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An Automorphic Theta Module for Quaternionic Exceptional Groups

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Abstract. We construct an automorphic realization of the global minimal representation of quaternionic exceptional groups, using the theory of Eisenstein series, and use this for the study of theta correspondences.

1 Introduction

Let $F = \mathbb{Q}$ be the rational number field, and let D be either a definite quaternion algebra $\mathbb{H}$, or a definite octonion algebra $\mathbb{O}$ over F. To such a D, there corresponds an exceptional group H of relative rank 4, and type E_7 or E_8 accordingly. In H, there is a reductive dual pair $G \times G'$. Here G is a split group of type G_2, and G' is the automorphism group of the Jordan algebra J of 3-by-3 hermitian matrices with coefficients in D, which is anisotropic over F. In this paper, we study the global theta correspondence which arises from the dual pair $G \times G'$. The local analogue of this correspondence has been studied in [MS] and [SG], and the reason for writing F in place of $\mathbb{Q}$ is the expectation that the results here hold for any totally real number field.

The first part of the paper is devoted to the construction of an automorphic realization of the global minimal representation Π of H, and follows the approach of [GRS1] for the split case. Let P be the Heisenberg parabolic subgroup of H, with modulus character δ_P. For a standard section $f_s \in \mathrm{Ind}_{P(\mathbb{A})}^{H(\mathbb{A})} \delta_P^{\frac{1}{2}+s}$, let $E(g, f_s)$ be the usual Eisenstein series. Then we have:

Theorem 1.1 *For any standard section f_s, $E(g, f_s)$ has at most a simple pole at $s = s_0$ (for a certain specific s_0). This pole is actually attained by some standard section. Moreover, the space Θ of automorphic forms spanned by the residues of $E(g, f_s)$ at $s = s_0$ is an irreducible square-integrable automorphic representation isomorphic to* Π.

The proof of this theorem follows the method of [GRS1]; the main difference being that, unlike the split case, the local representation Π_v may be non-spherical here for some place v. Though the above result is to be expected, we find it useful and necessary to have the details properly worked out, because of the importance of the automorphic theta module Θ in the theory of automorphic forms. For example, combined with the results of [Li1] and [Li2], the existence of Θ implies the non-vanishing of L^2-cohomology groups in low degrees of certain locally symmetric spaces. We remark also that an automorphic theta module was constructed in [R] for the rank 2 form of E_6 associated to a 9-dimensional division algebra.

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In the second part of the paper, we study the theta lifts of automorphic forms from G' to G using Θ. If π is an automorphic representation of G', let $\Theta(\pi)$ be its theta lift, which is a subspace of the space of automorphic forms on G. We characterize those π for which $\Theta(\pi)$ is non-zero and cuspidal in terms of period integrals over certain (anisotropic) subgroups of G', in the spirit of [GRS2] and [GrS]. More precisely, each totally real étale cubic algebra E determines a subgroup C_E of G'; also let C be the stabilizer in G' of a primitive idempotent in J. Then we have:

Theorem 1.2 *Let π be a non-trivial automorphic representation of G'. Then $\Theta(\pi)$ is non-zero and cuspidal if and only if the following two conditions hold:*

(i) π is not C-distinguished;
(ii) π is C_E-distinguished for some totally real étale cubic algebra E.

Essentially, the groups C_E are related to certain generic Fourier coefficients of the theta lift, whereas the group C is related to degenerate Fourier coefficients.

When $D = \mathbb{H}$, G' is an inner form of PGSp_6, and the functoriality principle predicts that an automorphic representation π of G' is a functorial lift from G if and only if its Spin L-function $L(s, \pi, \mathrm{Spin})$ has a pole at $s = 1$. On the other hand, in [MS] and [SG], the local theta correspondence is found to be a functorial lifting. In view of Theorem 1.2, one expects that $L(s, \pi, \mathrm{Spin})$ has a pole at $s = 1$ if and only if π is C_E-distinguished for some E. Can this be established independently?

General Notations

In this paper, F will denote the rational number field $\mathbb{Q}$. A place of F will be denoted by v, with F_v the corresponding completion of F. Let ζ_v be the local zeta factor of F at v. Hence,

$$\zeta_v(s) = \begin{cases} (1 - p^{-s})^{-1}, & \text{if } v = p \text{ is finite;} \\ \pi^{-\frac{s}{2}} \cdot \Gamma(\frac{s}{2}), & \text{if } v \text{ is real.} \end{cases}$$

The ring of adeles of F will be denoted by $\mathbb{A}$. As in the introduction, D will denote either a definite quaternion algebra $\mathbb{H}$, or a definite octonion algebra $\mathbb{O}$ over F. Let $\mathrm{SL}(D)$ be the group of norm 1 elements of D. Set $D_v := D \otimes F_v$, and let S be the finite set of places v where D_v is ramified. Hence, if $D = \mathbb{H}$, S contains the real place and has even cardinality, whereas for $D = \mathbb{O}$, S contains just the real place. For $v \notin S$, set:

$$\zeta_{D_v}(s) = \begin{cases} \zeta_v(s) \cdot \zeta_v(s-1), & \text{if } D = \mathbb{H}; \\ \zeta_v(s) \cdot \zeta_v(s-3), & \text{if } D = \mathbb{O}. \end{cases}$$

For any algebraic group H over F, we shall write H for $H(F)$ and H_v for $H(F_v)$. If v is finite, an element of $X^\bullet(H_v) \otimes_{\mathbb{Z}} \mathbb{C}$, where $X^\bullet(H_v)$ is the group of rational characters of H_v, gives rise to an unramified character of H_v, taking values in $\mathbb{C}^\times$. Hence we shall often identify an unramified character of H_v with an element of $X^\bullet(H_v) \otimes_{\mathbb{Z}} \mathbb{C}$. Similarly, an element of $X^\bullet(H) \otimes_{\mathbb{Z}} \mathbb{C}$ gives rise to an unramified character of $H(\mathbb{A})$, which is trivial on H.

Assume now that H is reductive. Let $P_0 = M_0 \cdot N_0$ be a fixed minimal parabolic subgroup, with modulus character $\delta_0 \colon P_0 \to \mathbb{R}_+^\times$. Let $A \subset M_0$ be a maximal split torus of H, Φ the set of roots of H relative to A, and Φ^+ the set of positive roots determined by P_0.

Let $\Delta \subset \Phi^+$ be the set of simple roots, and let $W := N_H(A)/M_0$ be the (relative) Weyl group. For any $\alpha \in \Phi$, $\alpha^\vee$ will denote the corresponding coroot, and U_α the corresponding root subgroup. Moreover, let $\langle \cdot, \cdot \rangle$ be the canonical pairing between the roots and the coroots. Sometimes $\langle \cdot, \cdot \rangle$ will also denote the Killing form of various Lie algebras. We hope that this will not cause any confusion.

For a standard parabolic subgroup $P = M \cdot N$ of H, let $\Delta_M \subset \Delta$ be the set of simple roots of its Levi factor M, Φ_M the corresponding root system and W_M the Weyl group of M. The opposite parabolic is denoted by $\overline{P} = M \cdot \overline{N}$, and the modulus character of P by δ_P. For each v, let K_v be a maximal compact subgroup of H_v, which is special if v is finite, so that the Iwasawa decomposition holds: $H_v = P_{0,v} \cdot K_v$. Then for almost all v, K_v is hyperspecial, and $K = \prod_v K_v$ is a maximal compact subgroup of $H(\mathbb{A})$.

Suppose that H is not split over F, but H_v is split. Then, in such cases, $P_{0,v}$ is no longer the minimal parabolic subgroup. Let B_v be a fixed Borel subgroup of H_v contained in $P_{0,v}$, with modulus character δ_{B_v}, and let $B_v \supset T_v \supset A_v$ be a maximal torus. If Φ^0 is the (absolute) root system of H_v relative to T_v, then we have a canonical map, $\Phi^0 \longrightarrow \Phi$, given by restriction of characters. For $\beta \in \Phi^0$ and $\alpha \in \Phi$, we write $\beta \mapsto \alpha$ if β restricts to α under the above map.

2 Quaternionic Exceptional Groups

In this section, we describe the groups which we will study in this paper. For more details, see [SG, Section 3].

Let J be the Jordan algebra of 3-by-3 hermitian matrices with coefficients in D. Then the dimension of J is:

$$d = \begin{cases} 15, & \text{if } D = \mathbb{H}; \\ 27, & \text{if } D = \mathbb{O}. \end{cases} \tag{2.1}$$

There is a cubic form det on J, giving rise to a symmetric trilinear form $(\cdot,\cdot,\cdot)$ normalized by $(X, X, X) = \det(X)$, and a symmetric bilinear trace form $(\cdot,\cdot)$, given by: $(X, Y) = \operatorname{Tr}(XY)$. Then, for $X, Y \in J$, we have the element $X \times Y \in J$, uniquely determined by:

$$(X \times Y, Z) = 3(X, Y, Z) \tag{2.2}$$

for all $Z \in J$. Recall that X has rank one if $X \neq 0$ but $X \times X = 0$. Equivalently, $X^2 = \operatorname{Tr}(X)X$. Note that if F is a totally real number field, as is our case, there are no rank one elements with trace zero.

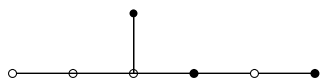
Let L be the algebraic group of linear transformations on J which preserve the determinant form det. Then

$$L \cong \begin{cases} \mathrm{SL}_3(D)/\mu_2, & \text{if } D = \mathbb{H}; \\ E_{6,2}^{sc}, & \text{if } D = \mathbb{O}. \end{cases} \tag{2.3}$$

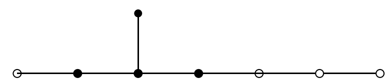
Here, $E_{6,2}^{sc}$ is a simply-connected group of type E_6 and relative rank 2.

Now associated to D is a simple adjoint algebraic group H of relative rank 4, and type E_7 (respectively E_8) if $D = \mathbb{H}$ (respectively $\mathbb{O}$). The Satake diagram of H is:

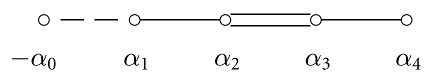
$E_7(\mathbb{H})$



$E_8(\mathbb{O})$



Moreover, the relative Dynkin diagram is of type F_4:



Here α_0 denotes the highest root. In particular,

$$\alpha_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4. \tag{2.4}$$

For simplicity, we shall represent a root $\sum_{i=1}^4 a_i\alpha_i$ as a 4-tuple (a_1, a_2, a_3, a_4). Let w_i be the simple reflection in W corresponding to the simple root α_i. If $w \in W$ has a minimal length expression $w = w_{i_1} \cdot w_{i_2} \cdot \cdots \cdot w_{i_k}$, then we shall write $w = (i_1, i_2, \ldots, i_k)$.

Let $P = M \cdot N$ be the Heisenberg maximal parabolic subgroup of H, which corresponds to the vertex α_1 in the relative Dynkin diagram. In particular, its unipotent radical N is a Heisenberg group with center Z, and the abelian group $V = N/Z$ has a natural structure of a symplectic vector space. Let $\overline{V} = \overline{N}/\overline{Z}$, where $\overline{Z}$ is the center of $\overline{N}$. Then there is a natural identification [MS, Section 6]

$$\overline{V} \cong F \oplus J \oplus J \oplus F. \tag{2.5}$$

The Levi factor M has derived group M^1 of type D_6 (respectively E_7) if $D = \mathbb{H}$ (respectively $\mathbb{O}$). The action of M^1 on $\overline{V}$ is the half-spin representation of dimension 32 if $D = \mathbb{H}$, and the 56-dimensional miniscule representation if $D = \mathbb{O}$. Moreover, the minimal non-trivial M-orbit Ω is the orbit of a highest weight vector, which can be chosen to be:

$$v_0 = (0, 0, 0, 1) \in F \oplus J \oplus J \oplus F. \tag{2.6}$$

Let Q be the stabilizer in M^1 of the line spanned by v_0. Then Q is a maximal parabolic subgroup of M^1, and is the intersection of M^1 with the maximal parabolic subgroup of H corresponding to the vertex α_2. It has an abelian unipotent radical $U \cong J$, and the derived group of its Levi factor is isomorphic to the group L introduced earlier in (2.3).

Now, we have [MS, Lemma 7.5]:

Lemma 2.7 *Q has 4 orbits on Ω, which are given by:*

$$\mathcal{O}_0 = \{(0,0,0,d) : d \in F^\times\},$$
$$\mathcal{O}_1 = \{(0,0,Y,d) : \operatorname{rank}(Y) = 1 \textit{ and } d \in F\},$$
$$\mathcal{O}_2 = \{\big(0, Y, 2B \times Y, (B,B,Y)\big) : \operatorname{rank}(Y) = 1 \textit{ and } B \in J\},$$
$$\mathcal{O}_3 = \{a\big(1, Z, Z \times Z, \det(Z)\big) : a \in F^\times \textit{ and } Z \in J\}.$$

Now note that the characters of the compact group $V \setminus V(\mathbb{A})$ can be parametrized by $\overline{V}$ as follows. Fix a non-trivial character $\psi = \prod_v \psi_v$ of $F \setminus \mathbb{A}$. The Killing form $\langle \cdot, \cdot \rangle$ induces a non-degenerate pairing of V with $\overline{V}$. Then, for $x \in \overline{V}$, the corresponding character ψ_x is given by:

$$\psi_x(n) = \psi(\langle x, n \rangle).$$

Similarly, the characters of N_v can be parametrized by $\overline{V}_v$ using the Killing form and ψ_v. Henceforth, we shall regard the elements of Ω as characters of $V \setminus V(\mathbb{A})$.

Finally, the modulus character of P is unramified, and is given by:

$$\delta_P = (2+d)\alpha_0, \tag{2.8}$$

when regarded as an element of $X^\bullet(A) \otimes_{\mathbb{Z}} \mathbb{C}$. Similarly, the modulus character of P_0, the minimal parabolic, is given by:

$$\delta_0 = (4+2d)\alpha_1 + (6+4d)\alpha_2 + \left(16 + \frac{16}{3}d\right)\alpha_3 + \left(12 + \frac{8}{3}d\right)\alpha_4. \tag{2.9}$$

3 Local Minimal Representation

We now summarize some important facts about the local minimal representation Π_v of H_v, and refer the reader to [GrW], [S], [R2] and [T] for more details.

For any $s \in \mathbb{C}$, consider the degenerate principal series representation:

$$I_v(s) = \operatorname{Ind}_{P_v}^{H_v} \delta_P^{\frac{1}{2}+s} \tag{3.1}$$

where for v real, $I_v(s)$ denotes the Harish-Chandra module of the corresponding smooth representation $I_v(s)^\infty$. Also, let

$$s_0 = \begin{cases} \frac{11}{34}, & \text{if } D = \mathbb{H}; \\ \frac{19}{58}, & \text{if } D = \mathbb{O}. \end{cases}$$

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Now we have:

Proposition 3.2

(i) *If v is the real place, then Π_v is a quotient of $I_v(s_0)$, and occurs with multiplicity one in $I_v(s_0)$.*

(ii) *If $v \notin S$, then Π_v is spherical and is the unique irreducible quotient of $I_v(s_0)$. Moreover, if ψ is a character of N_v, then ($v \neq 2$ if $D = \mathbb{O}$)*

$$\dim(\Pi_v)_{N_v,\psi} = \begin{cases} 1, & \text{if } \psi \in \Omega_v; \\ 0, & \text{otherwise.} \end{cases}$$

(iii) *If $v \in S$ is finite, then $I_v(s_0)$ has a unique irreducible quotient. If, further, $v \neq 2$, then $\dim(\Pi_v)_{N_v,\psi}$ is as given in (ii) above.*

Remarks In (i), we suspect, but do not know, that Π_∞ is the unique irreducible quotient of $I_\infty(s_0)$. We shall see later that in (iii), the unique irreducible quotient of $I_v(s_0)$ is exactly Π_v.

4 Eisenstein Series and Intertwining Operators

Now we can begin the construction of the automorphic theta module, *i.e.*, an embedding of $\Pi = \hat{\bigotimes}_v \Pi_v$ into the space of automorphic forms $\mathcal{A}(H)$ on H. The above results about the local minimal representations suggest that we should consider the global induced representation:

$$I(s) = \mathrm{Ind}_{P(\mathbb{A})}^{H(\mathbb{A})} \delta_P^{\frac{1}{2}+s} = \hat{\bigotimes_v} I_v(s), \tag{4.1}$$

where the restricted tensor product is formed using the unique K_v-spherical vector $\Gamma_{v,s}$, normalized by $\Gamma_{v,s}(1) = 1$. Indeed, as a corollary of Proposition 3.2, we have:

Proposition 4.2 *The global induced representation $I(s_0) = \hat{\bigotimes}_v I_v(s_0)$ has a unique irreducible quotient Π', with $\Pi'_\infty \cong \Pi_\infty$.*

The following lemma is straightforward:

Lemma 4.3

(1) *Let $\chi_s \colon P_0(\mathbb{A}) \longrightarrow \mathbb{C}^\times$ be the unramified character given by:*

$$\chi_s = \delta_P^{\frac{1}{2}+s} \cdot \delta_0^{-\frac{1}{2}}.$$

Then

$$I(s) \subset \mathrm{Ind}_{P_0(\mathbb{A})}^{H(\mathbb{A})} \chi_s \cdot \delta_0^{\frac{1}{2}} := I_0(\chi_s).$$

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(2) If H_v is split, let

$$\chi_s^0 = \delta_P^{\frac{1}{2}+s} \cdot \delta_{B_v}^{-\frac{1}{2}}.$$

Then

$$I_v(s) \subset \operatorname{Ind}_{B_v}^{H_v} \chi_s^0 \cdot \delta_{B_v}^{\frac{1}{2}}.$$

Now let $f_s = \bigotimes_v f_{v,s}$ be a standard section of $I(s)$. For $\operatorname{Re}(s)$ sufficiently large, we form the Eisenstein series:

$$E(g, f_s) := \sum_{\gamma \in P \backslash H} f_s(\gamma g), \quad g \in H(\mathbb{A}), \tag{4.4}$$

which admits a meromorphic continuation to the whole complex plane, and defines an automorphic form of H at a point s of holomorphy. We are interested in the analytic behaviour of $E(g, f_s)$ at $s = s_0$, which is the same as that of its constant term $E_{P_0}(g, f_s)$ along P_0. By a standard computation [MW, p. 92],

$$E_{P_0}(g, f_s) = \sum_{w \in \Psi} M(w, \chi_s)(f_s)(g) \tag{4.5}$$

where

$$\Psi = \{w \in W : w\Delta_M \subset \Phi^+\} \tag{4.6}$$

is the set of distinguished coset representatives for $P_0 \setminus H/P$, and for any $w \in W$,

$$M(w, \chi_s)\colon I_0(\chi_s) \longrightarrow I_0\big(w(\chi_s)\big) \tag{4.7}$$

is the standard intertwining operator, which, for $\operatorname{Re}(s)$ sufficiently large, is given by:

$$\begin{aligned} M(w, \chi_s)(f_s)(g) &= \prod_v M_v(w, \chi_s)(f_{v,s})(g_v) \\ &= \prod_v \int_{U_{w,v}} f_{v,s}(w^{-1} u_v g_v)\, du_v \end{aligned} \tag{4.8}$$

with

$$U_w := \prod_{\alpha > 0, w^{-1}\alpha < 0} U_\alpha. \tag{4.9}$$

Here, we have chosen a representative of $w \in W$ in H, and denoted it by w as well. The global operator does not depend on the choice of this representative, but the local operators $M_v(w, \chi_s)$ do. However, the local operators corresponding to two different choices differ up to multiplication by a non-vanishing entire function of s, so that their analytic properties are the same. In fact, by choosing the maximal compact subgroup K suitably, we can and do normalize this choice by requiring that $w \in K_v$ for all $v \notin S$.

For this normalized choice of w, there is a meromorphic function $c_v(w,s)$ such that

$$M_v(w,\chi_s)(\Gamma_{v,s}) = c_v(w,s)\Gamma_{v,w(\chi_s)} \tag{4.10}$$

where $\Gamma_{v,w(\chi_s)}$ is the normalized spherical vector in $I_{0,v}\big(w(\chi_s)\big)$. This function, which is called the c-function, was computed by Gindinkin-Karpelevich in the real case, and by Langlands [L] in the p-adic case. See also [R, Lemmas 6 and 7]. We have:

Proposition 4.11 *Suppose that H_v is split. Then*

$$c_v(w,s) = \prod_{\alpha>0, w\alpha<0} c_v(\alpha,s),$$

where, if α is a long root,

$$c_v(\alpha,s) = \frac{\zeta_v(\langle\chi_s,\alpha^\vee\rangle)}{\zeta_v(\langle\chi_s,\alpha^\vee\rangle+1)},$$

and if α is a short root,

$$\prod_{\beta\mapsto\alpha} \frac{\zeta_v(\langle\chi_s^0,\beta^\vee\rangle)}{\zeta_v(\langle\chi_s^0,\beta^\vee\rangle+1)}.$$

In the second case, the product is taken over all roots $\beta\in\Phi^0$ which project onto $\alpha\in\Phi$, and χ_s^0 is the unramified character of B_v defined in Lemma 4.3 (2).

5 An Automorphic Theta Module

Now let $f_s = \bigotimes_v f_{v,s}$ be a factorizable standard section, such that $f_{v,s}$ is the normalized spherical vector $\Gamma_{v,s}$ for all $v\notin S$. Then we want to understand the analytic properties of $E(g,f_s)$ at $s=s_0$. By (4.5), it suffices to understand the analytic properties of $M(w,\chi_s)(f_s)$, for $w\in\Psi$. The previous proposition allows us to evaluate

$$M_S(w,\chi_s)(f_s)(1) = \prod_{v\notin S} c_v(w,s) := c_S(w,s).$$

For $w\in\Psi$, which has cardinality 24, let:

$$\Phi_w = \{\alpha\in\Phi^+ : w\alpha<0\} \subset \Phi^+ \smallsetminus \Phi_M^+. \tag{5.1}$$

α	$\alpha^\vee$	$c_v(\alpha, s)$ for $D = \mathbb{H}$	$c_v(\alpha, s)$ for $D = \mathbb{O}$
(1,0,0,0)	(1,0,0,0)	$\zeta_v(17s + \frac{15}{2})/\zeta_v(17s + \frac{17}{2})$	$\zeta_v(29s + \frac{27}{2})/\zeta_v(29s + \frac{29}{2})$
(1,1,0,0)	(1,1,0,0)	$\zeta_v(17s + \frac{13}{2})/\zeta_v(17s + \frac{15}{2})$	$\zeta_v(29s + \frac{25}{2})/\zeta_v(29s + \frac{27}{2})$
(1,1,2,0)	(1,1,1,0)	$\zeta_v(17s + \frac{5}{2})/\zeta_v(17s + \frac{7}{2})$	$\zeta_v(29s + \frac{9}{2})/\zeta_v(29s + \frac{11}{2})$
(1,2,2,0)	(1,2,1,0)	$\zeta_v(17s + \frac{3}{2})/\zeta_v(17s + \frac{5}{2})$	$\zeta_v(29s + \frac{7}{2})/\zeta_v(29s + \frac{9}{2})$
(1,1,2,2)	(1,1,1,1)	$\zeta_v(17s - \frac{3}{2})/\zeta_v(17s - \frac{1}{2})$	$\zeta_v(29s - \frac{7}{2})/\zeta_v(29s - \frac{5}{2})$
(1,2,2,2)	(1,2,1,1)	$\zeta_v(17s - \frac{5}{2})/\zeta_v(17s - \frac{3}{2})$	$\zeta_v(29s - \frac{9}{2})/\zeta_v(29s - \frac{7}{2})$
(1,2,4,2)	(1,2,2,1)	$\zeta_v(17s - \frac{13}{2})/\zeta_v(17s - \frac{11}{2})$	$\zeta_v(29s - \frac{25}{2})/\zeta_v(29s - \frac{23}{2})$
(1,3,4,2)	(1,3,2,1)	$\zeta_v(17s - \frac{15}{2})/\zeta_v(17s - \frac{13}{2})$	$\zeta_v(29s - \frac{27}{2})/\zeta_v(29s - \frac{25}{2})$
(2,3,4,2)	(2,3,2,1)	$\zeta_v(34s)/\zeta_v(34s + 1)$	$\zeta_v(58s)/\zeta_v(58s + 1)$

Table 1: Long Roots

To use Proposition 4.11, we need to enumerate the set Φ_w and compute $c_v(\alpha, s)$. The values of $c_v(\alpha, s)$ are given in Table 1 for the 9 long roots in $\Phi^+ \smallsetminus \Phi_M^+$, and in Table 2 for the 6 short roots.

Let

$$c_S(\alpha, s) = \prod_{v \notin S} c_v(\alpha, s),$$

(5.2)

$$\zeta_S(s) = \prod_{v \notin S} \zeta_v(s).$$

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α	$\alpha^\vee$	$c_v(\alpha,s)$ for $D=\mathbb{H}$	$c_v(\alpha,s)$ for $D=\mathbb{O}$
(1,1,1,0)	(2,2,1,0)	$\zeta_{D_v}(17s+\frac{9}{2})/\zeta_{D_v}(17s+\frac{13}{2})$	$\zeta_{D_v}(29s+\frac{17}{2})/\zeta_{D_v}(29s+\frac{25}{2})$
(1,1,1,1)	(2,2,1,1)	$\zeta_{D_v}(17s+\frac{5}{2})/\zeta_{D_v}(17s+\frac{9}{2})$	$\zeta_{D_v}(29s+\frac{9}{2})/\zeta_{D_v}(29s+\frac{17}{2})$
(1,1,2,1)	(2,2,2,1)	$\zeta_{D_v}(17s+\frac{1}{2})/\zeta_{D_v}(17s+\frac{5}{2})$	$\zeta_{D_v}(29s+\frac{1}{2})/\zeta_{D_v}(29s+\frac{9}{2})$
(1,2,2,1)	(2,4,2,1)	$\zeta_{D_v}(17s-\frac{1}{2})/\zeta_{D_v}(17s+\frac{3}{2})$	$\zeta_{D_v}(29s-\frac{1}{2})/\zeta_{D_v}(29s+\frac{7}{2})$
(1,2,3,1)	(2,4,3,1)	$\zeta_{D_v}(17s-\frac{5}{2})/\zeta_{D_v}(17s-\frac{1}{2})$	$\zeta_{D_v}(29s-\frac{9}{2})/\zeta_{D_v}(29s-\frac{1}{2})$
(1,2,3,2)	(2,4,3,2)	$\zeta_{D_v}(17s-\frac{9}{2})/\zeta_{D_v}(17s-\frac{5}{2})$	$\zeta_{D_v}(29s-\frac{17}{2})/\zeta_{D_v}(29s-\frac{9}{2})$

Table 2: Short Roots

From the above results, one sees that when $D=\mathbb{H}$, $c_S(\alpha,s)$ is finite and non-zero at $s=s_0$ except possibly for the following α's:

α	$c_S(\alpha,s)$	Behaviour at $s=s_0$
$\alpha(0)=(1,3,4,2)$	$\zeta_S(-2)/\zeta_S(-1)$	Zero of order 1
$\alpha(1)=(1,2,4,2)$	$\zeta_S(-1)/\zeta_S(0)$	Pole of order $\lvert S\rvert-1$
$\alpha(2)=(1,2,3,2)$	$\big(\zeta_S(1)\cdot\zeta_S(0)\big)/\big(\zeta_S(3)\cdot\zeta_S(2)\big)$	Zero of order $\lvert S\rvert-2$

As for the case $D=\mathbb{O}$, $c_S(\alpha,s)$ is finite and non-zero at $s=s_0$ except for the following α's:

α	$c_S(\alpha,s)$	Behaviour at $s=s_0$
$\alpha(0)=(1,3,4,2)$	$\zeta_S(-4)/\zeta_S(-3)$	Zero of order 1
$\alpha(1)=(1,2,4,2)$	$\zeta_S(-3)/\zeta_S(-2)$	Pole of order 1

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Recall that:

$$c_v(w,s) = \prod_{\alpha\in\Phi_w} c_v(\alpha,s). \tag{5.3}$$

For those $w \in \Psi$, such that Φ_w does not contain any of the roots $\alpha(i)$, $i = 0,1,2$, the function $c_S(w,s) = \prod_{\alpha\in\Phi_w} c_S(\alpha,s)$ is thus finite and non-zero at $s = s_0$. The only w's not accounted for are:

$$\begin{aligned} w_0 &= (1,2,3,2,1,4,3,2,1,3,2,4,3,2,1),\\ w_{-1} &= (2,3,2,1,4,3,2,1,3,2,4,3,2,1),\\ w_{-2} &= (3,2,1,4,3,2,1,3,2,4,3,2,1). \end{aligned} \tag{5.4}$$

For these, we list the set Φ_w:

$$\begin{aligned} \Phi_{w_0} &= \Phi^+ - \Phi_M^+,\\ \Phi_{w_{-1}} &= \Phi_{w_0} - \{\alpha(0)\},\\ \Phi_{w_{-2}} &= \Phi_{w_0} - \{\alpha(0),\alpha(1)\}. \end{aligned}$$

Now we have:

Lemma 5.5

(i) $c_S(w_{-2},s)$ *is holomorphic at* $s = s_0$*; indeed it has a zero of order* $|S| - 2$ *if* $D = \mathbb{H}$.
(ii) $c_S(w_0,s)$ *is finite and non-zero at* $s = s_0$.
(iii) $c_S(w_{-1},s)$ *has a pole of order 1 at* $s = s_0$.

It remains now to analyze the finitely many terms $M_v(w,\chi_s)(f_{v,s})$ for $v \in S$. Using the functional equation for intertwining operators, we can write $M_v(w,\chi_s)$ as a product of simple intertwining operators, *i.e.*, those corresponding to simple reflections. Using the well-known analytic properties of intertwining operators for rank 1 groups, we deduce that if $w \neq w_i$, $i = 0,-1$, the integral defining $M_v(w,\chi_s)(f_{v,s})$ converges at $s = s_0$, for any choice of $f_{v,s}$. Thus, for $w \neq w_i$, $i = 0,-1$, $M(w,\chi_s)(f_s)$ is holomorphic at $s = s_0$.

Similarly, we deduce that, at $s = s_0$, $M_v(w_{-1},\chi_s)$ is holomorphic for all $v \in S$, whereas $M_v(w_0,\chi_s)$ is holomorphic for v finite, and can have a pole of order ≤ 1 at the real place. Now we have the following crucial lemma:

Lemma 5.6 *For* $v \in S$*, the intertwining operator*

$$M_v(w_{-1},\chi_{s_0})\colon I_v(s_0) \longrightarrow I_{0,v}\big(w_{-1}(\chi_{s_0})\big)$$

is not identically zero.

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Proof First, note that the double coset $P_v w_0 P_v = P_v w_0 N_v$ is open in H_v, and any element $g \in P_v w_0 P_v$ has a unique expression $g = p w_0 n$ with $p \in P_v$ and $n \in N_v$.

For v real, we let ϕ be a smooth real-valued non-negative function on N_v with compact support, and set

$$f_s(g) = \begin{cases} \delta(p)^{\frac{1}{2}+s}\phi(n), & \text{if } g = p w_0 n; \\ 0, & \text{otherwise.} \end{cases}$$

Then $f_s \in I_v(s)^\infty$ is an entire section, though not standard. By results of Vogan and Wallach [Wa, Chapter 10], $M_v(w_{-1}, \chi_{s_0})$ is an intertwining operator defined on $I_v(s_0)^\infty$, and is continuous with respect to the natural Fréchet topology on the smooth representations. Now, for $\mathrm{Re}(s)$ sufficiently large, and $g = w_{-1} w_0$,

$$\begin{aligned} M_v(w_{-1}, \chi_s)(f_s)(g) &= \int_{g^{-1}U_{w_{-1}}g} f_s(w_0 u)\, du \\ &= \int_{g^{-1}U_{w_{-1}}g} \phi(u)\, du \end{aligned}$$

since $g^{-1}U_{w_{-1}}g \subset N_v$. For a suitable choice of ϕ, we can certainly ensure that this last integral is non-zero. Since it is also independent of s, the meromorphic function

$$s \mapsto M_v(w_{-1}, \chi_s)(f_s)(g)$$

is constant and non-zero. Since $I_v(s_0)$ is dense in $I_v(s_0)^\infty$, we deduce that $M_v(w_{-1}, \chi_{s_0})$ must be non-zero on some K_v-finite vector. This proves the lemma for v real.

Now assume that $v \in S$ is finite. Let $f_s \in I(s)$ be defined by:

$$f_s(g) = \begin{cases} \delta(p)^{\frac{1}{2}+s}, & \text{if } g = p w_0 n \in P_v w_0 (N_v \cap K_v); \\ 0, & \text{otherwise.} \end{cases}$$

Then f_s is a standard section, and the same argument as above proves the lemma. ■

Corollary 5.7

(i) For $v \in S$ finite, $M_v(w_0, \chi_s)$ is holomorphic at $s = s_0$ and $M_v(w_0, \chi_{s_0})$ is not identically zero on $I_v(s_0)$.

(ii) For v real, $M_v(w_0, \chi_s)$ has a pole of order at most 1 at $s = s_0$, and this pole is attained for some vector in $I_v(s_0)$.

Proof This follows from the factorization

$$M_v(w_0, \chi_s) = M_v\big(w_1, w_{-1}(\chi_s)\big) \circ M_v(w_{-1}, \chi_s),$$

the lemma above and the well-known analytic properties of SL_2-intertwining operators. ■

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Now we have:

Theorem 5.8 *For any standard section $f_s \in I(s)$, the Eisenstein series $E(g, f_s)$ has at most a simple pole at $s = s_0$. Moreover, this pole is actually attained for some standard section f_s with $f_{v,s} = \Gamma_{v,s}$ for all $v \notin S$. Let*

$$\begin{aligned} \theta\colon I(s_0) &\to \mathcal{A}(H) \\ f &\mapsto \operatorname{Res}_{s=s_0} E(g, f_s) \end{aligned} \tag{5.9}$$

and let Θ be its image. Then Θ is an irreducible square-integrable automorphic representation isomorphic to Π.

Proof If the standard section f_s is such that $f_{v,s} = \Gamma_{v,s}$ for all $v \notin S$, then we have seen that $E(g, f_s)$ has at most a simple pole at $s = s_0$. Now we claim that this is true for any other standard sections as well. Suppose not; then for any decomposable f_s such that $E(g, f_s)$ has a pole at $s = s_0$ of order greater than 1, let

$$S_f := \{v \notin S : f_{v,s} \text{ is not spherical}\}.$$

Hence, S_f is non-empty. Choose f_s such that S_f is minimal, and suppose that $v_0 \in S_f$. Then consider:

$$\begin{aligned} I_{v_0}(s_0) &\to \mathcal{A}(H) \\ \phi &\mapsto lim_{s\to s_0}(s - s_0)^k E(g, \phi_s) \end{aligned}$$

where ϕ_s is the unique standard section satisfying:

$$\phi_{s_0} = \phi \otimes \Big(\bigotimes_{v \neq v_0} f_{v,s_0}\Big)$$

and $k > 1$ is the highest order of pole attained by $E(g, \phi_s)$ at $s = s_0$ for all such ϕ_s. Such a k exists since $I_{v_0}(s_0)$ has finite length. By the definition of k, this map is a non-zero H_{v_0}-intertwining map, and by the minimality of S_f, it vanishes on the spherical vector. But by Proposition 3.2(ii), $I_{v_0}(s_0)$ is generated by the spherical vector as a representation of H_{v_0}. With this contradiction, the claim is proved.

To see that the pole is actually attained for some sections, it remains to show that the poles of $M(w_{-1}, \chi_s)$ and $M(w_0, \chi_s)$ at $s = s_0$ do not cancel. For this, note that the residue of $M(w_i, \chi_s)(f_s)$ at $s = s_0$, when regarded as a function of the maximal split torus A, is the unramified character $\delta_0^{\frac{1}{2}} \cdot w_i(\chi_{s_0})$. Hence it suffices to see that these two unramified characters are different. One checks that if $D = \mathbb{H}$,

$$\begin{aligned} w_0(\chi_{s_0}) &= -11\alpha_1 - 24\alpha_2 - 36\alpha_3 - 20\alpha_4, \\ w_{-1}(\chi_{s_0}) &= -13\alpha_1 - 24\alpha_2 - 36\alpha_3 - 20\alpha_4, \end{aligned} \tag{5.10}$$

whereas if $D = \mathbb{O}$,

(5.11)
$$\begin{aligned} w_0(\chi_{s_0}) &= -19\alpha_1 - 42\alpha_2 - 60\alpha_3 - 32\alpha_4, \\ w_{-1}(\chi_{s_0}) &= -23\alpha_1 - 42\alpha_2 - 60\alpha_3 - 32\alpha_4. \end{aligned}$$

Hence, we see that the pole at $s = s_0$ is actually attained. Moreover, the above shows that all the cuspidal exponents of Θ have strictly negative coefficients. This implies, by Jacquet's criterion [MW, p. 74] that Θ is contained in $L^2\big(H \setminus H(\mathbb{A})\big) \cap \mathcal{A}(H)$.

Since Θ is square-integrable, it is semi-simple. Suppose that $\Pi_1 \subset \Theta$ is an irreducible summand. Then, for v finite, $(\Pi_1)_v$ is the unique irreducible quotient of $I_v(s_0)$, by Proposition 3.2(ii) and (iii). Moreover, for $v \notin S$, $(\Pi_1)_v \cong \Pi_v$. Now by a rigidity result of Kazhdan [R2, Proposition 57], this implies that $(\Pi_1)_v$ is also the minimal representation for all $v \in S$, in particular for v the real place. Thus, $\Pi_1 \cong \Pi$, and in view of Proposition 4.2, we deduce that Θ must be irreducible. This proves the theorem. ∎

Corollary 5.12 *For $v \in S$ finite, the minimal representation Π_v is the unique irreducible quotient of $I_v(s_0)$.*

Corollary 5.13 *For any non-zero $f \in \Pi \subset I(-s_0)$, let f_s be the unique standard section extending f. Then $E(g, f_s)$ is holomorphic at $s = -s_0$, and $E(g, f_{-s_0})$ generates Θ.*

Proof This follows from the functional equation of Eisenstein series. ∎

Remarks (i) It seems likely that there is exactly one automorphic realization for Π. As in [GRS1], this uniqueness statement would follow if the multiplicity one result for Jacquet modules in Proposition 3.2(ii) holds for all places v.

(ii) When D is an indefinite quaternion algebra, the data in this section allows one to conclude that the minimal representation for the corresponding group is automorphic as well.

6 Fourier Coefficients

In this section, we consider the Fourier coefficients of $\theta = \theta(\bigotimes_v f_v) \in \Theta$ along the unipotent radical N of the Heisenberg parabolic subgroup $P = M \cdot N$. Recall that N is a Heisenberg group with center Z, and $V = N/Z$.

Consider the constant term of θ along Z:

(6.1)
$$\theta_Z(g) = \int_{Z \setminus Z(\mathbb{A})} \theta(zg)\, dz.$$

Note that θ_Z is non-zero since the constant term of θ with respect to P_0 is non-zero. We consider its Fourier expansion along the compact group $V \setminus V(\mathbb{A})$. For a character ψ of $V \setminus V(\mathbb{A})$, the ψ-Fourier coefficient of θ_Z is:

(6.2)
$$\theta_\psi(g) = \int_{V \setminus V(\mathbb{A})} \theta_Z(ng)\psi(n)^{-1}\, dn.$$

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Let C_ψ be the stabilizer of ψ in M^1. If $\psi \in \Omega$, then C_ψ is a conjugate of the derived group of the maximal parabolic subgroup Q of M^1. As in [GRS1] and [GrS], we have the following two important properties of θ_ψ:

Proposition 6.3 *Suppose that ψ is non-trivial. For any non-zero $\theta \in \Theta$, θ_ψ is non-zero if and only if $\psi \in \Omega$.*

Proposition 6.4 *Suppose that $\psi \in \Omega$. Then for all $c \in C_\psi(\mathbb{A})$, we have:*

$$\theta_\psi(cg) = \theta_\psi(g).$$

Finally, we consider the constant term θ_N, regarded as an automorphic form on M^1. By computing the constant term $E_P(g, f_s)$ of $E(g, f_s)$ along P, we have:

Proposition 6.5 *As an automorphic form on M^1,*

$$\theta_N = c + \theta'$$

where c is a constant function, and θ' is contained in an automorphic realization of the global minimal representation of M^1.

One can further compute the constant term of $E_P(g, f_s)$ along the unipotent radical U of the maixmal parabolic Q in M^1. Then one finds that the constant term $(\theta_N)_U$ of θ_N along U is simply the sum of two characters of the Levi factor of Q. This reflects the fact that, for $v \in S$, L_v does not have a minimal representation.

7 Dual Pairs and Étale Cubic Algebras

We shall use the automorphic theta module Θ and the properties of its Fourier coefficients discussed above to study theta correspondence. In this section, we briefly describe the dual pair $G \times G'$.

Let $e \in J$ be the identity, and let $G' \subset L$ be the algebraic subgroup stabilizing e. Then G' is the automorphism group of the Jordan algebra structure on J, and

(7.1)
$$G' \cong \begin{cases} \mathrm{PU}_3(D), & \text{if } D = \mathbb{H}; \\ F_4^{cpt}, & \text{if } D = \mathbb{O}. \end{cases}$$

Here, $\mathrm{PU}_3(D)$ is the projective unitary group in three variables with coefficients in D. Note that G' acts naturally on J_0, the space of trace zero elements in J. Moreover, G'_∞ is compact, whereas for finite $v \in S$, G'_v has rank 1, and for $v \notin S$, G'_v is split. Let G be the split group of type G_2. Then $G \times G'$ is a reductive dual pair in H (see [MS] and [SG]).

There is an embedding of $G \times G'$ in H such that

(7.2)
$$(G \times G') \cap P = P_2 \times G'$$

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where $P_2 = L_2 \cdot U_2$ is the Heisenberg parabolic subgroup of G. Here, U_2 is a Heisenberg group with center Z, and $\overline{\mathbb{V}} := \overline{U}_2/\overline{Z}$ is the subspace $F \oplus Fe \oplus Fe \oplus F$ of $\overline{V}$. Moreover, $L_2 \cong \mathrm{GL}_2$, and its action on $\overline{\mathbb{V}}$ is isomorphic to $\det \otimes \mathrm{Sym}^3(F^2)^*$. Thus we can identify $\overline{\mathbb{V}}$ with the space of binary cubic forms, and the non-zero L_2-orbits are then parametrized by cubic F-algebras [Wr]. We list the non-zero orbits below:

(i) $S_0 = L_2 \cdot (0, 0, 0, 1)$; this is the orbit corresponding to $E_0 = F[\varepsilon]/\varepsilon^3$.
(ii) $S_1 = L_2 \cdot (0, 0, 1, 0)$; this is the orbit corresponding to $E_1 = F \oplus F[\varepsilon]/\varepsilon^2$.
(iii) For each étale cubic algebra E, there is a generic orbit S_E. Given E, let E^0 denote the two-dimensional space of trace zero elements. Then the norm form of E restricts to give a binary cubic form on E^0. This form is an element in the orbit S_E.

Since $\overline{\mathbb{V}}$ can be identified with the characters of $Z(\mathbb{A})U_2 \backslash U_2(\mathbb{A})$, we see that the L_2-orbits of non-trivial characters are parametrized by cubic algebras over F.

For $\psi_E \in S_E$, a non-zero orbit, let

$$\Omega_E = \{\psi \in \Omega : \psi|_{U_2(\mathbb{A})} = \psi_E\}.$$

Clearly, G' acts on Ω_E, which as a G'-set, depends only on E. As an example, the G'-set $\Omega_1 := \Omega_{E_1}$ can be identified as:

$$\Omega_1 = \{X \in J : \mathrm{rank}(X) = 1 = \mathrm{Tr}(X)\}.$$

Note that Ω_1 is clearly non-empty, but this is not always the case when E is étale. Indeed, we have (see [GrG, Proposition 1]):

Lemma 7.3

(1) Suppose that E is étale. Then Ω_E is non-empty if and only if E is totally real, in which case G' acts transitively on Ω_E. The algebraic subgroup of G' stabilizing an element of Ω_E is:

$$C_E = \begin{cases} \mathrm{Res}_{E/F}\big(\mathrm{SL}(D \otimes_F E)\big)/\mu_2, & \text{if } D = \mathbb{H}; \\ \mathrm{Spin}_8^{E,cpt}, & \text{if } D = \mathbb{O}. \end{cases}$$

(2) G' acts transitively on Ω_1, and the algebraic subgroup of G' stabilizing an element is:

$$C = \begin{cases} \big(\mathrm{SU}_2(D) \times \mathrm{SL}(D)\big)/\mu_2, & \text{if } D = \mathbb{H}; \\ \mathrm{Spin}_9^{cpt}, & \text{if } D = \mathbb{O}. \end{cases}$$

8 Cuspidality of Theta Lifts

Let $\pi = \hat{\bigotimes}_v \pi_v$ be an irreducible automorphic representation of G', and let α be an automorphic form in the space of π. Recall that the theta lift of α by $\theta \in \Theta$ is defined to be:

$$\beta(g) = \int_{G' \backslash G'(\mathbb{A})} \alpha(g')\theta(gg')\,dg'. \tag{8.1}$$

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Note that this integral converges because $G' \setminus G'(\mathbb{A})$ is compact. Moreover β is an automorphic form on G. Let $\Theta(\pi)$ be the subspace of $\mathcal{A}(G)$ spanned by all such β's. We would like to investigate conditions under which $\Theta(\pi)$ is non-zero and cuspidal, when π is a non-trivial automorphic representation.

In this section, we study the cuspidality question. Hence we need to compute the constant terms of β along the unipotent radicals of the two maximal parabolic subgroups of G. Recall that $P_2 = L_2 \cdot U_2$ is the Heisenberg parabolic of G. Let $P_1 = L_1 \cdot U_1$ be the other maximal parabolic of G. We introduce the following notations:

$$U_{12} = U_1 \cap U_2, \qquad V_{12} = U_{12}/Z,$$
$$V_1 = U_1/U_1 \cap U_2, \qquad V_2 = U_2/U_1 \cap U_2.$$

We first compute:

$$\begin{aligned}\beta_{U_{12}}(g) &= \int_{U_{12}\setminus U_{12}(\mathbb{A})} \beta(ug)\,du \\ &= \int_{V_{12}\setminus V_{12}(\mathbb{A})} \int_{G'\setminus G'(\mathbb{A})} \alpha(g') \cdot \theta_Z(vgg')\,dg'\,dv \\ &= \int_{G'\setminus G'(\mathbb{A})} \alpha(g')\Big(\theta_N(gg') + \sum_{\psi\in\mathcal{O}_0} \theta_\psi(gg')\Big)\,dg'\end{aligned} \tag{8.2}$$

where $\mathcal{O}_0$ is defined in Lemma 2.7. By Proposition 6.4, $\theta_\psi(gg') = \theta_\psi(g)$, for $\psi \in \mathcal{O}_0$. Hence,

$$\beta_{U_{12}}(g) = \int_{G'\setminus G'(\mathbb{A})} \alpha(g')\theta_N(gg')\,dg'. \tag{8.3}$$

It follows that $\beta_{U_2} = \beta_{U_{12}}$. Moreover, by Proposition 6.5, we know that, as an automorphic form on M^1, θ_N is equal to $c + \theta'$, where c is a constant function of M^1 and θ' lies in the global minimal representation of M^1. The integral of α against c is zero, since α is non-constant. On the other hand, note that $\mathrm{SL}_2 \times G'$ is a commuting pair in M^1, where SL_2 is the derived group of $L_2 \cong \mathrm{GL}_2$. Hence, $\beta_{U_{12}}$, regarded as a function on SL_2, is nothing but the theta lift of α by θ'. Moreover V_1 is nothing but the unipotent radical of a Borel subgroup of SL_2. It follows that, as functions on SL_2, β_{U_1} is simply the constant term of $\beta_{U_{12}}$ along V_1. Hence, we need to study the Fourier coefficients of $\beta_{U_{12}}$ along V_1.

For this, we consider the Fourier expansion of θ' along the unipotent radical $U \cong J$ of Q. The analogues of Propositions 6.3 and 6.4 hold for θ' [GrS]. In particular, the characters of $U \setminus U(\mathbb{A})$ can be parametrized by J, and

$$\theta' = \theta'_U + \sum_{X\in J:\,\mathrm{rank}(X)=1} \theta'_{\psi_X}. \tag{8.4}$$

Moreover, for X of rank 1,

$$\theta'_{\psi_X}(cg) = \theta'_{\psi_X}(g) \tag{8.5}$$

for $c \in C_{\psi_X}(\mathbb{A})$, the stabilizer of ψ_X in $L(\mathbb{A})$. Recall that L is the derived group of the Levi factor of Q. As we have noted before, θ'_U is a constant function of L and so its integral against α is zero. Further, since there are no rank one elements of J with trace zero, we deduce easily that:

$$\beta_{U_1}(g) = \int_{V_1 \backslash V_1(\mathbb{A})} \beta_{U_{12}}(vg)\, dv = 0.$$

On the other hand, for any non-trivial character ψ of $V_1 \setminus V_1(\mathbb{A})$, let

$$\Omega_\psi = \{X \in J : \mathrm{rank}(X) = 1 \text{ and } \psi_X|_{V_1(\mathbb{A})} = \psi\}.$$

This is a G'-homogeneous space isomorphic to Ω_1. Then the ψ-Fourier coefficient of $\beta_{U_{12}}$ is:

$$\begin{aligned}(\beta_{U_{12}})_{V_1,\psi}(g) &= \int_{G' \backslash G'(\mathbb{A})} \sum_{X \in \Omega_\psi} \alpha(g')\theta'_{\psi_X}(gg')\, dg' \\ &= \int_{C(\mathbb{A}) \backslash G'(\mathbb{A})} \theta'_{\psi_{X_0}}(gg') \cdot \left(\int_{C \backslash C(\mathbb{A})} \alpha(cg')\, dc \right) dg',\end{aligned}$$

where X_0 is an element of Ω_ψ, with stabilizer C. In conclusion, we have:

Proposition 8.6 *An element of $\Theta(\pi)$ is either cuspidal or is concentrated along the Heisenberg parabolic of G. Moreover, $\Theta(\pi)$ is cuspidal if and only if the linear functional*

$$P^C \colon \alpha \mapsto \int_{C \backslash C(\mathbb{A})} \alpha(c)\, dc$$

is identically zero on π, i.e., π is not C-distinguished.

Proof The sufficiency for the vanishing of P^C is clear. As for the necessity, we argue as in [GS, Section 5, Propostion 4.5] that if the period $P^C(\alpha)$ is non-zero, then we can choose θ such that $(\beta_{U_{12}})_{V_1,\psi}$ is non-zero. See also the proof of Proposition 9.3 below. ∎

9 Non-Vanishing of Theta Lifts

In this section, we investigate when $\Theta(\pi)$ is non-zero. To do this, we study the Fourier expansion of β along U_2. First note:

Lemma 9.1 *β is zero if and only if β_Z is zero.*

Proof Suppose that $\beta_Z = 0$. Let $Z_1 \supset Z$ be the center of U_1. Then certainly $\beta_{Z_1,\psi} = 0$ for any character ψ of $Z_1 \setminus Z_1(\mathbb{A})$ which is trivial on $Z(\mathbb{A})$. But L_1 acts transitively on the non-trivial characters of $Z_1 \setminus Z_1(\mathbb{A})$. This implies that $\beta = 0$, as required. ∎

Hence, to investigate the non-vanishing of β, it suffices to consider β_Z. Let ψ_E be a character of $U_2(\mathbb{A})$ in the L_2-orbit S_E. Then, as in the last section, the ψ_E-Fourier coefficient of β_Z is given by:

$$\beta_{\psi_E}(g) = \int_{C_E(\mathbb{A})\backslash G'(\mathbb{A})} \theta_{\tilde{\psi}_E}(gg')P^{C_E}(\alpha, g')\, dg' \tag{9.2}$$

where $\tilde{\psi}_E$ is an element of Ω_E with stabilizer C_E, and

$$P^{C_E}(\alpha, g') = \int_{C_E\backslash C_E(\mathbb{A})} \alpha(cg')\, dc.$$

Now we have:

Proposition 9.3 *Assume that $\Theta(\pi)$ is cuspidal. Then the linear functional $\mathcal{F}_E\colon \beta \mapsto \beta_{\psi_E}(1)$ is identically zero on $\Theta(\pi)$ unless E is étale and totally real, in which case it is non-zero if and only if the linear functional $\alpha \mapsto P^{C_E}(\alpha) := P^{C_E}(\alpha, 1)$ is non-zero on π.*

Proof By Proposition 8.6, the assumption that $\Theta(\pi)$ is cuspidal implies that $\beta_{\psi_{E_1}} = 0$ for all $\beta \in \Theta(\pi)$. Moreover, the fact that π is non-trivial implies that $\beta_{\psi_{E_0}} = 0$, since $C_{E_0} = G'$. Also, if E is étale but not totally real, Ω_E is empty by Lemma 7.3, so that $\beta_{\psi_E} = 0$.

By (9.2), it is clear that the vanishing of P^{C_E} implies that of $\mathcal{F}_E$. The proof of the converse is along the lines of [GrS, Section 5, Proposition 4.5]. So suppose that $P^{C_E}(\alpha)$ is non-zero. Note that $P^{C_E}(\alpha, g')$ descends to a smooth function α_E on $\Omega_E(\mathbb{A})$. Choose a neighbourhood $\mathcal{N} = \prod_v \mathcal{N}_v$ of $\tilde{\psi}_E \in \Omega_E(\mathbb{A})$ with $\mathcal{N}_v$ open compact for finite v. By shrinking $\mathcal{N}$, we can ensure that $\alpha_E(x) = \alpha_E(x_\infty)$, for all $x \in \mathcal{N}$. Note that for almost all v, $\mathcal{N}_v = \Omega_E(\mathbb{Z}_v) = G'(\mathbb{Z}_v) \cdot \tilde{\psi}_E$.

Similarly, the restriction of $\theta_{\tilde{\psi}_E}$ to $M^1(\mathbb{A})$ descends to a function on $C_{\tilde{\psi}_E}(\mathbb{A}) \setminus M^1(\mathbb{A})$. As in [GrS, Section 5, (3.11)], for suitable θ, this function can be written as a product $\prod_v f_v$ where f_v is some smooth function on Ω_v which, for almost all finite places v, is equal to a distinguished function f_v^0 (corresponding to the normalized spherical vector in Π_v). Now, for almost all places v, the restriction of f_v^0 to $\Omega_{E,v}$ is the characteristic function of $\Omega_E(\mathbb{Z}_v)$ (see [GrG, Proposition 2]). On the other hand, at the other finite places, it follows from [MS, Theorem 6.1] that f_v can be any locally constant, compactly supported function on Ω_v. Since $\Omega_{E,v}$ is a closed subset of Ω_v, we can choose f_v such that its restriction to $\Omega_{E,v}$ is the characteristic function of $\mathcal{N}_v$. Hence, we can take $f^\infty = \hat{\bigotimes}_{v\neq\infty} f_v$ such that its restriction to $\Omega_E(\mathbb{A}^\infty)$ is the characteristic function of $\mathcal{N}^\infty = \prod_{v\neq\infty} \mathcal{N}_v$. Thus, up to multiplication by a non-zero constant, we have:

$$\mathcal{F}_E(\beta) = \int_{\Omega_E(\mathbb{R})} f_\infty(x)\alpha_E(x)\, dx.$$

Now as in [GrS], the fact that $\Omega_E(\mathbb{R})$ is compact implies, by the Stone-Weierstrass theorem, that the restrictions of the functions f_∞ span a dense subspace of the space of smooth functions on $\Omega_E(\mathbb{R})$. Hence the above integral is non-zero for a suitable choice of f_∞. This completes the proof of the proposition. ∎

By Propositions 8.6 and 9.3, we have:

Theorem 9.4 *If π is a non-trivial automorphic representation of G', then $\Theta(\pi)$ is non-zero and cuspidal if and only if the following two conditions hold:*

(i) *π is not C-distinguished.*
(ii) *π is C_E-distinguished for some totally real étale cubic algebra E.*

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