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REDUCING TOWERS OF PRINCIPAL FIBRATIONS

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Consider a tower of principal fibrations

$$\begin{array}{ccccccc} B & \longleftarrow & E_2 & \longleftarrow \cdots & E_n & \longleftarrow & E_{n+1} \\ \downarrow & & \downarrow & & \downarrow & & \\ R_1 & & R_2 & & R_n & & \end{array}$$

That is, E_{i+1} is the pullback of $E_i \to R_i$ and the path fibration $PR_i \to R_i$. The question arises as to whether or not the tower can be shortened, that is, whether or not $E_{n+1} \to B$ is fiber homotopically equivalent to a nice fibration $E \to B$. If "nice" is taken to mean "principal" then sufficient conditions are known. They involve connectivity assumptions on the E_i. In this paper "nice" is taken to mean "D-relatively principal" for some space D. Relative principal fibrations are more general than principal fibrations. Their lifting properties are studied in [7]. They enjoy some but not all of the nice properties of principal fibrations. The assumptions on the tower above which imply that $E_{n+1} \to B$ is nice are weaker than the assumptions showing it to be principal—as expected, since the conclusion is weaker.

One application of the sufficient conditions is a kind of represetation theorem for certain fibrations. Suppose $F \to E \to B$ is a fibration, F, E, B, having the homotopy type of CW complexes, and $\Pi_i(F) = 0$ except possibly when $s \leq i < 2s - 1$. Then it is shown that $E \to B$ is a relatively principal fibration. No connectively assumptions are made on B. It follows that if $E \to B$ is any fibration with an n-connected fiber then the $2n$'th stage of its Moore-Postnikov factorization is a relatively principal fibration.

In the first section a twisted suspension operation is studied. In the

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second section this operation is used to give sufficient conditions for reducing a two stage tower. In the last section sufficient conditions are given for reducing an arbitrary tower and the above mentioned representation theorem is proved.

1. A Suspension Operation for Relatively Principal Fibrations.

First, we recall a few definitions from [6]. Let $\mathrm{Top}\,(u: C \to D)$ be the category of triples $(X, \check{x}, \hat{x})$ where $\check{x}: C \to X$, $\hat{x}: X \to D$, $\hat{x}\check{x} = u$ and all of this takes place in $\mathrm{Top} = \mathrm{Top}\,(\emptyset \to \mathrm{pt}) =$ category of topological spaces and continuous functions. Write $\mathrm{Top}\,(D = D)$ for $\mathrm{Top}\,(\mathrm{id}: D \to D)$. It has all of the basic properties of $\mathrm{Top}\,(\mathrm{pt} = \mathrm{pt}) =$ the category of pointed spaces and maps. In particular if $Z \in \mathrm{Top}\,(D = D)$ then there is a canonical principal fibration (path-loop fibration) $\Omega_D Z \to P_D Z \to Z$. (The properties of $\mathrm{Top}\,(C \to D)$ were established in my 1966 thesis [5] and outlined in the published abstract. A couple of years later similar notions were described by others.)

Now, let $X \in \mathrm{Top}\,(C \to D)$ and $f: X \to Z \in \mathrm{Top}\,(C \to D)$ where $C \to Z$ is $C \to D \to Z$. Then if $P = P_f \to X$ is the pullback of $P_D Z \to Z$ and f then it is called a D-relative principal fibration. Suppose that $L \in \mathrm{Top}\,(D = D)$. We wish to define a secondary operation $\rho: [Z, L]_D^D \to [P, \Omega_D L]_D^C$. The operation can be treated in a fairly direct manner. However, in the interest of clarity and unity (with an operation in [8]) we will start from an abstract level.

Consider the following data (Δ).

$$(\Delta) \qquad \{\delta_s: H_t \longrightarrow G\} \qquad S \xrightarrow{\beta} T \xrightarrow{\alpha} (U, u_0)$$

Here $\{\delta_s: H_t \to G\}$, $t \in T$, $s \in \beta^{-1}(t)$, is a family of group homomorphisms. S is a G-set, $\beta: S \to T$, $\alpha: T \to U$ are set maps, $u_0 \in U$, $\alpha^{-1}(u_0) = \beta(S)$; each $\beta^{-1}(t)$ is a G-subset of S and G acts transitively on it; $\delta_s(H_t) = G_s = \{g \in G \mid gs = s\} =$ the stability subgroup of s. $\gamma_s: G \to S$ is defined by $\gamma(g) = gs$. This situation is exactly the one occurring at the bottom of exact homotopy sequences. A prototype example can be obtained as follows. Let S be a G-set and $T = S/G$, $U = \{u_0\}$, $H_t = G_{s(t)}$ where $s(t)$ is a chosen element of $\beta^{-1}(t)$. For $s \in \beta^{-1}(t)$, $\delta_s: H_t \to G$ is defined by $\delta_s(g) = \bar{g} g \bar{g}^{-1}$ where $s = \bar{g} s(t)$.

Given the data (Δ), one can select $s \in S$ and form the sequence

$$(\Delta_s) \qquad \begin{array}{ccccccccc} H_t & \xrightarrow{\delta_s} & G & \xrightarrow{\gamma_s} & S & \xrightarrow{\beta} & T & \xrightarrow{\alpha} & U \\ 1 & & 1 & & s & & t = \beta s & & u_0 \end{array}$$

Then it is easily checked that this is an exact sequence of pointed sets.

A morphism $\Delta \to \Delta'$ is

$$\begin{array}{ccc} H(t) & \xrightarrow{\delta(s)} & G \\ \downarrow m_t & & \downarrow k \\ H'(t') & \xrightarrow{\delta(s)} & G' \end{array} \qquad \begin{array}{ccccc} S & \longrightarrow & T & \longrightarrow & U \\ \downarrow h & & \downarrow g & & \downarrow f \\ S' & \longrightarrow & T' & \longrightarrow & U' \end{array}$$

where the diagram is commutative, $gt = t'$, $hs = s'$, m_t and k are homomorphisms, and $f(u_0) = u_0'$. For each $s \in S$ there is induced a morphism $\Delta(s) \to \Delta(s')$.

1.1 Definition. Let $\Delta \to \Delta'$ be given. Let $t \in T$, $t' = gt$, $\alpha t = u_0$. Suppose $s' \in S'$ with $\beta s' = t'$. Define

$$\Gamma(s'\,;t) = \{g' \in G' \mid g's' \in h\beta^{-1}(t)\}\ .$$

1.2 Theorem. (1) *$\Gamma(s'\,;t)$ is a double coset of $(kG, G(s'))$, i.e., $g' \in \Gamma(s'\,;t)$ implies $\Gamma(s'\,;t) = (kG)g'(G(s'))$.*

(2) *$\Gamma(g's'\,;t)g' = \Gamma(s'\,;t)$, all $g' \in G'$*

(3) *kG normal in G' implies $\Gamma(g's'\,;t) = \Gamma(s'\,;t)$, all $g' \in k(G)$*

Proof. I'll prove (1) only. The interesting thing is that s' needn't be in the image of h. Pick s_0 with $\beta s_0 = t$ so $\beta^{-1}(t) = Gs_0$ and $h\beta^{-1}t = h(Gs_0) = (kG)hs_0$. Let $s_0' = hs_0$. Suppose $g', g'' \in \Gamma(s'\,;t)$ and $g's' = k(g_1)s_0'$, $g''s' = k(g_2)s_0'$. Thus $kg_2^{-1}g''s' = kg_1^{-1}g's'$, hence $(kg_1^{-1}g')^{-1}kg_2^{-1}g'' \in G(s')$ implying $g'' \in kg_2kg_1^{-1}g'G(s') \subset kGg'G(s)$. Conversely, $\bar g \in kGg'G(s')$ implies $g'^{-1}kg^{-1}\bar g \in G(S')$ (some g) implying $\bar g s' = kgg's' = kg\,kg_1s_0'$ and hence $\bar g \in \Gamma(s'\,;t)$.

As a first example we take up the operation of [8, Section 2]. Let $F \xrightarrow{f} E \xrightarrow{g} B$ be a fibration in Top (pt), $L \in \mathrm{Top}\,(D = D)$ and $h: B \to L \in \mathrm{Top}\,(\mathrm{pt} \to D)$, putting $F \to E \to B$ into Top (pt $\to D$). Consider

$$(F, F) \longrightarrow (E, F) \longrightarrow (B, \mathrm{pt}) \longrightarrow (L, L)\ .$$

Theorem 3.4 of [7] and the obvious naturality give the following situation where [ ] means $[\ ]^{\mathrm{pt}}$.

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$$\begin{array}{ccc} [(E,F);(\hat\Omega B,\hat\Omega \,\mathrm{pt})]_B & \longrightarrow & [(E,F);(\Omega L,\Omega L)]_D \\ \downarrow & & \downarrow \\ [(F,F);(\hat\Omega B,\hat\Omega \,\mathrm{pt})]_B & \longrightarrow & [(F,F);(\Omega L,\Omega L)]_D \end{array}$$

$$\begin{array}{ccccc} [(E,F);(P,\Omega T)]_D & \longrightarrow & [(E,F);(B,\mathrm{pt})]_D & \longrightarrow & [(E,F);(L,L)]_D \\ \downarrow & & \downarrow & & \downarrow \\ [(F,F);(P,\Omega T)]_D & \longrightarrow & [(F,F);(B,\mathrm{pt})]_D & \longrightarrow & [(F,F);(L,L)]_D \end{array}$$

Here $P = P_h$, $T = \check{k}^{-1}(d_0)$ where $\check{k}: L \to D$, ΩT is the ordinary loop space, ΩL means $\Omega_D L$ and $\hat\Omega$ means $\hat\Omega_D$. This simplifies to

$$\begin{array}{ccc} 0 \longrightarrow [E;\Omega L]_D & & [(E,F);(P,\Omega T)]_D \longrightarrow [(E,F);(B,\mathrm{pt})] \longrightarrow [E;L]_D \\ \downarrow & & \downarrow \\ 0 \longrightarrow [F;\Omega T] & & [F;\Omega T] \qquad \longrightarrow 0 \end{array}$$

Take $s' = * : F \to \Omega T$ and define $\sum : [(E,F);(B,\mathrm{pt})]_D \to [F;\Omega T]$ by $\sum(g) = \Gamma(s';g)$. It follows from 1.2 that $\sum g$ is a coset of $i^*[E,\Omega L]_D$ in $[F,\Omega T]$. This is the same definition as in [8].

Now we return to the situation at the beginning of the section. Let $f: X \to Z \in \mathrm{Top}\,(C \to D)$ and $h: Z \to L \in \mathrm{Top}\,(D = D)$. We have

$$(P_f, P_f) \to (X, P_f) \to (Z, D) \to (L, L)$$

From [7] we get the following.

$$\begin{array}{ccc} [(X,\ P_f);(\hat\Omega Z,D)]_Z & \longrightarrow & [(X,\ P_f);(\Omega L,\Omega L)] \\ \downarrow & & \downarrow \\ [(P_f,P_f);(\hat\Omega Z,D)]_Z & \longrightarrow & [(P_f,P_f);(\Omega L,\Omega L)] \end{array}$$

$$\begin{array}{ccccc} [(X,\ P_f);(P,\Omega L)]_D & \longrightarrow & [(X,P_f);(Z,D)]_D & \longrightarrow & [(X,\ P_f);(L,L)]_D \\ \downarrow & & \downarrow & & \downarrow \\ [(P_f,P_f);(P,\Omega L)]_D & \longrightarrow & [(X,P_f);(Z,D)]_D & \longrightarrow & [(P_f,P_f);(L,L)]_D \end{array}$$

This simplifies to the following.

$$\begin{array}{ccc} [(X,P_f);(\hat\Omega Z,D)]_Z & \longrightarrow & [X,\ \Omega L]_D \\ & & \downarrow \\ 0 & \longrightarrow & [P_f,\Omega L]_D \end{array}$$

$$\begin{array}{ccccc} [(X,P_f),(P,\Omega L)]_D & \longrightarrow & [(X,P_f);(Z,D)] & \longrightarrow & (X,L)]_D \\ \downarrow & & & & \\ [P_f,\Omega L]_D & \longrightarrow & 0 & & \end{array}$$

1.3 DEFINITION. Let $s': P_f \to D \to \Omega_D L$ be the composition of the structure maps. Define $\rho: [Z,L]_D^D \to [P_f,\Omega_D L]_D^C$ by $\rho(h) = \Gamma(s';f)$.

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It follows from Theorem 1.2 that $\rho(h)$ is a coset of $p^*[X; \Omega_D L]$ in $[P_f; \Omega_D L]_D$. More concretely, consider

$$\begin{array}{ccccc} P_f & \longrightarrow & X & \xrightarrow{f} & Z \\ \downarrow w & & \downarrow & & \downarrow h \\ \Omega_D L & \longrightarrow & D & \longrightarrow & L \end{array}$$

Let $H: X \to P_D L$ be a homotopy of $\check{k}\hat{x}$ to hf in $\mathrm{Top}\,(C \to D)$. Define $w(x, m) = (Ph)m - H(x)$. Then $w \in \rho(h)$ and it is, in fact, a typical element.

Now suppose that $\phi: \Pi \to \mathrm{Aut}\, G$ is a homomorphism where Π is a group and G is an abelian group. Suppose that $D \to K(\Pi, 1)$ is given, defining a local coefficient system G_ϕ on D and hence on P, X, and Z. Use these coefficients and form the following diagram.

$$\begin{array}{ccccc} \cdots H^t(X, C) & \longrightarrow & H^t(P, C) & \longrightarrow & H^{t+1}(X, P) \\ \uparrow & & \uparrow & & \uparrow \\ \cdots H^t(Z, C) & \longrightarrow & H^t(D, C) & \longrightarrow & H^{t+1}(Z, D) \end{array}$$

$$\begin{array}{ccccc} \longrightarrow H^{t+1}(X, C) & \longrightarrow & H^t(P, C) & \longrightarrow & \cdots \\ \uparrow & & \uparrow & & \\ \longrightarrow H^{t+1}(Z, C) & \longrightarrow & H^t(D, C) & \longrightarrow & \cdots \end{array}$$

Here, and elsewhere in cohomology, a "pair" (X, A) is to be interpreted as the mapping cone of whatever natural map $A \to X$ is indicated by the context.

1.4 DEFINITION. $R: H^{t+1}(Z, D; G_\phi) \to H^t(P, C; G_\phi)$ is defined by $R = S^{-1}\bar{f}^*$.

R is a secondary cohomology operation and its indeterminacy is $p^*H^{t-1}(X, C) \subset H^{t-1}(P, C)$. Now take $L' = L_\phi(G, t+1)$, the classifying space for local coefficient cohomology, and $L = D \times_K L'$. What follows is also valid if L' is replaced by a product over K of such spaces. We have

$$\begin{array}{ccc} H^{t+1}(Z, D) & \xrightarrow{R} & H^t(P, C) \\ \| & & \| \\ [(Z, D); (L, D)]_D & & \| \\ \| & & \| \\ [Z, L]_D^D & \xrightarrow{\rho} & [P, \Omega_D L]_D^C \end{array}$$

1.5 THEOREM. $R = \rho$.

Proof. Let $h: Z \to L$. The indeterminacies of $R(h)$ and $\rho(h)$ are the same so it suffices to find a common element. Consider the following diagram.

$$\begin{array}{ccccc} P & \longrightarrow & X & \xrightarrow{f} & Z \\ \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle h} \\ \Omega_D L & \longrightarrow & P_D L & \longrightarrow & L \end{array}$$

H is a null homotopy of hf (i.e. a homotopy of $\check{k}\hat{x}$ to hf) and w is the naturally induced map. It is convenient to think of the bottom row as a principal fibration. First of all, it is clear that R is natural for such maps of relative principal fibrations. Secondly, note that if $\lambda(t+1) \in H^{t+1}(L, D; G_\phi)$ is a fundamental class for L then $\lambda(t) \in R(\lambda(t+1))$. Hence $R(h) = R(h^*\lambda(t+1)) \supset w^*R(\lambda(t)) \ni w$. However, it is immediate from the defining diagram for ρ that $w \in \rho(h)$. Q.E.D.

This proof should be compared to the proof that $\sum = \sigma$ in [8]. With some slight awkwardness it would be possible to define a homotopy operation including both $\sum$ and ρ as special cases and prove a theorem which would specialize to both Theorem 1.5 and Theorem 3.1 of [8]. Both operations can be viewed as versions of the bracket operation of Section 5 of [6].

We are interested in finding sufficient conditions for $\rho(= R)$ to be onto. Now assume $\hat{z}: Z \to D$ is a fibration in Top (pt) and that its fiber is $(n-1)$-connected and that the map $\hat{x}: X \to D$ is b-connected.

1.6 THEOREM. *$t \leq \min(2n-3, n+b-1)$ implies R onto,*

$$R: H^t(Z, D; G_\phi) \longrightarrow H^{t-1}(P, C; G_\phi)\ .$$

Proof. In the defining diagram of R above it suffices to show $\bar{f}^*$ is isomorphic. Since $Z \to D$ is a fibration in Top (pt) so are $P_D Z \to Z$ and $P \to X$. We have

$$\begin{array}{ccc} P & \longrightarrow & P_D Z \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

The fibers in Top (pt) of $P \to X$ and $P_D Z \to Z$ are the same. It follows

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from the 3×3 lemma (Nomura [10]) that the "fibers" of $P \to P_D Z$ and $X \to Z$ are homotopically equivalent and so the relative Serre theorem [8] can be applied to $(X, P) \to (Z, P_D Z)$ and hence to $(X, P) \to (Z, D)$. It is easy to see that the Top(pt) "fiber" of $X \to Z$ is $\min(n-2, b)$-connected and $H^i(Z, D; —) = 0$ for $i < n$. The relative Serre theorem implies that $\bar{f}^*$ is isomorphic for $t + 1 \leq \min(2n - 2, b + n)$. Q.E.D.

Now suppose there is a commutative diagram

$$\begin{array}{ccccccl} P & \longrightarrow & X & \longrightarrow & Z & & \text{in Top}\,(C \to D) \\ \uparrow u_3 & & \uparrow u_2 & & \uparrow u_1 & & \\ P' & \longrightarrow & X' & \longrightarrow & Z' & & \text{in Top}\,(C' \to D') \end{array}$$

This gives (coefficients G_ϕ)

$$\begin{array}{ccc} H^{t+1}(Z, D) & \longrightarrow & H^t(P, C) \\ \downarrow u_1^* & & \downarrow u_3^* \\ H^{t+1}(Z', D') & \longrightarrow & H^t(P', C') \end{array}$$

So, in general, $u_3^* R(g) \subset R' u_1^*(g)$.

1.7 THEOREM. *Let $w \in H^t(P, C)$. Assume $u_3^* w \in R'(u_1^* g)$ for some g. Assume $u^* : H^{t+1}(X, P) \to H^{t+1}(X', P')$ is onto. Then $w \in R(g)$.*

Proof. $u^* f^* g = f'^* u_1^* g = \delta' u_3^* w = u^* \delta w$. Hence $f^* g = \delta w$ and $w \in R(g)$.

Consider now an ordinary principal fibration, in Top$(C \to \text{pt})$, $P(u) \to X \to Z$ where $u: X \to Z$. If a map $\hat{x}: X \to D$ is given then a twisted suspension operation $\rho: H^n(D \times Z, D; G) \to H^{n-1}(P, C; G)$ can be defined as follows. First consider $u' = u(\hat{x}, u): X \to D \times Z$ and form the D-relative principal fibration $P(u') \to X \to D \times Z$. Then $P(u') = P(u)$. (Here it might be better to write $P_D(u')$ instead of $P(u')$.) Now suppose $N \in \text{Top}(\text{pt})$ and set $L = D \times N$.

$$\begin{array}{ccc} [D \times Z; D \times N]_D^D & \xrightarrow{\rho} & [P; \Omega_D(D \times N)]_D^C \\ \| & & \| \\ [(D \times Z, D); (N, \text{pt})] & \longrightarrow & [P; \Omega N]^C \\ \| & & \| \\ H^n(D \times Z, D; G) & \longrightarrow & H^{n-1}(P, C; G) \end{array}$$

The last two rows give the twisted suspension operation in terms of the original data. The last row assumes $N = K(G, n)$. This transference technique can be generalized as follows. Given a K-relative principal fibration, a map $\hat{x}: X \to D$, and $N \in \mathrm{Top}(K = K)$, we get a twisted operation

$$\begin{array}{ccc} [(D \times_K Z, D); (N, \mathrm{pt})]_K^{\mathrm{pt}} & \longrightarrow & [P, \Omega_K N]_N^C \\ \| & & \| \\ H^n(D \times_K Z, D; G_\phi) & \longrightarrow & H^{n-1}(P, C; G_\phi) \end{array}$$

The last row assumes $N = L_\phi(G, n)$ and $K = K(\Pi, 1)$.

2. Reducing Two Story Towers.

Suppose that the following tower in $\mathrm{Top}(C \to D)$ is given.

$$\begin{array}{ccccc} B & \longleftarrow & E_2 & \longleftarrow & E_3 \\ \downarrow{\scriptstyle k_1} & & \downarrow{\scriptstyle k_2} & & \\ L_1 & & L_2 & & \end{array}$$

Here $L_i \in \mathrm{Top}(D = D)$ and E_i is the D-relative principal fibration induced by k_i. We are interested in finding $M \in \mathrm{Top}(D = D)$ and $f: B \to M$ so that $E_3 \to B$ is homotopically equivalent in $\mathrm{Top}(C \to B)$ to $P_f \to B$ (the D-relatively principal fibration induced by f). The particular example to keep in mind is the following one: $L_1' = L_\phi(G, n)$, $L_2' = L_\psi(H, t)$, $L_i = D \times_K L_i'$, $D \to K = K(\Pi, 1)$ is a fixed map, Π is a group, G and H are abelian groups and $\phi: \Pi \to \mathrm{Aut}(G)$ and $\psi: \Pi \to \mathrm{Aut}(H)$ are homomorphisms. The main homotopy theoretic result of this section is the following one.

2.1 THEOREM. *Assume* $L_2 = \Omega_D J$ *and* $k_2 \in \mathrm{Im}\,\rho: [L_1; J]_D^D \to [E_2; L_2]_D^C$. *Then there is an* $M \in \mathrm{Top}(D = D)$ *and* $f: B \to M \in \mathrm{Top}(C \to D)$ *such that* $P_f \to B$ *and* $E_3 \to B$ *are homotopically equivalent in* $\mathrm{Top}(C \to B)$.

The theorem will be deduced from a couple of lemmas. First consider the following diagram in $\mathrm{Top}(C \to D)$.

$$\begin{array}{ccccc} P & \longrightarrow & X & \longrightarrow & Z \\ \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ P' & \longrightarrow & X' & \longrightarrow & Z' \end{array} \qquad H: bf \sim f'a$$

Here P and P' are the induced D-relative principal fibrations. $H: X \to W_D Z'$ is a given homotopy and $w = w_H: P \to P'$ is defined by $w(x, m) = (a(x), (Pb)m + H(x))$. If $C = D = \mathrm{pt}$ then the properties of w are known

[Nomura, 9]. These known results can be generalized to the present setting without difficulty. In particular, the following lemma can be proved.

2.2 LEMMA. *If a and b are homotopy equivalences then so is w. If, in addition, $a = \mathrm{id}: X \to X$ then w is a homotopy equivalence in* $\mathrm{Top}\,(C \to X)$, *i.e., a fiber homotopy equivalence in* $\mathrm{Top}\,(C \to D)$.

Now consider the following commutative diagram in $\mathrm{Top}\,(C \to D)$.

$$\begin{array}{ccccc} P_k & \xrightarrow{w} & P_h & & \\ \downarrow & & \downarrow & & \\ P_f & \xrightarrow{p_2} & X & \xrightarrow{f} & Z \\ \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle h} & & \| \\ P_g & \xrightarrow{p_1} & Y & \xrightarrow{g} & Z \end{array}$$

Assume that $g: Y \to Z \in \mathrm{Top}\,(D = D)$ so that $P_g \in \mathrm{Top}\,(D = D)$, $gh = f$, $k = (h, \mathrm{id})$, so $hp_2 = p_1 k$ and $w = (p_2, Pp_1)$ here.

2.3 LEMMA. *The map w is a homotopy equivalence. In fact w is a fiber homotopy equivalence of $P_k \to X$ to $P_h \to X$.*

Proof. If $C = D = \mathrm{pt}$ this is a result of Nomura [9]. His proof carries over to the present setting without difficulty. The lemma can also be deduced from a general 3×3 lemma.

We will now combine the previous two lemmas to get a proof of the main theorem. Consider the following diagram.

$$\begin{array}{ccccccccccc} P_r & \xrightarrow{a} & P_r & \xrightarrow{w} & P_s & \xrightarrow{b} & P_f & & & & \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\ P_k & = & P_k & = & P_k & \longrightarrow & B & \longrightarrow & L & & \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle -r} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle f} & & \| & & \\ \Omega J & \xrightarrow{-1} & \Omega J & \xrightarrow{n} & P_p & \longrightarrow & P_u & \xrightarrow{p} & L & \xrightarrow{u} & J \end{array}$$

Here $L = L_1$, $k = k_1$, $r = k_2 \in \rho(u)$. r is defined by means of some null-homotopy of uk. Use the same null-homotopy to define f. n is the natural homotopy equivalence and $s = (f, \mathrm{id})$ and $n(-r) \sim s$ by H, say, as is readily checked; all other squares are strictly commutative; the

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map w is due to the homotopy H. b is a homotopy equivalence in $\mathrm{Top}\,(C \to B)$ by Lemma 2.3 and a and w are homotopy equivalences in $\mathrm{Top}\,(C \to P_k)$ by Lemma 2.2. Hence bwa is a homotopy equivalence in $\mathrm{Top}\,(C \to B)$. Take $M = P_u$. This completes the proof of Theorem 2.1.

If $C = D = \mathrm{pt}$, then 2.1 is related to Lemma 1.6 of Gershenson [2]. In order to apply 2.1 we need some conditions which guarantee that k_2 is in the image of ρ. For simplicity we consider only the specific situation described at the beginning of the section.

$$\begin{array}{ccccc} B & \longleftarrow & E_2 & \longleftarrow & E_3 \\ \downarrow & & \downarrow & & \\ D \times_K L_\phi(G, n) = L_1 & & D \times_K L_\psi(H, t) = L_2 & & \end{array}$$

Here $n \leq t$. The results below can, however, be stated in more general terms and proved by the same methods. In particular $L(G, n)$ can be replaced by $\Pi_K L(G_i, n_i)$ and $L(H, t)$ by $\Pi_K L(H_j, t_j)$. It is now assumed that all spaces involved have the homotopy type of CW complexes.

2.4 COROLLARY. *Assume* $B \to D$ *is* b*-connected and that* $t \leq \min(2n - 3, n + b - 1)$. *Then there is an* $M \in \mathrm{Top}\,(D = D)$ *and* $f: B \to M \in \mathrm{Top}\,(C \to D)$ *such that* $P_f \to B$ *and* $E_3 \to B$ *are homotopically equivalent in* $\mathrm{Top}\,(C \to B)$.

Proof. Theorem 2.1, 1.5, and 1.6 give this. Here

$$J = D \times_K L_\psi(H, t + 1) .$$

Now the above diagram is enlargened as follows.

$$\begin{array}{ccccc} B' & \longleftarrow & E_2' & & \\ \downarrow & & \downarrow & & \\ B & \longleftarrow & E_2 & \longleftarrow & E_3 \\ \downarrow & & \downarrow & & \\ L_1 & & L_2 & & \end{array}$$

E_2' is the pullback of $E_2 \to B$. We have

$$\begin{array}{ccccc} L_1 & \longleftarrow & B & \longleftarrow & E_2 \\ \| & & \uparrow u_2 & & \uparrow u_3 \\ L_1 & \longleftarrow & B' & \longleftarrow & E_2' \end{array}$$

and hence

$$\begin{array}{ccc} [L_1, J]^D_D & \xrightarrow{\rho} & [E_2, L_2]^C_D \\ \| & & \downarrow \\ [L_1, J]^D_D & \xrightarrow{\rho'} & [E'_2, L_2]^C_D \end{array}$$

Let k'_2 be the composition $E'_2 \to E_2 \to L_2$.

2.5 THEOREM. *Assume* $k'_2 \in \operatorname{Im} \rho'$ *and that* $t \leq n + b$ *where* $b =$ *connectivity of* $B' \to B$. *Then the conclusion of* 2.4 *is valid.*

Proof. It suffices, by 2.1, to show $k_2 \in \operatorname{Im} \rho$. Theorem 1.7 will be used for this purpose. Recall, Theorem 1.5, $\rho = R$ and $\rho' = R'$, so:

$$\begin{array}{ccc} H^{t+1}(L_1, D; H_\psi) & \longrightarrow & H^t(E_2, C; H_\psi) \\ \| & & \downarrow \\ H^{t+1}(L_1, D; H_\psi) & \longrightarrow & H^t(E'_2, C; H_\psi) \end{array}$$

and $k_2 \in H^t(E_2, C)$, $u_3^* k_2 = k'_2 \in \operatorname{Im} R'$. We must only establish that $u^*: H^{t+1}(B, E_2) \to H^{t+1}(B', E'_2)$ is a monomorphism. Since $L_1 \to D$ is a fibration in Top (pt), so are $E_2 \to B$ and $E'_2 \to B'$ and these last two have the same fiber. It follows from the 3×3 lemma in Top (pt) that $B' \to B$ and $E'_2 \to E_2$ have homotopically equivalent "fibers" and hence that the relative Serre theorem [8] can be applied to $(B', E'_2) \to (B, E_2)$. If it can be shown that $H^p(B, E_2; H^q(F; H)\tilde{}_\psi) = 0$ for $p < m$ or $0 < q < m'$ and $t + 1 \leq m + m'$ it will then follow that u^* is monomorphic. By assumption F is b-connected so $m' = b + 1$. Now consider $H^p(L_1, D; \Gamma) \to H^p(B, E_2; \Gamma)$ where Γ is any local coefficient system. Just as above we see that the "fibers" of $B \to L_1$ and $E_2 \to D$ are homotopically equivalent, so the relative Serre spectral sequence can be applied. But $H^p(L_1, D; \Gamma) = 0$ for $p < n$ so it follows that the same is true of $H^p(B, E_2; \Gamma)$. Hence $m = n$ and $u^*: H^i(B, E_2; H_\psi) \to H^i(B', E'_2; H_\psi)$ is isomorphic for $i \leq n + b - 1$ and monomorphic for $i \leq n + b + 1$. Q.E.D.

Some special cases are of interest. First note that if $B = B'$ then 2.5 reduces to 2.1. Next, take $B' = E_3$.

2.6 COROLLARY. *Suppose* $B' = E_3$, $k'_2 \in \operatorname{Im} \rho'$. *Then if* $t \leq 2n - 2$ *the conclusion of* 2.4 *is valid.*

Proof. Since $L_i \to D$ is a fibration in Top (pt), $i = 1, 2$, it follows that $E_3 \to B$ is also and is an extension of $K(G, n-1)$ by $K(H, t-1)$. Since $n \leq t$, we see that $E_3 \to B$ is $(n-2)$-connected and the corollary follows from Theorem 2.5.

Next, take $C = B' = D$. Then there is the following diagram in Top $(D = D)$.

$$\begin{array}{ccccc} D & \longleftarrow & \Omega_D L_1 & & \\ \downarrow & & \downarrow & & \\ B & \longleftarrow & E_2 & \longleftarrow & E_3 \\ \downarrow & & \downarrow & & \\ L_1 & & L_2 & & \end{array}$$

Here $\Omega_D L_1 = \Omega_D(D \times_K L(G, n)) = D \times_K L(G, n-1)$. Let k_2' be the composite $\Omega_D L_1 \to E_2 \to L_2$.

2.7 COROLLARY. *Suppose* $k_2' \in \operatorname{Im} \Omega_D : [L_1, J] \to [\Omega_D L_1, L_2]$. *Suppose also that* $B \to D$ *is* r*-connected and* $t \leq n + r - 1$. *Then there is an* $f : B \to M \in \text{Top}\,(D = D)$ *such that* $P_f \to B$ *and* $E_3 \to B$ *are homotopically equivalent in* Top $(D \to B)$.

Proof. Consider

$$\begin{array}{ccccc} L_1 & \longleftarrow & B & \longleftarrow & E_2 \\ \| & & \uparrow & & \uparrow \\ L_1 & \longleftarrow & D & \longleftarrow & \Omega_D L_1 \end{array}$$

It is clear from the definition of ρ that $\rho = \Omega_D$. Also, since $B \to D$ is a retraction and is r-connected it follows that $D \to B$ is $(r-1)$-connected. Hence 2.7 follows from 2.5.

Consider now a tower of ordinary principal fibrations is Top $(C \to \text{pt})$.

$$\begin{array}{ccccc} B & \longleftarrow & E_2 & \longleftarrow & E_3 \\ \downarrow & & \downarrow & & \\ R_1 & & R_2 & & \end{array}$$

Assume $\hat{b} : B \to D$ is given and $R_2 = \Omega R_2'$. Then a sufficient condition for $E_3 \to B$ to be D-relatively principal is that $r_2 \in \operatorname{Im} \rho : [(D \times R_1, D); (R_2', \text{pt})]$

$\to [E_2, R_2]$. ρ is the operation discussed at the end of Section 1. This follows from 2.1 because we can consider

$$\begin{array}{ccccc} B & \longleftarrow & E_2 & \longleftarrow & E_3 \\ \Big\downarrow{\scriptstyle (\hat{b}, r_1)} & & \Big\downarrow{\scriptstyle (\hat{b}, r_2)} & & \\ D \times R_1 & & D \times R_2 & & \end{array}$$

and it is a tower of D-principal fibrations so 2.1 applies. More generally, one can transfer from Top $(C \to K)$ to Top $(C \to D)$ and this is what was done implicitly in 2.4.

3. Reducing Towers

I want to give a version of 2.1 for higher towers. Consider the following tower of D-relatively principal fibrations.

$$\begin{array}{ccccccccc} B & \longleftarrow & E_2 & \longleftarrow & \cdots & \longleftarrow & E_n & \longleftarrow & E_{n+1} \\ \Big\downarrow{\scriptstyle k_1} & & \Big\downarrow{\scriptstyle k_2} & & & & \Big\downarrow & & \\ L_1 & & L_2 & & & & L_n & & \end{array}$$

In this section write $P(f)$ for P_f. For the operation ρ of 1.3 write $\rho(f:h)$ instead of $\rho(h)$. In the proof of Theorem 2.1 denote f by k_2'. Thus we have the diagram

$$\begin{array}{ccccc} E_3 & & & & \\ \Big\downarrow & & & & \\ E_2 & \xrightarrow{\ominus} & \Omega M_2 & \longrightarrow & P(u) \\ \Big\downarrow & & \nearrow{\scriptstyle k_2'} & & \Big\downarrow \\ B & & \xrightarrow{k_1} & & L_1 \longrightarrow M_1 \end{array}$$

where Ω means Ω_D and $\Omega M_2 = L_2$. The conclusion of 2.1 is that $E_3 \to B$ and $P(k_2')$ are homotopy equivalent in Top$(C \to B)$. Identify E_3 and $P(k_2')$ by the equivalence of the proof of Theorem 2.1. Consider the following statements (A_i) for $i \geq 2$.

$$(A_i) \qquad \begin{array}{ll} k_i \in \rho(k_{i-1}'; v_{i-1}) & \text{where } v_{i-1}: P_{i-1} \to M_{i-2}\,, \\ P_{i-1} = P(v_{i-2})\,, & \Omega M_{i-2} = L_{i-1}\,. \end{array}$$

If $i = 2$ interpret P_i as L_1.

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3.1 THEOREM. *Assume* (A_i) *for* $2 \leq i \leq n$. *Then* $E_{n+1} \to B$ *is equivalent to* $P(k'_n) \to B$ *in* Top $(C \to B)$.

Proof. If A_{i-1} is true then $P_{i-1} = P(v_{i-2})$ can be formed and $v_{i-1}: P_{i-1} \to M_{i-2}$ with $k_i \in \rho(k'_{i-2}; v_{i-2})$ can be selected. We can form k'_{i-1} and identify $E_{i-1} \to B$ with $P(k'_{i-1}) \to B$. Hence A_i makes sense. A_2 is true so 3.1 follows from 2.1 by induction.

From now on assume all spaces have homotopy type of CW complexes.

3.2 THEOREM. *Assume* $L_i = D \times_K L_\phi(H_i, t_i)$ *where* $D \to K = (\Pi, 1)$ *is given. Assume* $t_1 \leq t_2 \leq \cdots \leq t_n$ *and* $t_i \leq \min(2t_1 - 3, t_1 + b + 1)$ *where* $b =$ *connectivity of* $B \to D$. *Then there is an* $M \in$ Top $(D = D)$ *and* $f: B \to M \in$ Top $(C \to D)$ *with* $P(f) \to B$ *and* $E_{n+1} \to B$ *homotopically equivalent in* Top $(C \to B)$.

Proof. Assume (A_{i-1}) has been established. $H^j(L_1, D; —) = 0$, $j < t_1$, plus the Serre spectral sequence gives $H^j(P_m, D; —) = 0$, $j < t_1$, $m \leq i - 1$. The proof of Theorem 1.6 now gives (A_i). Q.E.D.

3.3 COROLLARY. *Assume* $L_i = D \times_K K_\phi(H_i, t_i)$, $t_1 \leq t_2 \leq \cdots \leq t_n \leq 2t_1 - 3$. *Then the conclusion of* 3.2 *is valid with* $D = B$.

Proof. $b = \infty$ in 3.2.

Note that in 3.2 and 3.3 we can take $L_i = D \times_K \Pi_K L_\phi(H_{i,j}, t_{i,j})$ provided $t_i = t_{i,1} \leq t_{i,2} \leq \cdots$. This last corollary is related to a result of Larmore [4].

3.4 COROLLARY. *Let* $p: E \to B$ *be a fibration in* Top (pt) *with fiber* $F = p^{-1}(b_0)$. *Assume* $\Pi_i(F) = 0$ *except possibly when* $s \leq i < 2s - 1$. *Then there is an* $M \in$ Top $(B = B)$ *and* $f: B \to M$ *such that* $P(f) \to B$ *and* $E \to B$ *are homotopically equivalent in* Top $(C \to B)$.

Proof. Let the diagram at the beginning of the section come from the Postnikov factorization of p (see Section 4 of [8]). Thus $L_1 = B \times_K L(\Pi_s F, s + 1), \cdots, L_n = B \times_K L(\Pi_{2s-2} F, 2s - 1)$, $n = s - 1$, $t_1 = s + 1$, and $t_i \leq 2s - 1 = 2t_1 - 3$. Hence 3.4 follows from 3.3.

Note that 3.3 is actually valid with $D = B'$ where $B \to B'$ is b-connected, $b \geq t_1 - 2$. So in 3.4 we can take $M \in$ Top $(B' = B')$ for such a B'. For example, if $\cdots \to B(j) \to \cdots \to B(1) = K(\Pi_1 B, 1)$ is the Postnikov system for B then $B' = B(t_1 - 2)$ is permissable in 3.3 and $B' = B(s - 1)$ in 3.4.

3.5 COROLLARY. *Let* $E \to B$ *be a fibration in* Top (pt) *with fiber* F *and* $\Pi_i F = 0$ *except possibly when* $s \leq i < 2s - 1$. *Assume* B *is* s*-connected. Then* $E \to B$ *is fiber homotopically equivalent to a principal fibration.*

Proof. $D =$ pt in the above comment.

Corollary 3.5 is known and is, in fact, a special case of a theorem of Ganea [1] and Hilton [3]. In order to generalize the Ganea-Hilton theorem we consider the following diagram in Top ($D = D$).

$$\begin{array}{ccccccc}
D & \longleftarrow & \bar{E}_2 & \longleftarrow \cdots & \bar{E}_n & \longleftarrow & \bar{E}_{n+1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
B & \longleftarrow & E_2 & \longleftarrow \cdots & E_n & \longleftarrow & E_{n+1} \\
\downarrow & & \downarrow & & \downarrow & & \\
L_1 & & L_2 & & L_n & &
\end{array}$$

The top row is obtained by pullback from the middle row. Let $\bar{k}_i$ be the composite $\bar{E}_i \to E_i \to L_i$.

3.6 THEOREM. *Assume* $B \to D$ *is* r*-connected,* $\bar{k}_i \in \operatorname{Im} \Omega_D : [\hat{E}_i; M_i] \to [\bar{E}_i, L_i]$, *where* $\Omega \hat{E}_i = \bar{E}_i$, *and* $t_i \leq t_1 + r + 1$, $1 \leq i \leq n$. *Then there is an* $f : B \to M \in$ Top ($D = D$) *such that* $P(f) \to B$ *and* $E_{n+1} \to B$ *are homotopically equivalent in* Top ($D \to B$). *Moreover,* $\Omega_D M = \bar{E}_{n+1}$.

Proof. This follows from 3.1 just as 3.7 followed from 2.1.

3.7 THEOREM. *Assume* $p : E \to B \in$ Top ($D = D$) *and is a fibration in* Top (pt) *with fiber* $F = p^{-1}(b_0)$. *Let* $\bar{E}$ *be the pullback of* p *and* $\check{b} : D \to B$. *Assume* $\bar{E} = \Omega_D Z$ *for some* $Z \in$ Top ($D = D$) *and* $B \to D$ *is* r*-connected. Assume* $\Pi_i F = 0$ *except possibly when* $s \leq i < s + r$. *Then there is an* $f : B \to Z$ *in* Top ($D = D$) *such that* $P(f) \to B$ *and* $E \to B$ *are homotopically equivalent in* Top ($D \to B$).

Proof. Let the above tower come from the Moore-Postnikov factorization of p. The $\bar{E}$-tower is then the Postnikov tower for $\bar{E}_{n+1} \to D$. However, this can also be obtained by applying Ω_D to the Postnikov tower for $Z \to D$. It follows that each $\bar{k}_i$ is indeed in the image of Ω_D (by "uniqueness" of Postnikov invariants). Here $t_1 = s + 1$ and $t_i \leq s + r = t_1 + r - 1$. The result now follows from 3.6. M can be taken to be Z because at each stage of the inductive construction the v_i can be taken to be the Postnikov invariant of Z.

The Ganea-Hilton result is the case $D = \mathrm{pt}$ of 3.7. Finally we describe a version of 3.1 which does not explicitly require Top $(D = D)$ language. Consider a tower of ordinary principal fibrations (in Top $C \to \mathrm{pt}$).

$$\begin{array}{ccccccccc} B & \longleftarrow & E_2 & \longleftarrow & \cdots & \longleftarrow & E_n & \longleftarrow & E_{n+1} \\ \downarrow{\scriptstyle r_1} & & \downarrow{\scriptstyle r_2} & & & & \downarrow & & \\ R_1 & & R_2 & & & & R_n & & \end{array}$$

Assume $\hat{b}: B \to D$ is given. For simplicity take $R_i = K(G_i, t_i)$.

$$(A_i) \quad \begin{array}{l} r_i \in \operatorname{Im} \rho : H^{t_i+1}(P_{i-1}, D; G_i) \to H^{t_i}(E_i, C; G_i) \quad \text{where } P_{i-1} = P(v_{i-2}) , \\ v_{i-2} : (P_{i-2}, D) \to (R_i', \mathrm{pt}) \quad \text{and} \quad \Omega R_i' = R_i . \end{array}$$

If follows from 3.1 that (A_i) for $2 \leq i \leq n$ gives $E_{n+1} \to B$ a D-relatively principal fibration (see the end of Section 2). There is a similar local coefficient formulation.

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