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GROUPS WITH THE SUBNORMAL JOIN PROPERTY

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1. Introduction and some preliminary results. A group G is said to have the *subnormal join property* (s.j.p.) if the join of two (and hence of finitely many) subnormal subgroups of G is always subnormal in G. Following Robinson [**6**], we shall denote the class of groups having this property by $\mathfrak{S}$. A particular subclass of $\mathfrak{S}$ is $\bar{\mathfrak{S}}$, consisting of those groups G in which the join of two subnormals is again subnormal in G and has defect bounded in terms of the defects of the constituent subgroups (for a more precise definition see Section 7 of [**6**]).

In [**16**], Wielandt showed that groups which satisfy the maximal condition for subnormal subgroups have the s.j.p. Many further results on groups with the s.j.p. were proved in [**6**] and [**7**]. In Sections 2 and 3 of this paper, it will be shown that several of these results can be exhibited as corollaries of a few rather more general theorems on the classes $\mathfrak{S}$, $\bar{\mathfrak{S}}$. At the same time, many new subclasses of $\mathfrak{S}$ and $\bar{\mathfrak{S}}$ are discovered. An example of a result in this area is Theorem 2.2, which states that an extension of a group having finite rank by a group in $\mathfrak{S}$ is again in $\mathfrak{S}$.

Since the publication of Robinson's papers in 1965, some important theorems have appeared on both the derived series and lower central series of a join of subnormal subgroups ([**11**], [**15**], [**4**]) followed by [**5**] which utilises these results to give a sufficient condition for a join J of two subnormal subgroups of a group G to be subnormal in G, namely that J/J' has finite rank. In [**13**] and [**17**], this hypothesis is weakened somewhat. By using Roseblade's theorem and these more recent results we are able here to improve on several of the existing theorems on groups with the s.j.p.

The following useful lemma, due to Lennox and Stonehewer and based on a result of P. Hall, is to date unpublished and a proof is therefore given here.

LEMMA 1.1. *Let H be a subgroup of the group G and suppose that there is an integer d such that, for every finitely generated subgroup F of H, there is a subgroup X of G with $X \lhd^d G$ and $F \leqq X \leqq H$. Then $H \lhd^d G$.*

Proof. For a subgroup S of the group G, let S_i denote the ith term of the normal closure series of S in G (see, for example, [**6**]) and suppose the hypotheses of the lemma are satisfied. We show that, for any integer i, if A is a finitely generated (f.g.) subgroup of H_i, then there is a f.g. subgroup B

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of H such that $A \leqq B_i$. Then, setting $i = d$ we obtain the result since, for f.g. A contained in H_d, there exist B, X such that $A \leqq B_d \leqq X_d = X \leqq H$ and so $H = H_d$.

The case $i = 1$ is easily seen to be true. Suppose $i > 1$ and let A be a f.g. subgroup of $H_i = H^{H_{i-1}}$. Then there are f.g. subgroups H^*, K of H, H_{i-1} respectively such that $A \leqq (H^*)^K$. By induction, we may suppose $K \leqq L_{i-1}$, for some f.g. subgroup L of H. Writing $B = \langle H^*, L\rangle$, we have

$$A \leqq H^*[K, H^*] \leqq B[B_{i-1}, B] = B_i,$$

as required.

When attempting to discover whether a given class of groups has the s.j.p. we shall usually be able to assume that the groups in question are soluble; this is because of our "reduction lemmas" (2.1 and 3.1) which utilise Roseblade's theorem [**11**] on derived series. It is then a short step to reduce to the case where our group G is a split extension $A]J$, where A is abelian and $J \leqq \text{Aut } A$ is a join of two subgroups H and K which are subnormal in G. Then proving that J is subnormal in G entails showing that J is nilpotent and hence that G is nilpotent. Since HA and KA are in any case nilpotent, this would certainly be the case if G had the property that the join of two subnormal nilpotent subgroups of G were always nilpotent. We are thus led in Section 4 to define a class $\mathfrak{X}_1$ of groups with this property and to consider how it is related to $\mathfrak{S}$. It is easily proved that $\mathfrak{X}_1 \subseteq \mathfrak{S}$ and, generally, classes shown to lie in $\mathfrak{S}$ are also seen to be contained in $\mathfrak{X}_1$. No example is known of a group in $\mathfrak{S}$ which does not lie in $\mathfrak{X}_1$, but it is not conjectured here that the two classes coincide. (A similar problem remains with regard to the classes $\mathfrak{S}$ and $\bar{\mathfrak{S}}$.)

General results are obtained for the class $\mathfrak{X}_1$ which compare with those deduced in Sections 2 and 3 for the class $\mathfrak{S}$. There is one problem unsolved for $\mathfrak{S}$, however, which we are able to deal with in the case of $\mathfrak{X}_1$. This is the question as to whether the class is closed with respect to forming finite direct products, and we shall see (Theorem 4.12) that $\mathfrak{X}_1 \times \mathfrak{X}_1 = \mathfrak{X}_1$.

The final section provides an example of a group which possesses a rather strong "finiteness property" but which does not lie in $\mathfrak{S}$. The construction is based on that of the examples of Zassenhaus and Hall (see [**18**] and [**6**]). It is proved that this condition, however, is sufficient to ensure that a join of two subnormal subgroups is ascendant.

Notation. Class notation will be employed where applicable. Thus $\mathfrak{N}$, $\mathfrak{A}$ denote the classes of nilpotent, abelian groups respectively, and $\mathfrak{F}_r$ consists of those groups having finite (Prüfer) rank. $\mathfrak{S}$ and $\bar{\mathfrak{S}}$ are as

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defined above, and other classes are defined as required. The notation for class products is the familiar one. Thus, for example, $\mathfrak{N}\mathfrak{A}$ denotes the class of groups which are nilpotent-by-abelian. Since the formation of products is a non-associative operation, bracketing is frequently required. If, in a particular instance there is associativity then brackets are of course omitted. Both of these cases are illustrated in the statement of Theorem 2.7.

The closure operations sn-, Q- are also used here. Thus for a given class $\mathfrak{X}$ of groups, sn$\mathfrak{X}$ (resp. Q$\mathfrak{X}$) denotes the class of groups which are subnormal subgroups (resp. homomorphic images) of $\mathfrak{X}$-groups. Standard notation is also adopted elsewhere, including that for the upper and lower central series, derived series and "repeated commutator subgroups" such as $[K_{,m} H]$.

This work consists of material included in the author's Ph.D. thesis, "Subnormality in Infinite Groups" (University of Wales, 1982).

Thanks are due to Drs. J. C. Lennox and S. E. Stonehewer for permission to include the result stated here as Lemma 1.1.

2. The class $\mathfrak{S}$. We begin this section with a reduction lemma which will prove to be very useful.

LEMMA 2.1. *Suppose* $\mathfrak{X} = \text{sn}\mathfrak{X} = \text{Q}\mathfrak{X}$ *is a class of groups such that soluble* $\mathfrak{X}$*-groups are in* $\mathfrak{S}$. *Then* $\mathfrak{X} \subseteq \mathfrak{S}$.

Proof. Suppose H, K are subnormal subgroups of defects m, n respectively in the $\mathfrak{X}$-group G. We must show that $J = \langle H, K\rangle$ is subnormal in G, and we proceed by induction on m. The case $m = 0$ is trivial, and if $m = 1$ we have

$$H \lhd G \quad \text{and} \quad J = HK \lhd^n G.$$

So we suppose $m > 1$ and assume the appropriate inductive hypothesis.

For each i, let H_i denote the ith term of the normal closure series of H in G. Then

$$L = H_{m-1} \lhd^{m-1} G$$

and, by induction, $J_0 = \langle L, K\rangle$ is subnormal in G and therefore lies in $\mathfrak{X}$. Let P be the permutizer of K in L. Then

$$H^K = H^{PK} \lhd \langle H, PK\rangle = J_1,$$

say, and so $J = H^K K$ is subnormal in J_1.

By Corollary B1 and Lemma 5 respectively of **[11]**, there is an integer a such that $L^{(a)} \leqq P$ and an integer b such that

$$J_0^{(b)} \leqq L^{(a)}K \subseteq PK \leqq J_1.$$

By **[12**, Lemma 3] and **[6**, Lemma 2.4], PK is subnormal in J_0. Then, since

$J_0/J_0^{(b)} \in \mathfrak{X}$, J_1 is subnormal in J_0 and the lemma is proved.

It is clear that the class $\mathfrak{S}$ is itself both sn- and Q-closed. Let max-sn (resp. min-sn) denote the class of groups G which satisfy the maximal (resp. minimal) condition for subnormal subgroups. Then

$$\mathfrak{M} = P(\text{max-sn} \cup \text{min-sn})$$

denotes the class of groups G which have a finite series each of whose factors satisfies either max-sn or min-sn. Robinson [**7**, Theorem 6.2] has proved that $\mathfrak{M}\mathfrak{S} = \mathfrak{S}$. Since soluble groups with max-sn or min-sn have finite rank, which is an "extension-closed" property, we may apply Lemma 2.1 to deduce that this result is a corollary of

THEOREM 2.2. $\mathfrak{F}_r\mathfrak{S} = \mathfrak{S}$.

Recall that a group G is a *Baer group* if every finitely generated subgroup is subnormal in G. That Baer groups are locally nilpotent is well known, as is

LEMMA 2.3. *A group G which is generated by subnormal nilpotent subgroups is a Baer group.*

Proof of Theorem 2.2. Suppose $G \in \mathfrak{F}_r\mathfrak{S}$. By 2.1, we may suppose G is soluble. Let $N \lhd G$ be such that N has rank r and $G/N \in \mathfrak{S}$, and let A be the penultimate term of the derived series of N. Then $A \lhd G$, and by induction on the derived length of N we may suppose $G/A \in \mathfrak{S}$.

Let H, K be subnormal in G and put $J = \langle H, K\rangle$. We show that J is subnormal in G.

Since JA is subnormal in G, we may assume $G = JA$. We may further assume that J is core-free in G and thus that $C_J(A) = 1$. Then $G = A]J$, $J \leqq \text{Aut } A$ and it follows that H and K are nilpotent (see [**2**]). G is thus a Baer group (by 2.3).

Let P be the maximal periodic subgroup of A. Then P is characteristic in A and thus normal in G, and A/P is torsion-free. Using [**9**, Lemma 6.37], we deduce that $[A_{,r} G] \leqq P$. Then $[A_{,r} JP] \leqq JP$ and so $JP \lhd^r G$. Replacing G by JP we may thus assume that A is periodic.

Let H_0, K_0 be arbitrarily finitely generated (f.g.) subgroups of H, K respectively and set $J_0 = \langle H_0, K_0\rangle$. Also, for an arbitrary prime p, let A_p denote the p-component of A. Since every f.g. subgroup of J is contained in some subgroup of the form J_0, it suffices to show that, for some integer d, independent of the choice of H_0, K_0 or p,

$$[A_{p,d}J_0] = 1.$$

Further, since $\langle a, J_0\rangle$ is nilpotent for each $a \in A_p$, $\langle a\rangle^{J_0}$ is finitely generated and we may therefore assume that A_p is finite (since the intersection of all characteristic subgroups of finite index in $\langle a\rangle^{J_0}$ is trivial).

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Let $C = C_{J_0}(A_p)$. Then $\bar{J}_0 = J_0/C$ is a finite p-group which may be viewed as a subgroup of Aut A_p, (action being of course by conjugation by elements of J_0). Hence, by [**9**, Lemma 7.44], $\bar{J}_0$ has bounded rank.

Since H and K are subnormal in G and nilpotent, H_0 and K_0 each have bounded defect in G and hence in J_0. By [**5**, Theorem A], therefore, $\bar{J}_0$ has bounded nilpotency class f, say. Also, for bounded integers b, c, we have

$$[A_{p,b}H_0] = [A_{p,b}K_0] = 1$$

and so

$$[A_{p,b}\bar{H}_0] = [A_{p,c}\bar{K}_0] = 1.$$

Thus $\bar{H}_0$ and $\bar{K}_0$ have defects at most $b + f$, $c + f$ respectively in $A_p]\bar{J}_0$, and, again by Theorem A of [**5**], $\bar{J}_0$ has suitably bounded defect d in $A_p]\bar{J}_0$.

Thus $[A_{p,d}J_0] = 1$, as required.

Using Lemma 2.1, we may now state

COROLLARY 2.4.

$$P(\text{max-sn} \cup \text{min-sn})\,\mathfrak{S} = \mathfrak{S}.$$

Also, denoting by $\mathfrak{G}^{sn}$ the class of groups G in which every subnormal subgroup is finitely generated, we have

COROLLARY 2.5. $\mathfrak{G}^{sn}\mathfrak{S} = \mathfrak{S}$.

Now the main result on $\mathfrak{S}$ from [**6**] is Theorem 5.2, which states that $\mathfrak{S}\mathfrak{G}^{sn} = \mathfrak{S}$. In view of 2.2 above, it is reasonable to ask whether $\mathfrak{S}\mathfrak{F}_r = \mathfrak{S}$. Although a similar result will be seen to hold for the (probably) more restricted class $\bar{\mathfrak{S}}$, this question will remain unanswered here.

For our next theorem we need to define another subclass of $\mathfrak{S}$.

Definition. Let $\mathfrak{S}^{\infty}$ denote the class of groups G in which the join of an arbitrary collection of subnormal subgroups is always subnormal.

Then it is not difficult to show that $\mathfrak{S}^{\infty}$ is properly contained in $\mathfrak{S}$ [**6**]. Also easily proved is

LEMMA 2.6. [**9**, Lemma 7.41]. *Every subgroup of a group G is subnormal in G if and only if G is a Baer group in the class $\mathfrak{S}^{\infty}$.*

The next result is then

THEOREM 2.7. $\mathfrak{S}^{\infty}(\mathfrak{F}_r\mathfrak{A}\mathfrak{F}_r) \subseteq \mathfrak{S}$.

Among the many consequences of this theorem, we note the following.

COROLLARY 2.8. [**6**, Theorem 8.5]. $\mathfrak{S}^{\infty}\mathfrak{A} \subseteq \mathfrak{S}$.

COROLLARY 2.9. $\mathfrak{S}^{\infty}\mathfrak{F}_r \subseteq \mathfrak{S}$.

COROLLARY 2.10. $\mathfrak{S}^{\infty}P(\text{max-sn} \cup \text{min-sn}) \subseteq \mathfrak{S}$.

The proof of Theorem 2.7 is postponed until Section 4.

Finally, it may be worth remarking here that $\mathfrak{F}_r$ is not contained in $\mathfrak{S}^{\infty}$. For example, denoting the nth prime by p_n, for each integer n, let

$$H_n = \langle a, b | a^{p_n^n} = b^{p_n^{n-1}} = 1, a^b = a^{1+p_n}\rangle$$

and define G to be the direct product of all such H_n. Then each H_n is metacyclic and nilpotent of class exactly n, and so G is a non-nilpotent Baer group of rank 2. However, it is easily shown that G has subgroups which are not subnormal.

Hence, by 2.6, G is not in $\mathfrak{S}^{\infty}$.

3. Groups with bounding functions. Let $\bar{\bar{\mathfrak{S}}}$ be defined as in [**6**] (see Section 1 above). Then the main result on $\bar{\bar{\mathfrak{S}}}$ from [**6**] (Theorem 7.2) states that an extension of an $\bar{\bar{\mathfrak{S}}}$-group by a group in which every subnormal subgroup is generated by at most n elements (for some fixed n) is again in $\bar{\bar{\mathfrak{S}}}$. Since we shall be proving a reduction lemma which serves for $\bar{\bar{\mathfrak{S}}}$ as Lemma 2.1 did for $\mathfrak{S}$, this result will be seen to follow from Theorem 3.6 below, which states that $\bar{\bar{\mathfrak{S}}}\mathfrak{F}_r = \bar{\bar{\mathfrak{S}}}$. It may indeed be the case that $\mathfrak{S}\mathfrak{F}_r = \mathfrak{S}$ also, but perhaps the discovery of an $\mathfrak{S}$-group which is not in $\bar{\bar{\mathfrak{S}}}$ would also shed some light on this problem.

For each integer d, let us denote by $\mathfrak{A}^d$ the class of soluble groups of derived length at most d. Then, as a companion result to Lemma 2.1, we have

LEMMA 3.1. *Suppose* $\mathfrak{X} = \text{sn}\mathfrak{X} = \text{Q}\mathfrak{X}$ *is a class of groups such that, given an arbitrary group* $G \in \mathfrak{X} \cap \mathfrak{A}^d$ *and subnormal subgroups* H, K *of defects* m, n *respectively in* G, *there exists* $f = f(m, n, d)$ *such that* $J = \langle H, K\rangle$ *is subnormal in* G *with defect at most* f. *Then* $\mathfrak{X} \subseteq \bar{\bar{\mathfrak{S}}}$.

What we are saying here is that, in order to prove that a suitable class $\mathfrak{X}$ lies in $\bar{\bar{\mathfrak{S}}}$, we may assume that a given $\mathfrak{X}$-group G is soluble and that the derived length of G is a "permissible" parameter in attempting to bound the defect of J in G.

To see that Lemma 3.1 holds, we need only follow through the proof of 2.1, with suitably amended induction hypothesis, and note that bounds exist at the appropriate stages (as indicated in [**6**], [**11**] and [**12**]).

Now $\bar{\bar{\mathfrak{S}}}$ is itself sn- and Q-closed [**6**, Lemma 7.1] and an examination of the proof of Theorem 2.2 shows that all integers arising may be bounded. Hence we may state immediately

THEOREM 3.2. $\mathfrak{F}_r\bar{\bar{\mathfrak{S}}} = \bar{\bar{\mathfrak{S}}}$.

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COROLLARY 3.3. $P(\text{max-sn} \cup \text{min-sn})\bar{\mathfrak{S}} \subseteq \bar{\bar{\mathfrak{S}}}$.

COROLLARY 3.4. $\mathfrak{G}^{\text{sn}}\bar{\bar{\mathfrak{S}}} = \bar{\bar{\mathfrak{S}}}$.

A class of groups easily shown to lie in $\bar{\bar{\mathfrak{S}}}$ is $\mathfrak{N}\mathfrak{A}$, that is, those groups G with G' nilpotent. (This is proved in [**6**].)

So a particular case of 3.2 is

COROLLARY 3.5. *If G is a group such that G' has a normal subgroup N of finite rank with G'/N nilpotent, then $G \in \bar{\bar{\mathfrak{S}}}$.*

In the opposite direction from 3.2 we may also prove

THEOREM 3.6. $\bar{\bar{\mathfrak{S}}}\mathfrak{F}_r = \bar{\bar{\mathfrak{S}}}$.

Proof. Suppose $H \lhd^m G$, $K \lhd^n G$, $J = \langle H, K \rangle$, where G has a normal $\bar{\bar{\mathfrak{S}}}$-subgroup N with G/N of rank r. By Lemma 3.1, we may suppose G is soluble of derived length d, say, and a further reduction (as in the proof of 2.2) allows us to assume that H, K are nilpotent of (bounded) class a, b say.

We proceed by induction on m. If $m = 0$, then $H = G$, and if $m = 1$, $J \lhd^n G$. So suppose $m > 1$ and assume the appropriate inductive hypothesis.

Let F be an arbitrary finitely generated (f.g.) subgroup of H^K. Then there are elements k_i of K such that

$$F \leqq H^{(K \cap N)\langle k_1, \ldots, k_r \rangle},$$

which by [**8**, Lemma 3.21] is contained in

$$L^{K \cap N}[H,_{n+b}K] = L^{K \cap N},$$

where L is generated by at most $t = t(r, n, b)$ conjugates of H by elements of K. Since each of these conjugates has defect $m - 1$ in H^G, we may employ the inductive hypothesis and a further induction on t to deduce that L is subnormal of bounded defect in H^G and hence in G. Now

$$L^{K \cap N} \lhd \langle L, K \cap N \rangle = (K \cap N)^L L$$

and so, by [**6**, Lemma 2.4], we need only bound the defect of $(K \cap N)^L$ in G (and apply 1.1). But $L/L \cap N$ has rank at most r and so we may proceed as before, considering an arbitrary f.g. subgroup of $(K \cap N)^L$. Since each conjugate of $K \cap N$ lies in $N \in \bar{\bar{\mathfrak{S}}}$, we deduce that the join M of a bounded number of such conjugates has bounded defect in G. Then

$$M^{L \cap N} \lhd \langle M, L \cap N \rangle \leqq N \in \bar{\bar{\mathfrak{S}}}$$

and the result follows.

As one immediate consequence of this theorem we have

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COROLLARY 3.7. $\mathfrak{N}\mathfrak{F}_r \subseteq \bar{\bar{\mathfrak{S}}}$.

Also of course there is

COROLLARY 3.8. (i) $\bar{\bar{\mathfrak{S}}}P(\text{max-sn} \cup \text{min-sn}) = \bar{\bar{\mathfrak{S}}}$.
(ii) $\bar{\bar{\mathfrak{S}}}\mathfrak{G}^{\text{sn}} = \bar{\bar{\mathfrak{S}}}$.

We now introduce yet another family of groups which are known to have the subnormal join property, namely the class $\mathfrak{B}$ of groups having bounded subnormality indices. Thus a group G is in $\mathfrak{B}$ if, for some integer i, every subnormal subgroup of G has defect at most i in G. $\mathfrak{B}$ is contained in $\mathfrak{S}^{\infty}$ (see [**9**, Vol. 1, p. 176]) and in $\bar{\bar{\mathfrak{S}}}$. We shall now see that, if we replace $\mathfrak{S}^{\infty}$ by $\mathfrak{B}$ in the statement of Theorem 2.7, we obtain a subclass of $\bar{\bar{\mathfrak{S}}}$. In view of 3.6 above, (a result for which we recall we have no parallel for the class $\mathfrak{S}$), it suffices to establish

THEOREM 3.9. $\mathfrak{B}(\mathfrak{F}_r\mathfrak{A}) \subseteq \bar{\bar{\mathfrak{S}}}$.

Proof. Suppose G is a group with a normal subgroup M contained in G' such that G'/M has rank r and $M \in \mathfrak{B}_s$ (that is, every subnormal subgroup of M has defect at most s). Let $H \triangleleft^m G$, $K \triangleleft^n G$, $J = \langle H, K \rangle$. We require a bound for the defect of J in G. By a now familiar argument we may suppose that G is a Baer group and thus, by Lemma 2.6,

$$J \cap M \triangleleft^s M.$$

For each i, let M_i denote the ith term of the normal closure series of $J \cap M$ in M. Since $M_i = (J \cap M)[M,_i(J \cap M)]$, it follows that each M_i is normalized by J, and so there is a chain of subgroups

$$J \leqq JM_{s-1} \leqq \ldots \leqq JM_1 \leqq JM.$$

By Theorem 3.6, $G/M \in \bar{\bar{\mathfrak{S}}}$ and so JM is subnormal of bounded defect in G. It suffices to prove, therefore, that for each i, JM_{i+1} is subnormal of bounded defect in JM_i.

Now

$$JM_{i+1} \cap M_i = M_{i+1}(J \cap M_i) = M_{i+1}$$

and so, using bars to denote factor groups modulo M_{i+1}, we have

$$\overline{JM_i} = \overline{M_i}]\bar{J}.$$

By [**13**, Theorem 1], $\bar{J}$ is subnormal in $\overline{JM_i}$ and has bounded defect, and the theorem is proved.

Clearly, Theorems 3.2, 3.6 and 3.9 may be combined to produce (using Lemma 3.1) quite a number of subclasses of $\bar{\bar{\mathfrak{S}}}$.

Finally, on the class $\mathfrak{B}$, we remark here that Theorem 4.16 below tells us that the direct product of two $\mathfrak{B}$-groups is again in $\mathfrak{B}$.

4. Joins of subnormal nilpotent subgroups.

Definition 4.1. We say that a group G has property $\mathscr{P}_1$ if the join of any two subnormal nilpotent subgroups of G is nilpotent, and a group G has property $\mathscr{P}_2$ if the join of any two subnormal nilpotent subgroups of G is subnormal in G.

Now each of the properties $\mathscr{P}_1$, $\mathscr{P}_2$ is clearly inherited by subnormal subgroups. However, neither passes to homomorphic images, for, while there are groups with neither property (see e.g. the Hall example in **[6]**), free groups possess both $\mathscr{P}_1$ and $\mathscr{P}_2$ in a rather trivial way.

To ensure the required closure properties, we introduce

Defintion 4.2. $\mathfrak{X}_1$ (resp. $\mathfrak{X}_2$) is the largest $\langle \text{sn}, \text{Q}\rangle$-closed class of groups G which have the property $\mathscr{P}_1$(resp. $\mathscr{P}_2$). (Thus $\mathfrak{X}_i$ is the so-called $\langle \text{sn}, \text{Q}\rangle$-*interior* of the class of groups with property $\mathscr{P}_i$).

Trivially, $\mathfrak{S}$-groups have property $\mathscr{P}_2$. Following our reduction method of Section 2 we may even state

THEOREM 4.3. (i) $\mathfrak{X}_1 \subseteq \mathfrak{S}$.
(ii) $\mathfrak{X}_2 = \mathfrak{S}$.

Defining $\overline{\mathscr{P}}_1$ (resp. $\overline{\mathscr{P}}_2$) in a similar fashion to $\mathscr{P}_1$ (resp. $\mathscr{P}_2$), with the additional requirement that the nilpotency class (resp. subnormal defect) of the join is bounded in terms of the nilpotency classes and defects of the constituent subgroups, we may introduce the classes $\overline{\mathfrak{X}}_1$ and $\overline{\mathfrak{X}}_2$, in an analogous way to 4.2, and state

THEOREM 4.4. (i) $\overline{\mathfrak{X}}_1 \subseteq \overline{\mathfrak{S}}$.
(ii) $\overline{\mathfrak{X}}_2 = \overline{\mathfrak{S}}$.

Our interest here is with the classes $\mathfrak{X}_1$, $\overline{\mathfrak{X}}_1$. In addition, a proof outstanding from Section 2, namely that of Theorem 2.7, will be given, using some of the results obtained for the class $\mathfrak{X}_1$.

The proofs of many of the theorems in this section are similar to some earlier ones. As a rule, therefore, sketch proofs only will be given.

The following rather obvious result is quite useful.

LEMMA 4.5. *Suppose* $\mathfrak{X} = \text{sn}\mathfrak{X} = \text{Q}\mathfrak{X}$ *is a subclass of* $\mathfrak{S}$ (*resp.* $\overline{\mathfrak{S}}$). *Then, to prove that* $\mathfrak{X} \subseteq \mathfrak{X}_1$ (*resp.* $\overline{\mathfrak{X}}_1$) *it suffices to show that a join of two subnormal nilpotent* $\mathfrak{X}$-*groups is nilpotent* (*of suitably bounded class*).

Now suppose that $G = \langle H, K\rangle$ is a join of two nilpotent subnormals and is contained in a suitable class $\mathfrak{X}$. Then G is certainly soluble (from **[14]** or **[11]**) and, for some abelian normal subgroup A of G, the action of G/A on A may occasionally be shown to be a nilpotent one. This is seen to be the case in the proof of Theorem 2.2 (and 3.2).

Thus we have

THEOREM 4.6. (i) $\mathfrak{F}_r\mathfrak{X}_1 = \mathfrak{X}_1$
(ii) $\mathfrak{F}_r\bar{\mathfrak{X}}_1 = \bar{\mathfrak{X}}_1$.

In the other direction, the method of proof of Theorem 3.6, with amended inductive hypothesis and supplemented by Lemma 4.5, yields

THEOREM 4.7. (i) $\mathfrak{X}_1\mathfrak{G}^{\mathrm{sn}} = \mathfrak{X}_1$.
(ii) $\bar{\mathfrak{X}}_1\mathfrak{F}_r = \bar{\mathfrak{X}}_1$.

One particular consequence of Theorems 4.6 (ii) and 4.7 (ii) that we shall require later is

COROLLARY 4.8. $\mathfrak{F}_r\mathfrak{A}\mathfrak{F}_r \subseteq \bar{\mathfrak{X}}_1$.

Next, we show that the class considered in Theorem 3.9 actually lies in $\bar{\mathfrak{X}}_1$.

THEOREM 4.9. $\mathfrak{B}(\mathfrak{F}_r\mathfrak{A}) \subseteq \bar{\mathfrak{X}}_1$.

Proof. Suppose $G = \langle H, K\rangle$, where H, K are subnormal in G of defects m, n and nilpotent of class a, b respectively. Suppose further that $N \lhd G$ is such that $N \in \mathfrak{B}_s$, $N \leqq G'$ and G'/N has rank r. By 4.5, it suffices to prove that G is nilpotent of bounded class.

Now G is a Baer group and so, by 2.6 and [**10**], N is nilpotent of class at most $f = f(s)$. For each $i = 1, \ldots, f$, let $\gamma_i = \gamma_i(N)$ and write

$$\bar{\gamma}_i = \gamma_i/\gamma_{i+1}.$$

If $C_i = C_G(\bar{\gamma}_i)$, then $N \leqq C_i$ and so $G_i = G/C_i$ has derived subgroup of rank at most r and acts on $\bar{\gamma}_i$ via conjugation by elements of G. Applying [**13**, Theorem 1] to the semi-direct product $P_i = \bar{\gamma}_i]G_i$, we deduce that G_i is subnormal of bounded defect d_i, say, in P_i. Thus $[P_{i,d_i}G_i] = 1$, that is,

$$[\gamma_{i,d_i}G] \leqq \gamma_{i+1}.$$

Then $[N_{,d}\,G] = 1$, where d is the sum of the d_i. Also, by 4.6 (ii), G/N is nilpotent of bounded class. The result follows.

With the aid of another lemma we shall be able to tackle the proof of Theorem 2.7. (The existence of a bound is given below, but is not necessary for our immediate purpose.)

LEMMA 4.10. *Suppose H, K are subnormal subgroups of the group G, with defects m, n respectively. Let $J = \langle H, K\rangle$ be nilpotent of class c, and suppose $N \lhd J$ is such that J/N, N' have finite rank r, s respectively. Then J is subnormal in G, with defect at most $d = d(m, n, c, r, s)$.*

Proof. By induction on m. The cases $m = 0, 1$ are elementary. Suppose $m > 1$ and let F be a finitely generated subgroup of H^K. Using [**8**, Lemma 3.21] we have $F \leqq L^{K\cap N}$, where L is generated by at most $t = t(r, c)$ conjugates of H by elements of K. By induction on m and a second

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induction on t, the defect of L in H^G and hence in G may be bounded. Then

$$L^{K \cap N} \lhd (K \cap N)^L L$$

and we need consider only $(K \cap N)^L$. Using [**8**, Lemma 3.21] again and applying [**13**, Theorem 1] (and another induction where necessary) to joins of subnormal subgroups of G contained in N, we arrive at a subgroup of the form $M^{L \cap N}$, where M is in N and has bounded defect in G. Then, again by [**13**, Theorem 1], $\langle M, L \cap N \rangle$ is subnormal and we deduce that H^K is subnormal, using 1.1. Finally, $J = H^K K$ and so the result follows.

Proof of Theorem 2.7. Suppose $M \lhd G$ is such that

$$G/M \in \mathfrak{F}_r \mathfrak{A} \mathfrak{F}_r \quad \text{and} \quad M \in \mathfrak{S}^\infty.$$

Reducing to the case where H, K are subnormal nilpotent subgroups of the Baer group G, we deduce that $J \cap M$ is subnormal in M (by 2.6) and then, as in the proof of 3.9, form a chain of subgroups

$$J \leqq JM_{t-1} \leqq \ldots \leqq JM_1 \leqq JM$$

for some integer t. Considering two successive subgroups JM_{i+1}, JM_i of this chain and factoring by M_{i+1} as before, we arrive at the situation where

$$\bar{J} \in \mathfrak{F}_r \mathfrak{A} \mathfrak{F}_r$$

for some image $\bar{J}$ of J. By Corollary 4.8. $\bar{J}$ is nilpotent. Lemma 4.10 now applies to give us that $\bar{J}$ is subnormal in $\overline{JM_i}$. Finally, JM is subnormal in G since $G/M \in \bar{\mathfrak{S}}$, and the proof is complete.

There is one obvious question that remains unanswered here, namely whether $\mathfrak{S}^\infty$ lies in $\mathfrak{X}_1$. Using Lemmas 2.3, 2.6 and 4.5, we may reformulate this as:

Must a group which is generated by two nilpotent subgroups and in which every subgroup is subnormal be nilpotent?

A further problem, and one which we are able to solve for the class $\mathfrak{X}_1$ but not for $\mathfrak{S}$, concerns direct products. Certainly the class $\mathfrak{S}^\infty$ is not closed with respect to forming direct products, since the class of groups having all subgroups subnormal is not thus closed [**3**]. A partial solution to the problem is provided by

THEOREM 4.11. $\mathfrak{X}_1 \times \mathfrak{S} = \mathfrak{S}$.

Proof. Let $G = A \times B$, where $A \in \mathfrak{X}_1$, $B \in \mathfrak{S}$, and suppose H, K are subnormal in G, with $J = \langle H, K \rangle$. We wish to show that J is subnormal in G. Now JA and JB are subnormal, and therefore so is the subgroup

$$G_0 = JA \cap JB = J(A \cap JB) = JC,$$

say. We also have

$$G_0 = J(B \cap JA) = JD$$

and, further,

$$JA = (JA \cap B)A, \quad JB = (JB \cap A)B$$

and so $G_0 = C \times D$, where C, D are subnormal in A, B and thus lie in $\mathfrak{X}_1$, $\mathfrak{S}$, respectively.

It suffices to prove that J is subnormal in G_0. Now $J \cap C \lhd J$ and $[J \cap C, D] = 1$ and so $J \cap C \lhd G_0$. Similarly $J \cap D \lhd G_0$. We may therefore assume that

$$J \cap C = 1 = J \cap D.$$

We shall show that G_0 is nilpotent. Then of course J is subnormal in G_0, as required.

Suppose $H \lhd^m G_0$. Then

$$[C_{,m} H] = 1, \quad [D_{,m} H] = 1$$

and so $[G_{0,m}H] = 1$ and H is nilpotent. Similarly K is nilpotent. Now

$$JD = \langle HD, KD \rangle = \langle HD \cap C, KD \cap C \rangle D = \langle H_1, K_1 \rangle D,$$

say, where $H_1 \simeq H_1D/D = HD/D \simeq H$, and $K_1 \simeq K$.

Similarly, $J \simeq J_1 = \langle H_1, K_1 \rangle$. Since C lies in $\mathfrak{X}_1$, J_1 and hence J is nilpotent. But

$$G_0 = C]J = D]J = C \times D$$

and so C, D and J are pairwise isomorphic.

Thus G_0 is nilpotent and the theorem is proved.

A similar method enables us to establish

THEOREM 4.12. $\mathfrak{X}_1 \times \mathfrak{X}_1 = \mathfrak{X}_1$.

Proof. Let $G = A \times B$, where A, B belong to $\mathfrak{X}_1$, and suppose H, K are subnormal subgroups of G, with $J = \langle H, K \rangle$. Let M be a normal subgroup of G such that HM/M, KM/M are nilpotent. It suffices to prove that JM/M is nilpotent. (We recall that $\mathfrak{X}_1 = Q\mathfrak{X}_1$.)

With the same notation as in the previous theorem, we have

$$J \leqq G_0 = JC = JD = C \times D,$$

where now both C and D are in $\mathfrak{X}_1$, and

$$JD = \langle H_1, K_1 \rangle \times D = J_1 \times D,$$

where H_1, K_1 are subgroups of C such that

$$H_1 \simeq H/H \cap D, \quad K_1 \simeq K/K \cap D.$$

Using bars to denote factor groups modulo M, we see that $\bar{H}_1$, $\bar{K}_1$ are nilpotent. Since $J_1 \simeq C$, we have that $\bar{C}$ and, similarly, $\bar{D}$ are nilpotent. So $\overline{G_0}$ is nilpotent and the result follows.

Clearly we may also state

THEOREM 4.13. (i) $\bar{\mathfrak{X}}_1 \times \bar{\mathfrak{S}} = \bar{\mathfrak{S}}$
(ii) $\bar{\mathfrak{X}}_1 \times \bar{\mathfrak{X}}_1 = \bar{\mathfrak{X}}_1$.

We can easily deduce a rather interesting consequence of Theorem 4.12 (resp. 4.13 (ii)). Firstly, we require the following (see [**1**]).

Definition 4.14. Let G_1, G_2 be given groups, and F their free product. For each integer $r = 1, 2, \ldots$, the rth *nilpotent product* of G_1 and G_2 is given by

$$F/[H, K_{,r-1}\, F].$$

Then we have

COROLLARY 4.15. *For each integer r, the* rth *nilpotent product G of two* $\mathfrak{X}_1$*-groups* (*resp.* $\bar{\mathfrak{X}}_1$*-groups*) *A and B is again in* $\mathfrak{X}_1$ (*resp.* $\bar{\mathfrak{X}}_1$).

Proof. Let G be as stated. Then

$$[A, B] \leqq Z_{r-1}(G) = N,$$

and $G/[A, B]$ is a direct product and hence in $\mathfrak{X}_1(\bar{\mathfrak{X}}_1)$.

Let $\bar{H}$, $\bar{K}$ be subnormal nilpotent subgroups of $\bar{G} = G/M$, say. Then $\bar{J}/\bar{N}$ is nilpotent and therefore $\bar{J}$ is nilpotent, as required.

We note that an arbitrary normal product of two $\mathfrak{X}_1$-subgroups is not necessarily contained in $\mathfrak{X}_1$. Indeed, P. Hall's example of a group not in $\mathfrak{G}$ (see also Section 5 here) may easily be shown to be the product of two normal metabelian subgroups, each of which is of course in $\mathfrak{X}_1$.

The final result of this section, and one which is easily obtained using the method of proof of Theorem 4.11 is

THEOREM 4.16. $\mathfrak{B} \times \mathfrak{B} = \mathfrak{B}$.

Proof. Suppose $G = A \times B$, where every subnormal subgroup of A, B has defect at most r in A, B respectively, and let H be a subnormal subgroup of G. We wish to show that the defect of H in G is bounded.

Now $G_0 = HB \cap HA$ is of the form $C \times D$, where C and D are subnormal in A and B respectively and hence share the above property with A and B. Also, $G_0 \triangleleft^r G$. Replacing G by G_0, we may further assume that

$$H \cap C = 1 = H \cap D.$$

Suppose $H \triangleleft^m G_0$. Then

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$$[C_{,m}\ H] = 1 = [D_{,m}\ H]$$

and so $[G_{0,m}\ H] = 1$. Thus H is nilpotent. But $H \simeq C \simeq D$. Then, by [**10**], C and D have nilpotency class at most $f = f(r)$. The same is true of G_0, and so

$$H \lhd^f G_0$$

and the proof is complete.

5. Groups with finite abelian section rank. In this final section we consider briefly a class of groups which in one way is the natural one to consider following a discussion of groups of finite rank. A group G is said to have *finite abelian section rank* if, for every abelian section A of G, the p-component of A has finite rank for each prime p and the factor group of A by its maximal periodic subgroup also has finite rank. We shall denote the class of groups with this property by $\mathfrak{F}_{r(ab)}$.

We shall prove

THEOREM 5.1. *There is a group G in $\mathfrak{F}_{r(ab)}$ which is not in $\mathfrak{S}$.*

There is something positive, however, that we can say about such groups:

THEOREM 5.2. *The join of two subnormal subgroups of an $\mathfrak{F}_{r(ab)}$-group G is ascendant in G.*

Proof. Proceeding exactly as in the proof of Lemma 2.1, with suitably amended inductive hypothesis, we reduce to the case where G is locally nilpotent. But $\mathfrak{F}_{r(ab)}$-groups which are locally nilpotent are hypercentral (see e.g. [**9**, Corollary to Theorem 6.36]) and it is well known that every subgroup of a hypercentral group is ascendant.

The question as to whether Theorem 5.2 holds for a join of an arbitrary (finite) number of subnormal subgroups is not answered here.

Proof of Theorem 5.1. Our example is based on those of Zassenhaus [**18**] and Hall [**6**], the main difference being that we require their basic construction for each prime p.

Let n be a positive integer, p a prime and let $\mathcal{S}$ be the set $\{1, \dots, n\}$. For $r = 1, \dots, n$, let V_r denote the set of r-tuples $(v_1, \dots, v_r)$ such that $1 \leqq v_1 < v_2 < \dots < v_r \leqq n$. Put

$$V = \bigcup_{r=1}^{n} V_r.$$

Now let A, B be elementary abelian p-groups with bases $\{a_X | X \in V\}$, $\{b_X | X \in V\}$ respectively.

Suppose $X' = (v_1, \dots, v_r)$ is any sequence of r distinct integers from $\mathcal{S}$, where $1 \leqq r \leqq n$, and π is a permutation which takes X' to $X \in V$, that is, π "orders" the v_i.

Then define

$$a_{X'} = (a_X)^{\text{sign}\pi}, \quad b_{X'} = (b_X)^{\text{sign}\pi}.$$

Further, if $X = (v_1, \ldots, v_r)$ and $i \in \mathscr{S}$, define

$$\begin{aligned} a_{X+i} &= a_{(v_1, \ldots, v_r, i)} && \text{if no } v_j = i, \\ &= 1 && \text{if some } v_j = i. \end{aligned}$$

Similarly define b_{X+i}.

Set $M = A \times B$ and define automorphisms α_i, β_i of M by

$$a_X^{\alpha_i} = a_X, \quad b_X^{\alpha_i} = b_X a_{X+i},$$

$$b_X^{\beta_i} = b_X, \quad a_X^{\beta_i} = a_X b_{X+i}.$$

for all a_X, b_X and for each $i = 1, \ldots, n$.

Let

$$H = \langle \alpha_i | i = 1, \ldots, n \rangle, \quad K = \langle \beta_i | i = 1, \ldots, n \rangle$$

and put

$$J = \langle H, K \rangle, \quad G = M]J.$$

Routine calculations (see **[6]**) show that J is nilpotent of class 2 and that H and K are each subnormal of defect 3 in G. Further, G has nilpotency class exactly n.

Now construct, for each integer n, a group G_n as above, where $p = p(n)$ is the nth prime in each case. Let G be the direct product of the G_n, and write $H = \underset{n}{Dr}\, H_n$, etc.

Then $H, K \lhd^3 G$, but $J = \langle H, K \rangle$ is not subnormal in G, else G would be nilpotent, which is clearly not the case.

We note finally that G is periodic and that each p-component of G is of course finite.

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