1-t)b) + f\big((1-t)a+tb\big)\big)\widetilde{\Delta}t \le \int_{\mathbb{T}_{[a,b]}} \big(f(a)+f(b)\big)\widetilde{\Delta}t.$$

Using arguments simiar to those used above, we can show that

$$\frac{1}{b-a}\Big(\int_a^b f(t)\Delta t + \int_a^b f(t)\nabla t\Big) \le f(a)+f(b). \qquad \blacksquare$$

4 Discrete Fractional Hermite–Hadamad Inequality

In this section we first present sufficient fundamental definitions and formulas so that the article is self-contained.

The ν−th order nabla fractional sum and the right fractional sum of f are defined, respectively, by

$$\nabla_a^{-\nu} f(t) = \sum_{s=a}^{t} \frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} f(s) \quad \text{and} \quad {}_b\Delta^{-\nu} f(t) = \sum_{s=t+\nu}^{b} \frac{(s-\sigma(t))^{(\nu-1)}}{\Gamma(\nu)} f(s),$$

where $\nu \in \mathbb{R}\backslash\{\dots,-2,-1,0\}$.

Let α be any real number. Then "t to the α rising" is defined to be

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)},$$

where $t \in \mathbb{R} \setminus \{\dots,-2,-1,0\}$, and $0^{\overline{\alpha}} = 0$.

Theorem 4.1 *Suppose $f:\mathbb{Z} \to \mathbb{R}$ is a convex function on $[a,b]_{\mathbb{Z}}$, where $a,b \in \mathbb{Z}$, $a < b$, and $a+b$ an even number. Then*

$$f\Big(\frac{a+b}{2}\Big) \le \frac{\Gamma(\alpha)}{2\beta(b-a)}\Big[{}_{b-1}\Delta^{-\alpha} f(t)\,|_{t=a-\alpha} + \nabla_{a+1}^{-\alpha} f(t)\,|_{t=b}\Big] \le \frac{f(a)+f(b)}{2}, \tag{4.1}$$

where

$$\beta = \int_{\mathbb{T}_{[a,b]}} \big((b-a)t + (\alpha-1)\big)^{\underline{\alpha-1}}\widetilde{\Delta}t$$

and α is a positive real number.

Proof Let $t \in \mathbb{T}_{[a,b]} \setminus \{0,1\}$. We define

$$x = ta + (1-t)b, \quad y = (1-t)a + tb.$$

This implies that $x,y \in [a,b]_{\mathbb{Z}}$ and $x+y$ is an even number. Since f is also convex on $[x,y]_{\mathbb{Z}}$ (or $[y,x]_{\mathbb{Z}}$), we have

$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x)+f(y)}{2}.$$

If we replace x and y in the inequality, we obtain

$$f\Big(\frac{a+b}{2}\Big) \le \frac{1}{2}\big[f(ta+(1-t)b) + f((1-t)a+tb)\big].$$

Multiplying each side by $((b-a)t+(\alpha-1))^{\underline{\alpha-1}}$ and integrating over $\mathbb{T}_{[a,b]}$, we have

$$\begin{aligned}
&\int_{\mathbb{T}_{[a,b]}} \big((b-a)t+(\alpha-1)\big)^{\underline{\alpha-1}} f\Big(\frac{a+b}{2}\Big)\widetilde{\Delta}t \\
&\quad = f\Big(\frac{a+b}{2}\Big)\int_{\mathbb{T}_{[a,b]}} \big((b-a)t+(\alpha-1)\big)^{\underline{\alpha-1}}\widetilde{\Delta}t \\
&\quad \le \frac{1}{2}\Big[\int_{\mathbb{T}_{[a,b]}} ((b-a)t+(\alpha-1))^{\underline{\alpha-1}} f(ta+(1-t)b)\widetilde{\Delta}t \\
&\qquad\qquad + \int_{\mathbb{T}_{[a,b]}} \big((b-a)t+(\alpha-1)\big)^{\underline{\alpha-1}} f\big((1-t)a+tb\big)\widetilde{\Delta}t\Big].
\end{aligned}$$

Next we claim that

$$\int_{\mathbb{T}_{[a,b]}} \big((b-a)t+(\alpha-1)\big)^{\underline{\alpha-1}} f\big((1-t)a+tb\big)\widetilde{\Delta}t = \frac{\Gamma(\alpha)}{b-a}\,{}_{b-1}\Delta^{-\alpha} f(t)\,|_{t=a-\alpha}.$$

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To prove our claim we define $g(t) = (t-a+(\alpha-1))^{\underline{\alpha-1}}$, $v(t) = \frac{t-a}{b-a}$ and $F(t) = g(t)f(t)$. Then we observe that

$$\begin{aligned} F\big(v^{-1}(t)\big) &= (gf)\big(v^{-1}(t)\big) = g\big(v^{-1}(t)\big)f\big(v^{-1}(t)\big) \\ &= \big(v^{-1}(t)-a+(\alpha-1)\big)^{\underline{\alpha-1}} f\big((1-t)a+tb\big) \\ &= \big((b-a)t+(\alpha-1)\big)^{\underline{\alpha-1}} f\big((1-t)a+tb\big). \end{aligned}$$

Next we use the substitution method (Theorem 3.3) for the integral. We have

$$\begin{aligned} &\int_{\mathbb{T}_{[a,b]}} \big((b-a)t+\alpha-1\big)^{\underline{\alpha-1}} f\big((1-t)a+tb\big)\widetilde{\Delta}t \\ &= \int_a^b F(t)v^{\Delta}(t)\Delta t = \frac{1}{b-a}\sum_{t=a}^{b-1}(t-a+\alpha-1)^{\underline{\alpha-1}}f(t) \\ &= \frac{\Gamma(\alpha)}{b-a}\, {}_{b-1}\Delta^{-\alpha}f(t)\,|_{t=a-\alpha}. \end{aligned}$$

This completes the proof of the claim.

Next we claim that

$$\int_{\mathbb{T}_{[a,b]}} \big((b-a)t+\alpha-1\big)^{\underline{\alpha-1}} f\big(ta+(1-t)b\big)\widetilde{\Delta}t = \frac{\Gamma(\alpha)}{b-a}\nabla_{a+1}^{-\alpha}f(t)\,|_{t=b}\,.$$

To prove this claim, we again use basic notation and results of dual time scales [5].

Indeed, we have

$$\begin{aligned} \int_{\mathbb{T}_{[a,b]}} \big((b-a)t+\alpha-1\big)^{\underline{\alpha-1}} f\big(ta+(1-t)b\big)\widetilde{\Delta}t &= \int_{\mathbb{T}_{[a,b]}} (F\circ v^{-1})(t)\widetilde{\Delta}t \\ &= \int_{-1}^{0} (F\circ v^{-1})^{*}(s)\widetilde{\nabla}s, \end{aligned}$$

where $F(t) = f(t)g(t)$, $g(t) = (b-t+(\alpha-1))^{\underline{\alpha-1}}$, $v(t) = \frac{t-b}{a-b}$, and $u(s) = -s$.

Here we have

$$(u^{-1}\circ v)(s) = u^{-1}(v(s)) = u^{-1}\Big(\frac{s-b}{a-b}\Big) = \frac{b-s}{a-b}.$$

This implies that $(u^{-1}\circ v)^{\nabla}(s) = \frac{1}{b-a} > 0 \Rightarrow u^{-1}\circ v$ is strictly increasing.

Using the substitution method (Theorem 3.3) for the last integral above, we have

$$\int_{-1}^{0}(F\circ v^{-1})^{*}(s)\nabla^{*}s = \int_{-1}^{0} F\big((u^{-1}\circ v)^{-1}\big)(s)\nabla^{*}s = \int_a^b F(t)\frac{1}{b-a}\nabla t.$$

Next, replacing $F(t)$ back in the integral, we have

$$\begin{aligned} \int_a^b F(t)\frac{1}{b-a}\nabla t &= \frac{1}{b-a}\sum_{s=a+1}^{b}(b-s+\alpha-1)^{\underline{\alpha-1}}f(s) \\ &= \frac{1}{b-a}\sum_{s=a+1}^{b}\big(b-\rho(s)\big)^{\overline{\alpha-1}}f(s) = \frac{\Gamma(\alpha)}{b-a}\nabla_{a+1}^{-\alpha}f(t)\,|_{t=b}, \end{aligned}$$

where we used the identity $(b-s+\alpha-1)^{\underline{\alpha-1}} = (b-s+1)^{\overline{\alpha-1}}$. This completes the proof of our second claim.

Hence, we obtain the following inequality:

$$f\Big(\frac{a+b}{2}\Big) \le \frac{\Gamma(\alpha)}{2\beta(b-a)}\Big[{}_{b-1}\Delta^{-\alpha} f(t)\,|_{t=a-\alpha} + \nabla_{a+1}^{-\alpha} f(t)\,|_{t=b}\Big].$$

To prove the other half of the inequality (4.1), we again use convexity of f on $[x,y]_{\mathbb{Z}}$. Hence we have

$$f(ta+(1-t)b) \le tf(a)+(1-t)f(b),$$
$$f((1-t)a+tb) \le (1-t)f(a)+tf(b).$$

Adding these two inequalities, we obtain

$$f\big(ta+(1-t)b\big) + f\big((1-t)a+tb\big) \le f(a)+f(b).$$

As before, we multiply each side of the inequality by $((b-a)t+(\alpha-1))^{\underline{\alpha-1}}$ and integrate over $\mathbb{T}_{[a,b]}$ to get

$$\begin{aligned}
&\int_{\mathbb{T}_{[a,b]}} \big((b-a)t+(\alpha-1)\big)^{\underline{\alpha-1}} f\big(ta+(1-t)b\big)\widetilde{\Delta}t \\
&\qquad + \int_{\mathbb{T}_{[a,b]}} \big((b-a)t+(\alpha-1)\big)^{\underline{\alpha-1}} f\big((1-t)a+tb\big)\widetilde{\Delta}t \\
&\le \int_{\mathbb{T}_{[a,b]}} \big((b-a)t+(\alpha-1)\big)^{\underline{\alpha-1}} \big(f(a)+f(b)\big)\widetilde{\Delta}t \\
&= \big(f(a)+f(b)\big)\int_{\mathbb{T}_{[a,b]}} \big((b-a)t+(\alpha-1)\big)^{\underline{\alpha-1}}\big)\widetilde{\Delta}t
\end{aligned}$$

Following the claims we proved above, we have

$$\frac{\Gamma(\alpha)}{2\beta(b-a)}\Big[{}_{b-1}\Delta^{-\alpha} f(t)\,|_{t=a-\alpha} + \nabla_{a+1}^{-\alpha} f(t)\,|_{t=b}\Big] \le \frac{f(a)+f(b)}{2}.$$

This completes the proof. ∎

We want to point out that if one chooses α as 1 in (4.1), then Theorem 3.2 becomes a corollary of Theorem 4.1. In Section 3, we gave the proof of the discrete Hermite–Hadamard inequality, since one might want to continue only in the direction of the discrete version of the inequality (1.1).

5 A Convex Function on a Time Scale

In this section, we extend and unify the definition of a convex function. Let $\mathbb{T}$ be any time scale and $a, b \in \mathbb{T}$ with $a < b$. By $[a,b]_{\mathbb{T}}$ we mean $[a,b] \cap \mathbb{T}$. We define

$$\mathbb{T}_{[a,b]} = \Big\{ u \mid u = \frac{t-b}{a-b} \text{ for } t \in [a,b]_{\mathbb{T}} \Big\}.$$

Definition 5.1 $f\colon \mathbb{T} \to \mathbb{R}$ is called convex on $\mathbb{T}$ if for every $x, y \in \mathbb{T}$ with $x < y$ the inequality

$$f\big(\lambda x + (1-\lambda)y\big) \le \lambda f(x) + (1-\lambda) f(y)$$

is satisfied for all $\lambda \in \mathbb{T}_{[x,y]}$.

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Note that if $\mathbb{T} = \mathbb{R}$, then the set $\mathbb{T}_{[x,y]}$ is equal to the real interval $[0,1]$. Hence, the above definition coincides with the definition of a real valued convex function defined on $\mathbb{R}$. If $\mathbb{T} = \mathbb{Z}$, then the set $\mathbb{T}_{[x,y]}$ is a subset of the real interval $[0,1]$. In this case, Definition 5.1 coincides with the definition of a real valued convex function defined on $\mathbb{Z}$, namely Definition 2.1.

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