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A singular perturbation problem with a turning point

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We consider the equation

$$\varepsilon y'' + p(x)y' + q(x)y = 0 ,$$

where ε is a small positive parameter and p vanishes in the interval. Two asymptotic forms of solution are obtained and a rigorous estimate is made of the difference between the exact solutions and the asymptotic forms.

1. Introduction

The equation

$$\varepsilon y'' + p(x)y' + q(x)y = 0 , \tag{1.1}$$

where ε is a small positive parameter, has been studied extensively and the asymptotic properties of the solutions are well understood when p does not change sign. Cases when p has a simple zero have been considered by Ackerberg and O'Malley [1], but they have made the assumption that p and q are analytic functions. Here we show that quite general results can be obtained for the solution in the interval $0 \le x \le 1$ under the much weaker conditions that q be continuously differentiable and p twice continuously differentiable. We assume $p(0) = 0$, $p'(0) = -1$ and p is strictly negative for $0 < x \le 1$. We use the notation $q(0) = b$. The continuation of the solutions through the origin is also considered.

Interest in this equation developed from the related problem of the solution of the von Kármán similarity equations for a rotating fluid. The

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angular velocity satisfies an equation of the same form as (1.1) with the parameter ε corresponding to the inverse of the Reynold's number. The change in sign of the coefficient p corresponds to a reversal of the axial velocity in the fluid, which is usually associated with a change in sign of the angular velocity. The understanding of the flow between counter-rotating discs is the eventual aim of this work, but it has not yet been achieved.

Some preliminary forms of approximate solutions are obtained in order to explain the choice of variables used below. Near the origin the approximate equation is

$$\varepsilon y'' - xy' + by = 0 . \tag{1.2}$$

With $\eta = x\varepsilon^{-\frac{1}{2}}$, we obtain

$$\frac{d^2y}{d\eta^2} - \eta \frac{dy}{d\eta} + by = 0 . \tag{1.3}$$

The solutions of this equation are the parabolic cylinder functions, which also play an important role in the solution of the approximate equation for larger values of the independent variable.

Away from the origin, the outer or slowly varying approximate solution is given by

$$p(x)y' + q(x)y = 0 .$$

The solution of this equation is

$$Y_0(x) = x^b \exp\left\{- \int_0^x \left[\frac{q(\tau)}{p(\tau)} + \frac{b}{\tau}\right] d\tau\right\} . \tag{1.4}$$

To obtain the rapidly varying boundary layer solution, we use a change of variable to

$$t = - \frac{1}{\varepsilon} \int_0^x p(\tau)d\tau . \tag{1.5}$$

Equation (1.1) becomes

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} + \varepsilon(q-p')y = 0 ,$$

and the approximate solution is

$$y = Ae^t .$$

It must be remembered that since t is a stretched variable this solution is valid only over a small interval, that is, over an interval in which x changes by a small quantity. The important consequence of this solution for our purposes is that the variable t is an appropriate one for the rapidly varying solution, while x is appropriate for the slowly varying solution. We note also that for $x \to 0$, $2t \sim \frac{x^2}{\varepsilon} = \eta^2$, which suggests that $t^{\frac{1}{2}}$ might be a more suitable variable to use.

More useful forms of approximate solutions valid over the whole interval $0 \le x \le 1$ are derived in the next section.

2. Construction of approximate solutions

The independent variable $\xi(x)$ which is suggested by the preliminary forms of the approximate solutions is given by

$$\varepsilon\xi^2 \equiv X^2 = -2\int_0^x p(\tau)d\tau . \tag{2.1}$$

Let

$$\xi_1 = \xi(1) . \tag{2.2}$$

Equation (1.1), in terms of ξ, becomes

$$L(y) \equiv \frac{d^2y}{d\xi^2} + P(\xi;\ \varepsilon)\frac{dy}{d\xi} + Q(\xi;\ \varepsilon)y \equiv \frac{d^2y}{d\xi^2} - \xi\frac{dy}{d\xi} + by - \varepsilon^{\frac{1}{2}}\left(\frac{1}{X} + \frac{p'X}{p^2}\right)\frac{dy}{d\xi} + \left(b - \frac{X^2q}{p^2}\right)y = 0 . \tag{2.3}$$

As before $b = q(0)$, $p'(0) = -1$ and from (2.1) we have

$$X \sim x \quad \text{for} \quad x \to 0 ,$$

so that

$$b - \frac{X^2q}{p^2} \to 0 \quad \text{as} \quad x \to 0 ,$$

that is

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$$b - \frac{X^2 q}{p^2} \sim Cx \quad \text{for} \quad x \to 0$$

for some constant C, since q, p, X are continuously differentiable functions of x.

Similarly, $\left(\frac{x}{X} + \frac{p'Xx}{p^2}\right)$ vanishes at $x = 0$ and, since p', p and X are continuously differentiable functions of x, we have that $\left(\frac{1}{X} + \frac{p'X}{p^2}\right)$ is bounded.

A "two-variable" method is used to determine the approximate solutions. These are of the form

$$w(\xi;\, \varepsilon) = \alpha(x)u(\xi) ,$$

where

$$u'' - \xi u' + bu = 0 , \tag{2.4}$$

and the derivatives of w are given by formulae such as

$$w' = \alpha u' + \frac{dx}{d\xi}\alpha' u . \tag{2.5}$$

For the sake of abbreviation we write

$$w(\xi) \equiv w(\xi;\, \varepsilon) .$$

Substitution into (2.3) gives

$$L(\alpha u) = \varepsilon^{\frac{1}{2}}\left[\frac{2X\alpha'}{p} + \left(\frac{1}{X} + \frac{p'X}{p^2}\right)\alpha\right]u' - \left[\frac{X^2\alpha'}{p} - \left(b - \frac{X^2 q}{p^2}\right)\alpha\right]u - \varepsilon\left[\frac{X^2\alpha''}{p^2} - \frac{X^2 p'\alpha'}{p^3} - \frac{\alpha'}{p} + \frac{X\alpha'}{p}\right]u . \tag{2.6}$$

There is no α which would make this expression vanish, but we may choose α so that the expression obtained when u and u' are replaced by their asymptotic forms vanishes. If we were to use an iterative method to find more accurate approximate solutions, this choice of α would prevent the development of "resonance" in the next approximation.

We derive two approximate solutions w_1, w_2 where

$$w_1(\xi) = \alpha_1(x)u_1(\xi) , \quad w_2(\xi) = \alpha_2(x)u_2(\xi) . \tag{2.7}$$

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u_1 and u_2 are solutions of (2.4), chosen so that

$$u_1 = C_1\xi^{b}\left[1 + O\left(\xi^{-2}\right)\right] ,$$

(2.8)

$$u_2 = C_2\xi^{-b-1}e^{\frac{1}{2}\xi^2}\left[1 + O\left(\xi^{-2}\right)\right] , \text{ for } \xi \to \infty .$$

The asymptotic forms of the derivatives of u_1 and u_2 may be shown to be

$$u_1' = bC_1\xi^{b-1}\left[1 + O\left(\xi^{-2}\right)\right] ,$$

(2.9)

$$u_2' = C_2\xi^{-b}e^{\frac{1}{2}\xi^2}\left[1 + O\left(\xi^{-2}\right)\right] .$$

u_1 and u_2 are scaled so that their Wronskian is $e^{\frac{1}{2}\xi^2}$.

With u replaced by u_1 , and α by α_1 , the term in equation (2.6) which dominates is

$$\left[\frac{X^2\alpha_1'}{p} - \left(b - \frac{X^2q}{p^2}\right)\alpha_1\right]u_1 .$$

To make this vanish, α_1 is chosen to be

(2.10) $$\alpha_1(x) = X^{-b}Y_0(x) ,$$

where Y_0 is the outer solution given by equation (1.4). Note that Y_0 has the factor x^b so that α_1 is bounded and non-zero at the origin as well as over the remainder of the interval. Also, X and Y_0 are twice continuously differentiable and so is α_1 , provided we make the further restrictions that p is three times continuously differentiable and q twice, in the neighbourhood of $x = 0$.

With u_2 and α_2 , the dominant terms in (2.6) are

(2.11) $$\varepsilon^{\frac{1}{2}}\left[\frac{2X\alpha_2'}{p} + \left(\frac{1}{X} + \frac{p'X}{p^2}\right)\alpha_2\right]u_2' - \left[\frac{X^2\alpha_2'}{p} - \left(b - \frac{X^2q}{p^2}\right)\alpha_2\right]u_2$$
$$= \left[\frac{X^2}{p}\alpha_2' + \left(b + 1 + \frac{p'X^2}{p^2} - \frac{X^2q}{p^2}\right)\alpha_2\right]u_2 + O\left(\xi^{-2}\right)u_2 .$$

α_2 is then given by

$$\alpha_2' + \left(\frac{(b+1)p}{X^2} + \frac{p'}{p} - \frac{q}{p}\right)\alpha_2 = 0 ,$$

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from which we obtain

$$\alpha_2 = \frac{X^{b+1}}{pY_0} ,$$

and again we note that α_2 has two continuous derivatives for $0 \le x \le 1$, and does not vanish in the interval, under the same conditions as α_1.

We now have two approximate solutions w_1 and w_2 with

$$L(w_1) = \varepsilon^{\frac{1}{2}} g_1(x) u_1'(\xi) + \varepsilon f_1(x) u_1 = F_1(\xi) , \tag{2.12}$$

$$L(w_2) = \varepsilon^{\frac{1}{2}} g_2(x) u_2'(\xi) + h_2(x) u_2(\xi) + \varepsilon f_2(x) u_2(\xi) = F_2(\xi) , \tag{2.13}$$

where

$$g_1(x) = \frac{2X\alpha_1'}{p} + \left(\frac{1}{X} + \frac{p'X}{p^2}\right)\alpha_1 , \tag{2.14}$$

$$f_1(x) = -\left[\frac{X^2\alpha_1''}{p^2} - \frac{X^2 p'\alpha_1'}{p^3} - \frac{\alpha_1'}{p} + \frac{X\alpha_1'}{p}\right] , \tag{2.15}$$

$$h_2(x) = -\left[\frac{X^2\alpha_2'}{p} - \left(b - \frac{X^2 q}{p^2}\right)\alpha_2\right] , \tag{2.16}$$

and $g_2(x)$, $f_2(x)$ are defined in a similar way to g_1, f_1 but with α_1 replaced by α_2.

It is clear that the functions g_1, g_2, f_1, f_2 and h_2 are all bounded and $O(1)$ for $\varepsilon \to 0$. We also have, since h_2 is continuously differentiable and vanishes at the origin,

$$|h_2(x)| < Kx \tag{2.17}$$

for some constant K.

Further, because of our choice of α_2, we have

$$|\varepsilon^{\frac{1}{2}} g_2 u_2 + h_2 u_2| < K\xi^{-2} u_2(\xi) \tag{2.18}$$

for $\xi > \xi_0$, where ξ_0 is some number independent of ε and greater than the largest zero of u_2.

Note that the quantities on the left-hand sides of (2.17), (2.18) may be simultaneously small since x may be small while ξ is large.

3. Existence of solutions

Our task now is to show that, in the interval $0 \le \xi \le \xi_1$, there exist exact solutions of (2.3) to which w_1 and w_2 are approximations. This is done by the construction of a suitable integral equation.

Since w_1 and w_2 are twice differentiable functions of ξ, there is a second order linear operator of the form

$$M(w) = w'' + P_1(\xi, \varepsilon)w' + Q_1(\xi, \varepsilon)w \,, \tag{3.1}$$

such that

$$M(w_1) = M(w_2) = 0 \,. \tag{3.2}$$

Equations (2.12), (2.13) may be written in the form

$$w_1'' + Pw_1' + Qw_1 = F_1 \,,$$

$$w_2'' + Pw_2' + Qw_2 = F_2 \,,$$

and we have, by our definition of the operator M,

$$w_1'' + P_1w_1' + Q_1w_1 = 0 \,,$$

$$w_2'' + P_1w_2' + Q_1w_1 = 0 \,.$$

These two pairs of equations give

$$(P-P_1)w_1' + (Q-Q_1)w_1 = F_1 \,,$$

$$(P-P_1)w_2' + (Q-Q_1)w_2 = F_2 \,,$$

so that

$$P_2 \equiv P - P_1 = -\frac{w_2F_1+w_1F_2}{W(w_1,\, w_2)} \,, \tag{3.3}$$

$$Q_2 \equiv Q - Q_1 = \frac{w_2'F_1-w_1'F_2}{W(w_1,\, w_2)} \,, \tag{3.4}$$

where $W(w_1, w_2)$ is the Wronskian of w_1, w_2.

Equation (2.3) may be expressed in the form

$$M\big(y(\xi;\, \varepsilon)\big) = -P_2y' - Q_2y \,. \tag{3.5}$$

In view of (3.2) we can use w_1 and w_2 to construct a suitable Green's function for the operator M and obtain an integral equation equivalent to (3.5) The appropriate integral equation is

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(3.6) $$y(\xi) = Aw_1(\xi) + Bw_2(\xi) + Ty(\xi) ,$$

where T is the integral operator given by

(3.7) $$Ty(\xi) = -w_1(\xi)\int_0^{\xi} [W(\eta)]^{-2}w_2(\eta)\Big\{[w_1(\eta)F_2(\eta) - w_2(\eta)F_1(\eta)]y'(\eta) + [w_2'(\eta)F_1(\eta) - w_1'(\eta)F_2(\eta)]y(\eta)\Big\}d\eta$$
$$- w_2(\xi)\int_{\xi}^{\xi_1} [W(\eta)]^{-2}w_1(\eta)\Big\{[w_1(\eta)F_2(\eta) - w_2(\eta)F_1(\eta)]y'(\eta) + [w_2'(\eta)F_1(\eta) - w_1'(\eta)F_2(\eta)]y(\eta)\Big\}d\eta ,$$

where

$$W(\eta) = W\big(w_1(\eta), w_2(\eta)\big) ,$$

and ξ_1 is defined by (2.2).

Let $v_1(\xi)$, $v_2(\xi)$ be two continuous functions defined by

$$v_1(0) = v_2(0) = 1 ,$$
$$v_1'(\xi) = v_2'(\xi) = 1 \quad \text{for} \quad 0 \le \xi \le \xi_0 ,$$
$$v_1'(\xi) = \xi^{b-1} ,$$
$$v_2'(\xi) = \xi^{-b}e^{\frac{1}{2}\xi^2} \quad \text{for} \quad \xi_0 \le \xi \le \xi_1 .$$

We consider the function spaces S_1 and S_2, where S_1 consists of functions satisfying, for some constant K which depends on the function,

$$|f(\xi)| < Kv_1(\xi) , \quad |f'(\xi)| < Kv_1'(\xi) ,$$

and similarly, S_2 consists of the differentiable functions satisfying

$$|f(\xi)| < Kv_2(\xi) , \quad |f'(\xi)| < Kv_2'(\xi) .$$

After some lengthy calculations using (2.17), (2.18), we can show that T is a contractive mapping in both S_1 and S_2 with contraction factors that are $O\left(\varepsilon^{\frac{1}{3}}\right)$, and hence that there exist exact solutions y_1, y_2 of equation (2.3) with

$$
\begin{aligned}
y_1(\xi;\ \varepsilon) &= w_1(\xi) + O\left(\varepsilon^{\frac{1}{3}}\right)v_1(\xi) \ , \\
y_1'(\xi;\ \varepsilon) &= w_1'(\xi) + O\left(\varepsilon^{\frac{1}{3}}\right)v_1'(\xi) \ , \\
y_2(\xi;\ \varepsilon) &= w_2(\xi) + O\left(\varepsilon^{\frac{1}{3}}\right)v_2(\xi) \ , \\
y_2'(\xi;\ \varepsilon) &= w_2'(\xi) + O\left(\varepsilon^{\frac{1}{3}}\right)v_2'(\xi) \ .
\end{aligned}
\tag{3.9}
$$

4. Continuation of solutions through the origin

One of the most important extensions of the problem originally posed is the continuation of the solutions through the zero of the coefficient p . Clearly, with similar conditions on the coefficients, similar results can be obtained for negative x as for positive x . We need to relate the asymptotic forms of solution for negative x to those for positive x .

Suppose firstly that b is neither zero nor a positive integer. Then a reference to Miller [3, pp. 687, 689], for example, tells us that the solution $u_1(\xi)$ of (2.4) satisfying the first of (2.8) is exponentially large for large negative ξ , that is

$$u_1 = C_3(-\xi)^{-b-1}e^{\frac{1}{2}\xi^2}\left[1 + O\left(\xi^{-2}\right)\right] \quad \text{for} \quad \xi \to -\infty \ .$$

Similarly $u_2(\xi)$ may be chosen to satisfy (2.4), the second of (2.8) and

$$u_2 = C_4(-\xi)^{b}\left[1 + O\left(\xi^{-2}\right)\right] \quad \text{for} \quad \xi \to -\infty \ ,$$

so that the roles of u_1 and u_2 are reversed on passing through the origin. Let two approximate solutions for $x < 0$ be $\beta_1(x)u_1(\xi)$, $\beta_2(x)u_2(\xi)$, where β_1 and β_2 are chosen in the same manner as α_2 and α_1 , respectively. Suppose also that $\beta_1(0) = \alpha_1(0)$, $\beta_2(0) = \alpha_2(0)$.

Let $z_1(\xi;\ \varepsilon)$, $z_2(\xi;\ \varepsilon)$ be exact solutions of (2.3) to which β_1u_1 and β_2u_2 approximate. By (3.9) and similar equations for z_1, z_2 we have

$$y_1(0;\ \varepsilon) = \alpha_1(0)u_1(0) + O\left(\varepsilon^{\frac{1}{3}}\right) = z_1(0;\ \varepsilon) + O\left(\varepsilon^{\frac{1}{3}}\right) ,$$

$$y_1'(0;\ \varepsilon) = \alpha_1(0)u_1'(0) + O\left(\varepsilon^{\frac{1}{3}}\right) = z_1'(0;\ \varepsilon) + O\left(\varepsilon^{\frac{1}{3}}\right) ,$$

$$y_2(0;\ \varepsilon) = z_2(0;\ \varepsilon) + O\left(\varepsilon^{\frac{1}{3}}\right) ,$$

$$y_2'(0;\ \varepsilon) = z_2'(0;\ \varepsilon) + O\left(\varepsilon^{\frac{1}{3}}\right) .$$

Clearly, y_1 and y_2 may be continued into negative values of ξ, and z_1 and z_2 may each be expressed as a linear combination of y_1 and y_2. Using their relations at the origin we have

$$y_1(\xi;\ \varepsilon) = z_1(\xi;\ \varepsilon) + O\left(\varepsilon^{\frac{1}{3}}\right)z_1(\xi;\ \varepsilon) + O\left(\varepsilon^{\frac{1}{3}}\right)z_2(\xi;\ \varepsilon) , \tag{4.1}$$

$$y_2(\xi;\ \varepsilon) = z_2(\xi;\ \varepsilon) + O\left(\varepsilon^{\frac{1}{3}}\right)z_1(\xi;\ \varepsilon) + O\left(\varepsilon^{\frac{1}{3}}\right)z_2(\xi;\ \varepsilon) . \tag{4.2}$$

In terms of the prototype function v_2,

$$y_1(\xi;\ \varepsilon) = z_1(\xi;\ \varepsilon) + O\left(\varepsilon^{\frac{1}{3}}\right)v_2(-\xi) .$$

We cannot use z_2 or $\beta_2 u_2$ as an approximation to y_2 for $\xi < 0$, since the term $O\left(\varepsilon^{\frac{1}{3}}\right)z_1(\xi;\ \varepsilon)$ on the right of (4.2) dominates for large negative ξ. However, in this case we can continue z_2 to positive values of ξ and obtain, for $\xi > 0$,

$$z_2(\xi;\ \varepsilon) = y_2(\xi;\ \varepsilon) + O\left(\varepsilon^{\frac{1}{3}}\right)v_2(\xi) .$$

Thus there are two solutions y_1, z_2 with y_1 exponentially large for negative x and z_2 exponentially large for positive x.

These and their approximations allow us to obtain solutions to two-point boundary value problems where the conditions are applied on opposite sides of the origin. The solutions give a boundary layer at each boundary point with exponentially small values in the interior.

In the case when b is a positive integer, $u_1(\xi)$ is a polynomial, while $u_2(\xi)$ is exponentially large for both large positive and large negative ξ. Equations (4.1) and (4.2) still hold but y_1 cannot be approximated by z_1 (or $\beta_1 u_1$) because the term $O\left(\varepsilon^{\frac{1}{3}}\right)z_2(\xi;\ \varepsilon)$ on the right of (4.1) dominates for large negative ξ. Without the ability to extend the approximate form for y_1 through the origin we cannot find approximate solutions to two-point boundary value problems such as those

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discussed by Ackerberg and O'Malley [1] in Part 3 of their paper.

A useful example here is the equation

$$\varepsilon y'' - xy' + (b+x)y = 0 . \qquad (4.5)$$

The transformations

$$\xi = \frac{x}{\varepsilon^{\frac{1}{2}}} ,$$

$$y = e^{\frac{\xi^2}{4}} z ,$$

yield

$$\frac{d^2 z}{d\xi^2} + \left[b + \varepsilon + \frac{1}{2} - \left(\frac{\xi - 2\varepsilon^{\frac{1}{2}}}{2}\right)^2\right] z = 0 , \qquad (4.6)$$

whose solutions are parabolic cylinder functions. If $b + \varepsilon$ is a positive integer, then there is a solution of (4.5) which is only algebraically large for both positive and negative x. If b is a positive integer, any solution of (4.5) is exponentially large for positive x or negative x or both. This contradicts the results in Parts 3a, 3b of the paper by Ackerberg and O'Malley.

5. Remarks

For an interval extending to ∞, the approximate solutions w_1 and w_2 are not valid, in that they do not approximate to the exact solutions y_1 and y_2. It is the terms containing $f_1(x)$ and $f_2(x)$ in (2.12), (2.13) which cause the trouble and it is these terms, as shown by (2.6), which contain α_1'' and α_2''. To obtain approximate solutions valid over larger intervals than $[0, \xi_1]$, it would be necessary to continue the two-variable process to higher order terms. The same type of behaviour is shown in problems of non-linear oscillations and in problems involving adiabatic invariants, where it is possible to trade off accuracy in the approximation for a larger interval of validity. That is, the approximate solutions become less accurate as the interval is extended until, after a sufficiently large time, the errors become comparable with the solution itself. There are, however, examples such as the asymptotic solutions of Bessel's equation where the first approximation to the solution is valid over an infinite interval.

The method used here of deriving approximate solutions by a two-variable procedure and then using them to obtain an integral equation is also applicable in problems which have been dealt with by other methods such as those described by Erdélyi [2] and Mitropol'skiĭ [4].

The restrictions on p and q that were introduced in Section 2 (after (2.10)) may be avoided by the use of a messy modification to the method. The third continuous derivative of p and the second of q are necessary only to ensure that α_1 and α_2 have bounded second derivatives in the neighbourhood of the origin, but the particular forms of α_1 and α_2 are chosen to satisfy asymptotic conditions for large ξ. Instead of adding these extra conditions on p and q, we could use alternative approximate solutions given by

$$\begin{aligned} w_1(\xi) &= \alpha_1(x)u_1(\xi) & \xi_0 < \xi < \xi_1 , \\ &= \beta_1(\xi;\ \varepsilon)u_1(\xi) & 0 < \xi < \xi_0 , \\ w_2(\xi) &= \alpha_2(x)u_2(\xi) & \xi_0 < \xi < \xi_1 , \\ &= \beta_2(\xi;\ \varepsilon)u_2(\xi) & 0 < \xi < \xi_0 , \end{aligned}$$

where β_1 and β_2 are chosen so that $\beta_1, \frac{d\beta_1}{dx}, \frac{d^2\beta_1}{dx^2}, \beta_2, \frac{d\beta_2}{dx}, \frac{d^2\beta_2}{dx^2}$ have the same values as $\alpha_1, \alpha_1', \alpha_1'', \alpha_2, \alpha_2', \alpha_2''$, respectively, at ξ_0. It is also specified that $\frac{d^2\beta_1}{dx^2}, \frac{d^2\beta_2}{dx^2}$ are $O\left(\varepsilon^{-\frac{1}{2}}\right)$, which is possible since α_1'' and α_2'' are of this order at $\xi = \xi_0$.

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