9

Bounded Analytic Functions and the Cauchy Transform

General references for this chapter are [110, 182, 219, 321, 378, 415]. Verdera has a recent survey in [426]. Many of the topics are continued in the higher-dimensional case in Chapter 10.

9.1 Removable Sets and Menger Curvature

In 1888 Painlevé [375] proved that any compact subset E of the plane with one-dimensional Hausdorff measure zero is removable for bounded analytic functions. This was before the existence of Hausdorff measures, but the condition simply means that E can be covered with finitely many discs, the sum of whose diameters is arbitrarily small.

The removability means that any bounded complex analytic function in $U \setminus E$, where U is an open set containing E, has an analytic extension to U. It is easy to see by the Cauchy integral formula that it is enough to consider $U = \mathbb{C}$ and then by Liouville's theorem E is removable if and only if every bounded analytic function in the complement of E must be constant. The problem of finding a geometric characterization of removability is called Painlevé's problem.

In 1947 Ahlfors [1] characterized removable sets in terms of the *analytic capacity* γ:

$$\gamma(E) = \sup\left\{ \lim_{|z|\to\infty} |z(f(z) - f(\infty))| \colon |f| \leq 1, f \text{ analytic in } \mathbb{C} \setminus E \right\}.$$

Then $\gamma(E) = 0$ if and only if E is removable. However, this is still a complex analytic characterization.

Here is an easy proof of Painlevé's theorem: Let $f \colon \mathbb{C} \setminus E \to \mathbb{C}$ be analytic with $|f| \leq 1, f(\infty) = 0$, and let $z \in \mathbb{C} \setminus E$. Let $\varepsilon > 0$ and cover E with discs

B_j, $j = 1, \ldots, k$ such that $\sum_{j=1}^{k} d(B_j) < \varepsilon$ and z is outside them. Let Γ be the boundary of their union. Then by the Cauchy integral formula,

$$f(z) = -\frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} \, d\mathcal{H}^1 \zeta.$$

For small ε this is bounded by $C(z)\varepsilon$, so $f(z) = 0$.

What can we say if we only have $\mathcal{H}^1(E) < \infty$? From the above argument we still get that

$$f(z) = -\int_{\Gamma_j} \frac{f(\zeta)}{\zeta - z} \, d\mathcal{H}^1 \zeta,$$

where the Γ_j are surrounding E with $\mathcal{H}^1(\Gamma_j) < \pi\mathcal{H}^1(E) + 1$ and $d(\Gamma_j, E) \to 0$. Some subsequence of $f\mathcal{H}^1 \llcorner \Gamma_j$ converges weakly to a complex measure on E. Looking at this a bit more closely, one finds that this measure is absolutely continuous with respect to $\mathcal{H}^1 \llcorner E$. This leads to the representation

$$f(z) = C_E(\varphi)(z) := \int_E \frac{\varphi(\zeta)}{\zeta - z} \, d\mathcal{H}^1 \zeta, \ z \in \mathbb{C} \setminus E,$$

where φ is a bounded complex-valued Borel function on E and $C_E(\varphi)$ is the *Cauchy transform* of φ.

So we can rephrase Painlevé's problem, at least for sets with finite length, in terms of the Cauchy transform: when can we put a bounded non-trivial Borel function on E whose Cauchy transform is bounded? How does this relate to rectifiability? For $\mathcal{H}^1$ almost all points $z \in E$ with $\varphi(z) \neq 0$, the integral $\int_E |\zeta - z|^{-1} |\varphi(z)| \, d\mathcal{H}^1 \zeta = \infty$, so $C_E(\varphi)(z)$ is not defined on E. Thus $C_E(\varphi)(z)$ can be bounded when z is near E only due to cancellation, and the cancellation comes from the symmetries of E: for many points $z \in E$, if $\zeta \in E$, the symmetric point $2z - \zeta$ should be close to E. Heuristically, rectifiable sets have such symmetries and purely unrectifiable sets fail to have them. We have a theorem of David [136]:

Theorem 9.1 *Let $E \subset \mathbb{C}$ be compact with $\mathcal{H}^1(E) < \infty$. Then E is removable for bounded analytic functions if and only if E is purely 1-unrectifiable.*

Although the above discussion may give some indication why this could be true, completing the proof has been a long and difficult journey. It was finished when David in [136] showed that purely unrectifiable sets are removable. But let us first look at the other direction. So we should show that if E is rectifiable with $0 < \mathcal{H}^1(E) < \infty$, then there is a bounded non-constant analytic function in its complement. We may assume that E is a subset of a C^1 graph Γ, even with a small Lipschitz constant. If E is a line segment, finding φ is easy: if φ is a smooth function on E vanishing at the endpoints, then $C_E(\varphi)$ is bounded by a direct computation. If E is a subset of the graph Γ of a $C^{1,\alpha}$ function for some $\alpha > 0$, then one can still construct φ, see [219, Theorem I.7.1]. But when Γ is only C^1, nobody knows how to construct it, that is, without using Hahn–Banach or something like that. However, it is known to exist by duality methods involving the Hahn–Banach theorem or some of its equivalent forms, see [110, 378, 415]. To do this one considers C_Γ as a singular integral operator on Γ. We shall discuss this and other singular integrals more in Chapter 10. The final step needed was Calderón's theorem, [85], saying C_Γ is bounded in $L^2(\Gamma)$. More precisely,

$$\int_\Gamma |C_{\Gamma,\varepsilon}(g)|^2 \, d\mathcal{H}^1 \lesssim \int_\Gamma |g|^2 \, d\mathcal{H}^1 \text{ for } g \in L^2(\Gamma),$$

with constant independent of ε, where

$$C_{\Gamma,\varepsilon}(g)(z) = \int_{\{\zeta\in\Gamma:\ |\zeta-z|>\varepsilon\}} \frac{g(\zeta)}{\zeta - z} \, d\mathcal{H}^1\zeta, \ z \in \Gamma. \tag{9.1}$$

This only gives us non-zero functions g such that $C_{\Gamma,\varepsilon}(g) \in L^2(\Gamma)$ uniformly in ε, but then duality methods based on the Hahn–Banach theorem can be used to produce φ for which $C_\Gamma(\varphi)$ is bounded in $\mathbb{C} \setminus E$.

Calderón [85] proved the L^2-boundedness of the Cauchy transform on Lipschitz graphs with small Lipschitz constant in 1977. In 1982 Coifman, McIntosh and Meyer [115] proved this for general Lipschitz graphs. Since then many people have given different proofs. David [133] proved that the Cauchy transform is bounded on a rectifiable curve Γ if and only if Γ is AD-1-regular, that is, uniformly 1-rectifiable.

To prove the converse statement of Theorem 9.1 we need to show that if E is not removable, then it contains a rectifiable subset of positive measure. We can again start with a bounded complex-valued Borel function φ on E such that $C_E(\varphi)$ is bounded in $\mathbb{C} \setminus E$. There are three main steps in the proof.

(1) Modify $\mathcal{H}^1 \llcorner E$ and φ to a finite Borel measure μ and a bounded Borel function g such that $\mu \sim \mathcal{H}^1 \llcorner E$ on some $F \subset E$ with $\mu(F) > 0, \mu(B(z,r)) \leq r$ for $z \in \mathbb{C}, r > 0$, the real part of $g \geq \delta > 0$ on $\mathbb{C}$ and the Cauchy transform $C_\mu g \in BMO(\mu)$.

This was done in [143], except that we only derived L^2 estimates for $C_\mu g$, the BMO estimates were then (later, although the papers appeared in different order) proved in [136]. For the modifications we needed to construct generalized dyadic cubes for non-doubling measures similar to those mentioned in Section 5.5. Then $BMO(\mu)$ refers to BMO defined with these cubes. The construction of μ and g is done by stopping time arguments similar to those used by Christ in [111] in the AD-regular case.

Before explaining step (2), let us define that C_μ is bounded in $L^2(\mu)$ if the truncated *Cauchy transforms* $C_{\mu,\varepsilon}$,

$$C_{\mu,\varepsilon}g(z) = \int_{|\zeta - z|>\varepsilon} \frac{g(\zeta)}{\zeta - z}\, d\mu\zeta, \varepsilon > 0,$$

are uniformly bounded in $L^2(\mu)$. Furthermore, we say that C_E is bounded in $L^2(E)$ if this holds for $\mu = \mathcal{H}^1 \llcorner E$.

(2) Use a $T(b)$-theorem to prove that the singular integral operator C_μ is bounded in $L^2(\mu)$. Again, a suitable $T(b)$-theorem in the AD-regular case was proved in [111]. But in the present general non-doubling situation, no such result was known and David proved it in [136]. Roughly speaking, it says that the existence of $b = g$ as in step (1) implies the L^2-boundedness of C_μ.

(3) Recall from (3.2) the Menger curvature $c(z_1, z_2, z_3) = 1/R$, where R is the radius of the circle passing through z_1, z_2 and z_3. The final step (which was completed first) is based on its relation to the Cauchy kernel $1/z$, which is provided by the following remarkable formula of Melnikov [334]:

$$\sum_\sigma \frac{1}{(z_{\sigma(1)} - z_{\sigma(3)})\overline{(z_{\sigma(2)} - z_{\sigma(3)})}} = c(z_1, z_2, z_3)^2 \tag{9.2}$$

for distinct points $z_1, z_2, z_3 \in \mathbb{C}$, where σ runs through the six permutations of $\{1, 2, 3\}$. See [415, Section 3.2] for the easy elementary proof. Define the curvature of μ by

$$c^2(\mu) = \iiint c(z_1, z_2, z_3)^2\, d\mu z_1\, d\mu z_2\, d\mu z_3.$$

In [335], Melnikov and Verdera integrated (9.2) with respect to μ, wrote $|C_{\mu,\varepsilon}1(z)|^2$ as a double integral and used Fubini's theorem six times to get

$$\int \left|C_{\mu,\varepsilon}1\right|^2 d\mu = \frac{1}{6}c^2_\varepsilon(\mu) + O(\mu(\mathbb{C})), \tag{9.3}$$

where in $c^2_\varepsilon(\mu)$ the integration is performed over the triples with mutual distances bigger than ε. Since the left-hand side is bounded by step (2), so is the right-hand side, whence $c^2(\mu) < \infty$, and the proof is completed by the David–Legér Theorem 3.18.

The formula (9.2) is remarkable firstly because it relates to the Cauchy kernel a non-negative quantity which vanishes on lines, and only on lines. This alone would be very useful. Secondly, this quantity has a concrete geometric meaning.

The above argument gives

Theorem 9.2 *Let $E \subset \mathbb{C}$ be $\mathcal{H}^1$ measurable with $\mathcal{H}^1(E) < \infty$. If C_E is bounded in $L^2(E)$, then E is* 1*-rectifiable.*

Of course, the converse also holds in the sense that if E is 1-rectifiable, it contains subsets F with $\mathcal{H}^1(E \setminus F)$ arbitrarily small for which C_F is bounded in $L^2(F)$.

Nazarov, Treil and Volberg, see [365, 366], proved a little later than David a $T(b)$ theorem which also completes the proof of Theorem 9.1. Their method was quite different. They used random translations of the standard dyadic squares, finding them in a good position with large probability. This method has turned out to be very useful in many later developments.

The formula (9.3) was used by Melnikov and Verdera in [335] to give yet another proof for the L^2-boundedness of the Cauchy transform on Lipschitz graphs.

If in the above sketch of the proof we can start with a real-valued φ, then the modified function g will be positive and we can immediately go from step (1) to step (3). This was done in [143], and it resulted to the analogue of Theorem 9.1 for Lipschitz harmonic functions. We shall discuss them in higher dimensions in Chapter 10.

It is easy to see that sets of Hausdorff dimension bigger than 1 are not removable. Hence after David's theorem, only the case with dimension 1 and infinite measure remained open. The complete solution of Painlevé's problem was given by Tolsa in [409]:

Theorem 9.3 *Let $E \subset \mathbb{C}$ be compact. Then E is not removable if and only if there is $\mu \in \mathcal{M}(E)$ such that $\mu(B(z,r)) \le r$ for $z \in \mathbb{C}$ and $r > 0$ and $c^2(\mu) < \infty$.*

Notice that when $\mathcal{H}^1(E) < \infty$, Tolsa's criterion is equivalent to rectifiability by Theorems 3.18 and 5.2, with the latter applied to Lipschitz graphs.

By results of Melnikov [334] and Tolsa [410], we also have a quantitative version of Theorem 9.3 in terms of the analytic capacity, see also Theorems 4.14 and 6.1 in [415].

Theorem 9.4 *Let $E \subset \mathbb{C}$ be compact. Then*

$$\gamma(E) \sim \sup\left\{\mu(E) \colon \mu \in \mathcal{M}(E), c^2(\mu) \le \mu(E), \mu(B(z,r)) \le r \text{ for } z \in \mathbb{C}, r > 0\right\}.$$

For AD-regular sets, we have

Theorem 9.5 *If $E \subset \mathbb{R}^2$ is closed and AD-1-regular, then E is uniformly 1-rectifiable if and only if C_E is bounded in $L^2(E)$.*

As mentioned above, the boundedness on uniformly rectifiable sets is due to David [133]. The converse was proved in [327] using (9.3), the above-mentioned results of Christ and Theorem 5.4, with $p = 2$, of David and Semmes. It also gave Theorem 9.1 for AD-regular sets.

Jaye and Nazarov gave in [256] a different proof for Theorem 9.5 using reflectionless measures, see some comments on them at the end of Section 10.2. Their proof does not use Menger curvature but relies on other special properties of the Cauchy kernel.

For general measures μ on $\mathbb{C}$ with linear growth, $\mu(B(z,r)) \leq r$ for $z \in \mathbb{C}, r > 0$, the boundedness of C_μ in $L^2(\mu)$ is equivalent to $c^2(\mu \llcorner B) \lesssim \mu(2B)$ for all discs B, see [415, Theorem 3.5]. Hence, recalling Theorem 5.4 and the notation (6.1), the following result of Azzam and Tolsa [42] extends Theorem 9.5:

Theorem 9.6 *Let $\mu \in \mathcal{M}(\mathbb{C})$ with $\mu(B(z,r)) \leq r$ for $z \in \mathbb{C}, r > 0$. Then*

$$c^2(\mu) + \mu(\mathbb{C}) \sim \int_0^\infty \int \beta_\mu^{1,2}(x,r)^2 \frac{\mu(B(x,r))}{r} \, d\mu x \frac{1}{r} \, dr + \mu(\mathbb{C}). \tag{9.4}$$

Recall their closely related rectifiability characterization in Theorem 6.2. For the corresponding result for the Riesz kernels R_{n-1} in $\mathbb{R}^n$, see Theorem 10.5.

Combining Theorem 9.6 with Theorems 9.3 and 9.4, we have a β characterization of removable singularities of bounded analytic functions and of analytic capacity.

9.2 Projections

The validity of Theorem 9.1 was conjectured by Vitushkin in [431]. Actually he formulated his question for general compact sets E: is E removable if and only if $\mathcal{H}^1(P_L(E)) = 0$ for almost all lines L through the origin? By the Besicovitch projection Theorem 3.13, this is equivalent to pure unrectifiability when E has finite measure. In general the answer is no. I showed in [320] that the projection condition is not conformally invariant, whereas the removability obviously is. This did not tell us which of the implications is false, but soon afterwards Jones and Murai constructed in [265] a non-removable set with zero projections. An easier construction with Menger curvature was done by Joyce and Mörters in [266]. The other direction is still open. Because of Tolsa's Theorem 9.3, this can be stated purely as a geometric measure theory problem: if $\mathcal{H}^1(P_L(E)) > 0$ for positively many lines L, is it then possible to construct $\mu \in \mathcal{M}(E)$ such that $\mu(B(z,r)) \leq r$ for $z \in \mathbb{C}$ and $r > 0$ and $c^2(\mu) < \infty$? Chang and Tolsa proved in [88] a partial result in this direction.

Dabrowski and Villa [129] showed that the projection condition (5.6) implies that $\gamma(E) > 0$. Moreover, they then proved the quantitative estimate $\gamma(E) \sim d(E)$. For these results, they used their analyst's salesman theorem for sets satisfying (5.6) and Theorems 9.4 and 9.6.

9.3 Principal Values

From Theorem 9.2, we get a characterization of rectifiability in terms of the boundedness of the Cauchy transform. Now we give a characterization in terms of the convergence properties of the Cauchy transform.

Theorem 9.7 *Let $E \subset \mathbb{C}$ be $\mathcal{H}^1$ measurable with $\mathcal{H}^1(E) < \infty$. Then E is 1-rectifiable if and only if the finite limit*

$$\lim_{\varepsilon \to 0} \int_{\{\zeta \in E:\, |\zeta - z| > \varepsilon\}} \frac{1}{\zeta - z} \, d\mathcal{H}^1 \zeta$$

exists for $\mathcal{H}^1$ almost all $z \in E$.

The convergence for rectifiable sets was proved in [326]; in [425] Verdera gave a different Hahn–Banach proof. The other, more difficult, direction was proved by Tolsa in [408]. Under the assumption of positive lower density, it was proved in [322]. Both papers contain more general results for measures. We say that $\mu \in \mathcal{M}(\mathbb{C})$ has a *principal value* at z if the finite limit

$$C\mu(z) := \lim_{\varepsilon \to 0} \int_{\{\zeta:\, |\zeta - z| > \varepsilon\}} \frac{1}{\zeta - z} \, d\mu\zeta$$

exists. Define the maximal transform

$$C^*\mu(z) := \sup_{\varepsilon > 0} \left| \int_{\{\zeta:\, |\zeta - z| > \varepsilon\}} \frac{1}{\zeta - z} \, d\mu\zeta \right|.$$

Proving and using a variant of a $T(b)$ theorem of Nazarov, Treil and Volberg, see [365] or [366], Tolsa proved first that if $\Theta^{*1}(\mu, z) < \infty$ and $C^*\mu(z) < \infty$ for μ almost all $z \in \mathbb{C}$, then C_μ is L^2-bounded on a set of large μ measure. Then he concluded that by (9.3), there are compact sets E_i such that $\mu(\mathbb{C} \setminus \cup_{j=1}^{\infty} E_j) = 0$ and $c^2(\mu \llcorner E_j) < \infty$. Applying this and Theorem 3.18 to $\mu = \mathcal{H}^1 \llcorner E$, Theorem 9.7 follows, even with the existence of principal values replaced by the finiteness of the maximal function.

In [322] I proved:

Theorem 9.8 *Let $\mu \in \mathcal{M}(\mathbb{C})$. If $\Theta^1_*(\mu, z) > 0$ and $C\mu(z)$ exists for μ almost all $z \in \mathbb{C}$, then μ is 1-rectifiable.*

The proof uses tangent measures, recall Section 4.3. The assumptions imply that for μ almost all $z \in \mathbb{C}$ every $\nu \in \operatorname{Tan}(\mu, z)$ is *symmetric* which means that

$$\int_{B(z,r)} (\zeta - z) \, d\nu\zeta = 0 \text{ for } z \in \operatorname{spt} \nu, r > 0.$$

Then up to discrete measures the symmetric measures were characterized. Every non-discrete symmetric measure is either a constant multiple of the two-dimensional Lebesgue measure or a constant multiple of the one-dimensional Lebesgue measure on a line or a countable sum of constant multiples of the one-dimensional Lebesgue measures on parallel lines. Theorem 9.8 was deduced from this.

Tolsa extended also Theorem 9.8 in [411]:

Theorem 9.9 *Let $\mu \in \mathcal{M}(\mathbb{C})$. If $\Theta^{*1}(\mu, z) > 0$ and $C^*\mu(z) < \infty$ for μ almost all $z \in \mathbb{C}$, then μ is 1-rectifiable.*

The proof involves similar ingredients as [408], but considerable extra difficulties are caused by the weaker density assumptions. The paper contains more detailed information about measures with finite Cauchy maximal transform.

For $\mu \in \mathcal{M}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $r > 0$, define

$$C_\mu^m(x,r) = r^{-m-1} \int_{B(x,r)} (y - x)\, d\mu y.$$

It vanishes on $\operatorname{spt}\mu$ for all $r > 0$ if and only if μ is symmetric. The finiteness in the following theorem is a kind of approximate symmetry condition.

Theorem 9.10 *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ with $0 < \Theta^{*m}(\mu, x) < \infty$ for μ almost all $x \in \mathbb{R}^n$. Then μ is m-rectifiable if and only if $\int_0^\infty |C_\mu^m(x,r)|^2/r\, dr < \infty$ for μ almost all $x \in \mathbb{R}^n$.*

The sufficiency of this condition for rectifiability follows from the work of Mayboroda and Volberg [332], while Villa proved the necessity in [428]. Villa also proved an analogous result for uniform rectifiability and considered more general kernels.

9.4 Square Functions

In [147], David and Semmes proved the following:

Theorem 9.11 *Let $E \subset \mathbb{C}$ be AD-1-regular. For $f \in L^2(E)$ and $z \in \mathbb{C} \setminus E$ define*

$$F(z) = \int_E \frac{f(w)}{z - w}\, d\mathcal{H}^1 w.$$

Then E is uniformly rectifiable if and only if

$$\int_{\mathbb{C}} |F'(z)|^2 d(z, E)\, dz \lesssim \int_E |f|^2\, d\mathcal{H}^1 \text{ for all } f \in L^2(E).$$

See Theorem I.2.41 in [147], where also an $(n-1)$-dimensional version in $\mathbb{R}^n$ is given with the Cauchy transform replaced by the Riesz transform R_{n-1}.

Here is a related result with harmonic functions:

Theorem 9.12 *Let $\Omega \subset \mathbb{R}^n$ be a corkscrew domain (see Section 11.1) with AD-$(n-1)$-regular boundary. Then $\partial\Omega$ is uniformly rectifiable if and only if*

$$\int_{\Omega\cap B(a,r)} |\nabla u(x)|^2 d(x,\partial\Omega)\, dx \lesssim r^{n-1} |\sup\{u(x):\, x \in \Omega\}|^2$$

for all $a \in \partial\Omega$, $r > 0$ and for every bounded harmonic function u in Ω.

The 'only if' direction was proved by Hofmann, Martell and Mayboroda [238] and the 'if' direction by Garnett, Mourgoglou and Tolsa [222]. These papers show that this estimate is also equivalent to an approximation property of harmonic functions. Related results were proven by Hofmann and Tapiola [240] and Bortz and Tapiola [78].

9.5 Other Related Kernels

The Menger curvature trick (9.2) is particular to the Cauchy kernel $1/z$. We shall now discuss some positive results and counterexamples with other kernels.

For $\Omega\colon S^1 \to \mathbb{C}$, define

$$k_\Omega(z) = \Omega(z/|z|)/|z|,\ z \neq 0.$$

Huovinen proved in [244] that Theorem 9.8 remains valid for k_{Ω_k}, with $\Omega_k(z) = z^k/|z|^k$, where k is an odd positive integer. Assuming additionally that the lower density is finite, it holds for finite linear combinations of such kernels. Now the tangent measures satisfy the Ω-symmetricity $\int_{B(z,r)} \Omega(\zeta - z)\, d\nu\zeta = 0$ for $z \in \operatorname{spt}\nu, r > 0$. Huovinen characterized the supports of such non-discrete measures. They are either unions of lines or the whole plane $\mathbb{C}$. Observe that the cancellation for z^k does not only come from $-z$, but from $k-1$ other points too. Thus, when $k \geq 3$, in addition to flat tangent measures, also certain finite sums of up to k flat measures on lines through the origin are possible tangent measures. These are called spike measures by Jaye and Merchán, and they cause new problems as compared to the case $k = 1$. Anyway, Huovinen was able to show that under positive lower density and the existence of principal values, only flat measures occur as tangent measures almost everywhere.

Jaye and Merchán [255] proved Huovinen's Ω_k result assuming positive and finite upper density almost everywhere. So we have:

Theorem 9.13 *Let k be an odd positive integer and* $\mu \in \mathcal{M}(\mathbb{C})$. *Suppose that* $0 < \Theta^{*1}(\mu, z) < \infty$ *for* μ *almost all* $z \in \mathbb{C}$ *or* $\Theta^1_*(\mu, z) > 0$ *for* μ *almost all* $z \in \mathbb{C}$. *If the finite limit*

$$\lim_{\varepsilon \to 0} \int_{|\zeta - z| > \varepsilon} \frac{(\zeta - z)^k}{|\zeta - z|^{k+1}} \, d\mu\zeta$$

exists for μ *almost all* $z \in \mathbb{C}$, *then* μ *is 1-rectifiable.*

New methods are required; for $k \geq 3$, the Menger curvature is not available and tangent measures do not seem to work either. To deal with this, Jaye and Merchán combined Tolsa's and Huovinen's methods. They introduced α numbers in the spirit of (5.4), but now minimizing the distance to spike measures. They showed that positive and finite upper density and the existence of principal values imply that these α numbers tend to zero. This alone does not imply rectifiability, but combined with L^2-boundedness it does. The needed L^2-boundedness again follows by $T(b)$ theorems. The proof is also based on the earlier work [253] and [254] of these authors.

In [429], Villa proved the rectifiability result for odd bi-Lipschitz functions $\Omega\colon S^1 \to S^1$ with constant close to 1 under positive lower density and finite upper density. Then the AD-1-regular Ω-symmetric measures are 1-flat.

In [245], Huovinen considered kernels satisfying the standard Calderón–Zygmund conditions and some additional cancellation conditions. Let $K_t(z) = Re(z)/|z|^2 - tRe(z)^3/|z|^4, t \in \mathbb{R}$. When $t = 1$, he proved that there exist compact purely 1-unrectifiable AD-regular sets such that the principal values exist almost everywhere and the operator is L^2-bounded on some subset of positive measure. This phenomenon is caused by the cancellation coming from the coordinate axis. Jaye and Nazarov [257] found for $t = 3/4$, a compact purely 1-unrectifiable set with positive and finite 1-measure for which the operator is L^2-bounded. This set is not AD-regular but has the interesting property that, despite the L^2-boundedness, the principal values do not exist. In fact, their kernel was much simpler, $\bar{z}/z^2$, but its real part is $4K_t$. Mateu and Prat [316] gave an example in higher dimensions.

Chousionis, Mateu, Prat and Tolsa [105] considered the kernels $Re(z)^k/|z|^{k+1}$ for positive odd integers k. They proved for them the analogues of Theorems 9.2 and 9.5. They did it by relating to these kernels a sum of permutations as in (9.2) and showed that it is non-negative and has properties similar to the Menger curvature. This allowed them to prove the analogue of the David–Legér Theorem 3.18. Chunaev [113] did the same for a larger class of kernels including K_t as above for certain parameters t for which the permutation sum is non-negative. But for some t this sum takes both positive and negative values.

Even for a range of such t, Chunaev, Mateu and Tolsa [114] managed to prove analogous results.

David [137] and Chousionis [96] considered some self-similar fractals and constructed Calderón–Zygmund kernels for which the operators are bounded on these fractals. The kernels are defined to fit the self-similarities of the sets.