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ON THE RAMSEY NUMBER $r(F, K_m)$ WHERE F IS A FOREST

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The graphs considered here are finite and have no loops or multiple edges. In particular, K_m denotes the complete graph on m vertices. For any graph G, $V(G)$ and $E(G)$ denote, respectively, the vertex and edge sets of G. A *forest* is a graph which has no cycles and a *tree* is a connected forest. The reader is referred to [**1**] or [**4**] for the meaning of terms not defined in this paper.

A *2-coloring* of the graph K_n consists of the assignment to each edge of K_n of one of the colors blue and red. Equivalently, the two graphs B and R are said to form a 2-coloring of K_n if $V(B) = V(R) = V(K_n), E(B) \cap E(R) = \emptyset$, and $E(B) \cup E(R) = E(K_n)$. The graph B consists of *all* the edges of K_n which are colored blue, and R consists of *all* the edges colored red. If that is the case we write $K_n = B \dotplus R$. Given any two graphs G and H their *Ramsey number* $r(G, H)$ is the smallest integer n such that given any 2-coloring $K_n = B \dotplus R$, either $B \supseteq G$ or $R \supseteq H$. Reference [**2**] contains a survey of the known results regarding this parameter, in addition to an extensive bibliography on the subject. It is our purpose here to determine the value of $r(F, K_m)$ where F is an arbitrary forest. We begin by restating a theorem due to Chvátal [**3**].

THEOREM (Chvátal). *If T is a tree on n vertices, then*

$$r(T, K_m) = (n - 1)(m - 2) + n.$$

The method used by Burr [**2**] to prove Chvátal's theorem can be applied to yield an upper bound for the Ramsey number of some very large classes of graphs. In [**5**], Lick and White defined a *k-degenerate* graph to be a graph G which has the property that for any induced subgraph H of G, $\delta(H) \leqq k$ where $\delta(H)$ is the minimum degree of any vertex of H in H. In the same paper k-degenerate graphs were characterized as those graphs which could be reduced to K_1 by the successive removal of points of degree not greater than k. It is easily seen that every graph is k-degenerate for some non negative integer k and that a graph is 1-degenerate if and only if it is a forest. Relative to this classification of graphs we have the following theorem. (It has in the meantime been brought to the author's attention that Burr has independently proved a somewhat stronger version of this theorem. While Burr's proof predates the one given here, it has not yet been published.)

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THEOREM 2. *If G is a k-degenerate graph, $k > 0$, with p vertices then*

(1) $$r(G, K_m) \leqq k^{m-1} + (p - 1) \sum_{i=0}^{m-2} k^i.$$

Proof. Inequality (1) is easily verified in the case $p = 1$ or $m = 1$ (in the latter case we understand $\sum_{i=0}^{-1} k^i$ to be zero). We fix k and proceed by induction on the parameters p and m. Thus, fixing G and K_m we assume that for any graph H

$$r(H, K_{m'}) \leqq k^{m'-1} + (|H| - 1) \sum_{i=0}^{m'-2} k^i$$

whenever $|H| + m' < p + m$. Set $r = k^{m-1} + (p - 1) \sum_{i=0}^{m-2} k^i$ and assume that $K_r = B \dotplus R$ is a 2-coloring in which $B \not\supseteq G$ and $R \not\supseteq K_m$. We go on to derive a contradiction. Since G is k-degenerate, there is a vertex $v \in V(G)$ of degree $k' \leqq k$. Moreover, $G - v$ is also k-degenerate and has only $p - 1$ vertices. It follows from the induction hypothesis that either $B \supseteq G - v$ or $R \supseteq K_m$. As the latter was assumed not to be the case, we have $B \supseteq G - v$. Let $\{v_1, v_2, \ldots, v_{k'}\}$ be all the vertices of G adjacent to v (in G). If any vertex u of K_r-$(G - v)$ has the property that $uv_i \in B$ for all $i = 1, 2, \ldots, k'$, then by adding that vertex to $G - v$ we obtain a copy of G in B which cannot be. Hence for each $u \in V[K_r - (G - v)]$ there exists a v_i such that $uv_i \in R$. In other words, if V_i is the set of all vertices of $K_r - (G - v)$ which are joined to v_i by an edge in R, then $\bigcup_{i=1}^{k'} V_i = V[K_r - (G - v)]$. As $K_r - (G - v)$ has $k^{m-1} - ⱷ) + 1) \sum_{i=1}^{m-2} k^i$ vertices and $k' \leqq k$, it follows that for some $i = i_0$,

$$k|V_{i_0}| \geqq k^{m-1} + (p - 1) \sum_{i=1}^{m-2} k^i, \quad \text{so}$$

$$|V_{i_0}| \geqq k^{m-2} + (p - 1) \sum_{i=0}^{m-3} k^i.$$

By the induction hypothesis $|V_{i_0}| \geqq r(G, K_{m-1})$. So if G_{i_0} is the subgraph of K_r induced by V_{i_0}, then $G_{i_0} \cap B \supseteq G$ or $G_{i_0} \cap R \supseteq K_{m-1}$. The first alternative contradicts our assumption that $B \not\supseteq G$, so $G_{i_0} \cap R \supseteq K_{m-1}$. However, $v_{i_0} \notin V(G_{i_0} \cap R)$ and $uv_{i_0} \in R$ for all $u \in V(G_{i_0} \cap R)$. Hence v_{i_0} and the copy of K_{m-1} in $G_{i_0} \cap R$ span a copy of K_m completely contained in R. Having derived this contradiction the proof is concluded.

It was noted above that the family of 1-degenerate graphs is the collection of all forests, including totally disconnected graphs. For this class of graphs it is possible to find the exact value of $r(G, K_m)$. Again the method goes back to Burr's proof of Chvátal's theorem. We begin with a lemma which extends this theorem to what one might call "balanced" forests.

LEMMA. *If F is a forest which consists of k trees on n vertices each, then*

$$r(F, K_m) = (n - 1)(m - 2) + nk.$$

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Proof. For any two positive integers i, j we define jK_i to be j disjoint copies of K_i. For any two graphs G and H, $G \cup H$ is defined by $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. We first show that $r(F, K_m) \geqq (n-1)(m-2) + nk$. To show this it will suffice to exhibit a 2-coloring $K_{(n-1)(m-2)+nk-1} = B \dotplus R$ in which

(1) $B \not\supseteq F$ and $R \not\supseteq K_m$.

In fact, set $B = K_{nk-1} \cup (m-2)K_{n-1}$. The number of vertices of B is $(n-1)(m-2) + nk - 1$. Since F has exactly nk vertices, $K_{nk-1} \not\supseteq F$. On the other, each K_{n-1} component of B is too small to contain a component of F. Hence $B \not\supseteq F$. If we set R to be the complement of B then $K_{(n-1)(m-2)+kn-1} = B \dotplus R$. The graph R, however, is complete $(m-1)$-partite and so $R \not\supseteq K_m$. Thus

(2) $r(F, K_m) \geqq (n-1)(m-2) + nk$.

The reverse inequality is proved by induction on k. For $k = 1$ the lemma reduces to Chvátal's theorem. Assume that the lemma has been proved for all forests with $k - 1 (k > 1)$ components each of which is a tree on n vertices. We write $K = K_{(n-1)(m-2)+nk}$ and suppose that $K = B \dotplus R$ is a 2-coloring of K. The lemma will be proved if we show that whenever $R \not\supseteq K_m$, B necessarily contains F. Suppose, therefore that $R \not\supseteq K_m$. Let T be any component of F. Since $|V(K)| = (n-1)(m-2) + nk > (n-1)(m-2) + n$ we may apply Chvátal's theorem to K and conclude that since $R \not\supseteq K_m$, we must have $B \supseteq T$. Let $K - T$ denote the subgraph of K spanned by the vertices in $V(K) - V(T)$. Then $K - T$ is a complete graph and

(3) $$|V(K-T)| = |V(K) - V(T)| = (n-1)(m-2) + - nk\ n$$
$$= (n-1)(m-2) + n(k-1).$$

The reader may easily convince himself that

(4) $K - T = [(K-T) \cap B] \dotplus [(K-T) \cap R]$.

In fact B and R induce a 2-coloring on any complete subgraph of K. Let $F - T$ be defined in a manner analogous to $K - T$. The graph $F - T$ is clearly a forest with $k - 1$ components each of which is a tree on n vertices. In view of (3) and (4) the induction hypothesis may be applied to obtain that

$$(K-T) \cap B \supseteq F - T \quad \text{or} \quad (K-T) \cap R \supseteq K_m.$$

However $R \supseteq (K-T) \cap R$ and we have assumed that $R \not\supseteq K_m$. We therefor conclude that the first alternative holds, that is

$$(K-T) \cap B \supseteq F - T.$$

We recall that $B \supseteq T$. It now follows that

$$B \supseteq (T \cap B) \cup [(K-T) \cap B] \supseteq T \cup (F-T) \cong F.$$

Hence the proof of the lemma is concluded.

We now proceed to the general case where F is an arbitrary forest. For any such forest we define $k_i(F)$ to be the number of components of F which have exactly i vertices. The order of the largest component of F is denoted by $n(F)$.

THEOREM. *If F is an arbitrary forest then*

$$r(F, K_m) = \operatorname*{Max}_{1 \leqq j \leqq n(F)} \left\{ (j-1)(m-2) + \sum_{i=j}^{n(F)} ik_i(F) \right\}.$$

Proof. As was done in the lemma, we first prove that

$$(5) \quad r(F, K_m) \geqq \operatorname*{Max}_{1 \leqq j \leqq n(F)} \left\{ (j-1)(m-2) + \sum_{i=j}^{n(F)} ik_i(F) \right\}.$$

Suppose that the maximum in (5) is assumed for $j = j_0$ and set $p_0 = \sum_{i=j_0}^{n(F)} ik_i(F)$. The value of the maximum then becomes $(j_0 - 1)(m - 2) + p_0$. We modify slightly the 2-coloring used in the proof of the lemma to obtain a 2-coloring of $K = K_{(j_0-1)(m-2)+p_0-1}$. Define $B = K_{p_0-1} \cup (m-2)K_{j_0-1}$ and let R be the complement of B so that $K = B \dotplus R$. To see that $B \not\supseteq F$ we concentrate on F_{j_0} — the subforest of F which consists of all the trees of F which have j_0 or more vertices. By counting vertices we see that $K_{p_0-1} \not\supseteq F_{j_0}$. Again K_{j_0-1} is too small to contain any component of F_{j_0}. Therefore $B \not\supseteq F_{j_0}$ and so $B \not\supseteq F$. As before, R is $(m-1)$-partite and so $R \not\supseteq K_m$.

To complete the proof, suppose that $K_r = B \dotplus R$ where

$$r = \operatorname*{Max}_{1 \leqq j \leqq n(F)} \left\{ (j-1)(m-2) + \sum_{i=j}^{n(F)} ik_i(F) \right\}.$$

Assume further that $R \not\supseteq K_m$. We shall demonstrate, by construction, that $B \supseteq F$. As before let F_j be the subforest of F consisting of all the component trees of F with at least j vertices where $1 \leqq j \leqq n(F)$. Clearly $F_{j+1} \subseteq F_j$ and $F_j - F_{j+1}$ consists of $k_j(F)$ trees each with exactly j vertices. Using descending induction we show that $B \supseteq F_j$ for all $j \geqq 1$. For the sake of simplicity we now write n and k_i for $n(F)$ and $k_i(F)$ respectively.

It follows from the maximality of r that $r \geqq (n-1)(m-2) + nk_n$. The lemma therefore allows us to conclude that since $R \not\supseteq K_m$, we must have $B \supseteq F_n$. Assume now that $B \supseteq F_{j+1}$. Since F_{j+1} has $\sum_{i=j+1}^{n} ik_i$ vertices, $K_r - F_{j+1}$ has $r - \sum_{i=j+1}^{n} ik_i$ vertices. However, from the definition of r we know that

$$r \geqq (j-1)(m-2) + \sum_{i=j}^{n} ik_i = (j-1)(m-2) + jk_j + \sum_{i=j+1}^{n} ik_i.$$

Thus,

$$r - \sum_{i=j+1}^{n} ik_i \geqq (j-1)(m-2) + jk_j.$$

Hence, by the lemma, $r - \sum_{i=j+1}^{n} ik_1 \geqq r(F_j - F_{j+1}, K_m)$ and so, since

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$R \not\supseteq K_m$, $(K_r - F_{j+1}) \cap B$ contains a copy of $F_j - F_{j+1}$. We have

$$B \supseteq F_j.$$

By induction we conclude that $B \supseteq F_1 = F$ and thus the proof of the theorem is completed.

The following corollary shows that for fixed F and sufficiently large m, $r(F, K_m)$ is a linear function of m.

COROLLARY. *If F is a forest with $n = n(F)$, $k_i = k_i(F)$ and*

$$(6) \quad m \geqq 2 + \operatorname*{Max}_{1 \leqq j < n} \left\{ \frac{1}{n-j} \sum_{i=j}^{n-1} ik_i \right\}$$

then $r(F, K_m) = (n-1)(m-2) + nk_n$.

Proof. Condition (6) is equivalent to

$$m - 2 \geqq \frac{1}{n-j} \sum_{i=j}^{n-1} ik_i, \quad 1 \leqq j < n$$

or

$$(n-j)(m-2) + nk_n \geqq \sum_{i=j}^{n} ik_i, \quad 1 \leqq j \leqq n$$

or

$$(n-1)(m-2) + nk_n \geqq (j-1)(m-2) + \sum_{i=j}^{n} ik_i, \quad i \leqq j \leqq n$$

or

$$(n-1)(m-2) + nk_n = \operatorname*{Max}_{1 \leqq j \leqq n} \left\{ (j-1)(m-2) + \sum_{i=j}^{n} ik_i \right\} = r(F, K_m).$$

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