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FIRST ORDER THEORY OF COMPLETE STONEAN ALGEBRAS
(BOOLEAN-VALUED REAL AND COMPLEX NUMBERS)

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ABSTRACT. We axiomatize the theory of real and complex numbers in Boolean-valued models of set theory, and prove that every Horn sentence true in the complex numbers is true in any complete Stonean algebra, and provable from its axioms.

1. **Introduction.** In [6] we introduced *complete Stonean algebras*, to describe axiomatically abelian algebras of normal operators in a Hilbert space. Our approach is based on Scott and Solovay's description of complex numbers in Boolean-valued models of set theory [14] and extends Takeuti's application of Boolean-valued models to normal operators [15]. The axiomatization in [6] describes algebraic as well as topological properties of complete algebras of mutually commuting normal operators. It is proved in [6] that every complete Stonean algebra is isomorphic to the set of all complex numbers in the Boolean-valued model V^B, for some complete Boolean algebra B. Examples of complete Stonean algebras include:

(a) **C**, the complex numbers
(b) all measurable functions on [0, 1] modulo = a.e.
(c) all Borel functions on [0, 1] modulo = on comeager sets
(d) all normal operators affiliated with a given abelian von Neumann algebra.

The purpose of this note is to study the first order theory of complete Stonean algebra, or equivalently, the first order theory of the structure of real and complex numbers in a Boolean-valued model.

2. **Boolean-valued complex numbers.** Let B be a complete Boolean algebra, and let V^B be the corresponding Boolean-valued model of set theory. Without loss of generality, we assume that $\|x = y\| = 1$ only if $x = y$. In V^B, consider the set $\mathbf{C}^B$ of all complex numbers, with addition $+$, multiplication $\cdot$, complex conjugation $*$, zero 0 and unit 1.

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We also denote by $\mathbf{C}^B$ the set of all $x \in V^B$ such that $\|x \in \mathbf{C}^B\| = 1$, and endow it with the operations $+, \cdot, *$ in the obvious way:

$$x + y = z \quad \text{if } \|x + y = z\| = 1$$
$$x \cdot y = z \quad \text{if } \|x \cdot y = z\| = 1$$
$$x^* = z \quad \text{if } \|x^* = z\| = 1.$$

The structure

$$S = (\mathbf{C}^B, +, \cdot, *, 0, 1)$$

is called *B-valued complex numbers*, or a *complete Stonean algebra*.

3. **The axiomatic theory** $\mathscr{S}$. We consider the language $\mathscr{L} = \{+, \cdot, *, 0, 1\}$ of rings with involution. Let $\mathscr{S}$ be the following set of axioms for $\mathscr{L}$:

(S1) S is a commutative ring with 1
(S2) S is algebraically closed and of characteristic 0
(S3) S is von Neumann regular, i.e. $\forall x \exists y\ xyx = x$
(S4) $(a + b)^* = a^* + b^*$, $(ab)^* = a^*b^*$, $a^{**} = a$
(S5) the reals of S (i.e. $a^* = a$) are real-closed.

Let S be a model of $\mathscr{S}$, and let B be the algebra of all projections, i.e. $B = \{a \in S : a^2 = a\}$. B is a Boolean algebra. If S is a complete Stonean algebra, then B is complete, and S is isomorphic to $\mathbf{C}^B$.

Let (S_B) = the set of sentences indicating (i) the number $n \in \mathbf{N} \cup \{\infty\}$ of atoms of B, and (ii) whether or not for every nonzero $b \in B$ there is an atom a in B such that $a \leqq b$.

Let (*) be the sentence stating that the set of all atoms of B has a least upper bound:

$$(*)\ \exists c\ (c \text{ is the least upper bound of the atoms}).$$

We consider the theory

$$\mathscr{S}_B = \mathscr{S} + (S_B) + (*).$$

THEOREM 1. *For any Boolean algebra B,* $\mathscr{S}_B$ *is a complete theory.*

COROLLARY. *For any complete Stonean algebra S, the first order theory of S is a decidable, complete axiomatizable theory.*

PROOF. $\mathscr{S}_B$ is model-complete and has a prime model. This can be proved in an analogous way to the classical result of A. Tarski and A. Robinson, that the first order theory of complex and real numbers is decidable.

In fact, this result follows from the work of A. Carson [2], [3], L. Lipschitz and D. Saracino [8], A. Macintyre [11] and L. van den Dries [16]. □

We remark that if one defines the partial ordering of the reals of S by

$$a \leqq b \text{ if and only if } \exists x\ xx^* = b - a$$

then the reals of S form a real closed commutative regular f-ring as defined in [8].

In the particular case when B is the trivial Boolean algebra $\{0, 1\}$, S_B becomes

$$\text{(S0)} \qquad \forall a \in b \quad (a = 0 \vee a = 1).$$

The theory

$$\mathscr{S}_0 = \mathscr{S} + \text{(S0)}$$

is the complete theory of complex numbers $(\mathbf{C}, +, \cdot, *, 0, 1)$.

4. **Boolean-valued models and generic formulas.** Boolean-valued models are a generalization of ordinary (two-valued) models. Let B be a complete Boolean algebra. A *B-valued model* for a language $\mathscr{L} = (P, \ldots, f, \ldots)$ is

$$(M, \|x = y\|, \|R(\vec{x})\|, \ldots, f, \ldots)$$

where $\|x = y\|$ is a function from $M \times M$ into B, $\|R(\vec{x})\|$ is a function with values in B, and f is an operation on M, such that

(a) $$\|x = x\| = 1$$
$$\|x = y\| = \|y = x\|$$
$$\|x = y\| \cdot \|y = z\| \leqq \|x = z\|$$

(b) $$\|x_i = y_i\| \cdot \|R(\ldots x_i \ldots)\| \leqq \|R(\ldots y_i \ldots)\|$$

(c) $$\|x_i = y_i\| \leqq \|f(\ldots x_i \ldots) = f(\ldots y_i \ldots)\|$$

(d) $$\text{if } \|x = y\| = 1 \text{ then } x = y.$$

For every formula φ of $\mathscr{L}$, one defines, by induction on the complexity of φ, the B-value of φ:

$$\|\varphi(a_1, \ldots, a_n)\| \in B \quad (a_1, \ldots, a_n \in M).$$

All theorems of predicate calculus have B-value 1.

We shall deal with the following type of Boolean-valued models. Let M be a set in the Boolean-valued universe V^B, and assume that

$$\|M \text{ is a model for } \mathscr{L}\| = 1.$$

If we identify M with the set of all $x \in V^B$ such that $\|x \in M\| = 1$, and if $R, f \in V^B$ are the relations and operations of M in V^B, then

$$\text{(4.1)} \qquad (M, \|x = y\|, \|R(\vec{x})\|, \ldots, f, \ldots)$$

is a B-valued model for $\mathscr{L}$.

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It is a well-known fact in the theory of Boolean-valued models of set theory that a B-valued model M so obtained has the following property: for any formula φ of $\mathcal{L}$, and any $\vec{x} \in M$, there is some $a \in M$ such that

$$\|\varphi(a, \vec{x})\| = \|\exists z \varphi(z, \vec{x})\|. \tag{4.2}$$

We call a B-valued model M that satisfies (4.2) *full*.

Given a B-valued model M, we make M into a two-valued model as follows:

$$R(\vec{x}) \text{ if and only if } \|R(\vec{x})\| = 1 \quad (\vec{x} \in M). \tag{4.3}$$

Thus the construction of B-valued complex numbers in section 2 is a special case of the construction (4.3), for the full B-valued model $\mathbf{C}^B$ obtained by (4.1) from the complex numbers of V^B.

DEFINITION. STRONGLY GENERIC AND GENERIC FORMULAS.

(a) Every atomic formula is strongly generic.
(b) If φ and ψ are strongly generic formulas, then

$$\varphi \wedge \psi, \exists x \varphi, \forall x \varphi$$

are strongly generic.
(c) Every strongly generic formula is generic.
(d) If φ and ψ are generic formulas, then

$$\varphi \wedge \psi, \exists x \varphi, \forall x \varphi$$

are generic.
(e) If φ is strongly generic then $\neg\varphi$ is generic.
(f) If φ is strongly generic and ψ is generic then $\varphi \rightarrow \psi$ is generic.

THEOREM 2. *Let M be a full B-valued model.*
(a) *If φ is strongly generic formula, then*

$$\|\varphi(\vec{x})\| = 1 \textit{ if and only if } M \vDash \varphi(\vec{x}) \quad (\vec{x} \in M).$$

(b) *If φ is a generic formula, then*

$$\|\varphi(\vec{x})\| = 1 \textit{ implies } M \vDash \varphi(\vec{x}) \quad (\vec{x} \in M).$$

PROOF. By induction on the complexity of φ.

(a) If φ is atomic then the statement follows from (4.3). For the conjunction, we have

$$\|\varphi \wedge \psi\| = 1 \text{ if and only if } \|\varphi\| = 1 \text{ and } \|\psi\| = 1.$$

Similarly,

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$$||\forall x \varphi|| = 1 \text{ if and only if } \forall x \in M ||\varphi(x)|| = 1.$$

For the existential quantifier, we use fullness (4.2):

$$||\exists x \varphi|| = 1 \text{ if and only if } \exists x \in M ||\varphi(x)|| = 1.$$

(b) The argument for $\wedge$, $\forall$ and $\exists$ is similar to (a).

If φ is the negation of a strongly generic formula ψ and if $||\varphi|| = 1$, then $||\psi|| = 0 \neq 1$, and by (a), M does not satisfy ψ. Hence $M \models \varphi$.

Finally, consider $\varphi \rightarrow \psi$ where φ is strongly generic and ψ is generic, and assume that $||\varphi \rightarrow \psi|| = 1$. In order to show that $M \models \varphi \rightarrow \psi$, assume that $M \models \varphi$. By (a), $||\varphi|| = 1$, and therefore $||\psi|| = 1$. By the induction hypothesis on ψ, M satisfies ψ. □

Prof. M. Takahashi as well as the referee of this paper kindly pointed out to me that any generic formula is logically equivalent to a Horn formula:

DEFINITION [4; p. 328] HORN FORMULAS.

(a) A basic Horn formula is a disjunction

$$\theta_1 \vee \ldots \vee \theta_m,$$

where at most one of the formulas θ_i is an atomic formula, the rest being negations of atomic formulas.

(b) A Horn formula is built up from basic Horn formulas with the connectives $\wedge$, $\exists$ and $\forall$.

PROPOSITION. *Any generic formula is logically equivalent to a Horn formula, and vice versa.*

PROOF. By induction on formula length. □

5. **Transfer theorem.**

THEOREM. (a) *Any Horn sentence true in the structure* $(\mathbf{C}, +, \cdot, ^*, 0, 1)$ *is true in any complete Stonean algebra.*

(b) *Any Horn sentence true in* $(\mathbf{C}, +, \cdot, ^*, 0, 1)$ *is a theorem of* $\mathscr{S}$.

REMARK. All the axioms of φ are Horn sentences.

PROOF. (a) Let S be a complete Stonean algebra and let σ be a generic sentence true in $\mathbf{C}$. By the representation [6], S is isomorphic to $\mathbf{C}^B$, for some complete Boolean algebra B. The sentence σ, true in $\mathbf{C}$, is a theorem of the complete theory $\mathscr{S}_0$. By absoluteness of $\mathscr{S}_0$ for (Boolean-valued) models of set theory, $\mathscr{S}_0 \vdash \sigma$ holds in V^B, and so σ is true in the complex numbers in V^B. In other words, the B-valued model $\mathbf{C}^B$ has the property

$$||\sigma|| = 1.$$

Since σ is generic, the Theorem (b) in section 3 gives

$$\mathbf{C}^B \models \sigma,$$

and so σ is true in S.

(b) Let σ be a Horn sentence true in $\mathbf{C}$, and let S be a model of $\mathscr{S}$. Let B be the Boolean algebra of projections of S, and let X be the Stone space of B. Let F be the algebraic closure of the field of the (complex) rationals and let P be the ring of all locally constant functions $X \to F$. By results and methods from [3] and [16], P is an elementary submodel of S. Furthermore (by induction on formula length), σ holds in P. Hence σ holds in S.

6. **Examples.** The transfer theorem sheds light on the well known phenomenon of operator theory, namely that normal operators can be often treated as complex numbers. An example of a generic sentence true in $\mathbf{C}$ is the existence of a square root:

$$\forall a[a \geqq 0 \rightarrow \exists x\,(x \cdot x = a)\,].$$

On the other hand, the existence of an inverse,

$$\forall a[a \neq 0 \rightarrow \exists x\,(x \cdot a = 1)\,]$$

is not generic. Compare this to the fact that every positive operator in a Hilbert space has a square root, but not every nonzero operator is invertible.

Also, the partial order $\leqq$ is not linear in general, while it is linear for real numbers. The linearity condition

$$\forall a \forall b(a \leqq b \vee b \leqq a)$$

is not generic.

Another example of a generic sentence is the following:

(6.1) For every $n \times n$ real symmetric matrix M there exists a unitary $n \times n$ matrix U such that $U^T M U$ is a diagonal matrix.

Since (6.1) is true in $\mathbf{R}$, it follows that such diagonalization is true for any matrix of commuting self-adjoint operators. This is a special case of Kadison's theorem [7]. We refer the reader to [13] for an alternate approach (using intuitionistic logic).

We give one more example, a result from [17]:

(6.2) Let α be a real number, $0 \leqq \alpha \leqq 1/2$, and let a be a self-adjoint element of a von Neumann algebra whose spectrum is included in $[-1, 2\alpha - 1] \cup [1 - 2\alpha, 1]$. Then $a = \alpha u_1 + (1 - \alpha)u_2$ for some unitaries u_1 and u_2.

The following statement, in the language of $\mathscr{S}$, is obviously true in $\mathbf{C}$:

$$\text{(6.3)} \qquad [a^* = a \text{ and } (1 - 2\alpha)^2 \leqq a^*a \leqq 1]$$

$$\rightarrow \exists u_1 \exists u_2 [u_1^* u_1 = u_2^* u_2 = 1 \text{ and } a = \alpha u_1 + (1 - \alpha) u_2].$$

Since it is generic, it is also true in the complete Stonean algebra generated by a, and (6.2) follows.

7. **Generalizations.** The Transfer theorem remains valid if we expand the language of *-rings to the language of *-algebras, by adjoining constant symbols for all complex numbers (scalars). A complete axiomatization of the theory of $\mathbf{C}^B$ in this language is obtained by adding obvious axioms for scalar multiplication, as well as the diagram of **C**.

A generalization fails, however, if we wish to include the norm. The fact that complete Stonean algebras have in general elements of infinite norm provides a counterexample. We also note that if B is atomless then the theory of $\mathbf{C}^B$ in a language that includes the norm is undecidable: it is then possible to define the set of all scalars, which in turn, by Macintyre's argument [10], makes it possible to define the set of all natural numbers, and interpret an undecidable theory.

The Transfer theorem can also be generalized to allow infinitary generic formulas (i.e. allow infinite conjunctions). An example is the sentence

"there exists a transcendental element".

8. **Boolean-valued models and sheaves.** The referee has pointed out that the Transfer theorem can be derived using sheaves over Boolean spaces. In particular, a related result is proved in [1], Section 3, and our Transfer theorem can be obtained from that result, using representations [12] and [5], and using results from [3] and [11].

We wish to conclude by comparing the representations of a complete Stonean algebra by a sheaf of fields on one hand, and by a Boolean-valued model on the other.

Let B be a complete Boolean algebra, and let $R = \mathbf{R}^B$ be the ring of all B-valued real numbers. Let X be the Stone space of B; each $u \in X$ is an ultrafilter on B.

In the representation of R by continuous global sections of a sheaf over X, R is identified with all continuous functions on X, with values in K_u, $u \in X$. Each K_u is a field, in fact K_u is the quotient of R by the congruence relation $\|a = b\| \in u$. If u is nonprincipal then K_u is non-Archimedean; thus in general K_u need not be isomorphic to **R**.

On the other hand, Theorem B of [6] gives a representation of $R = \mathbf{R}^B$ by the space of all continuous functions $f: X \rightarrow \mathbf{R} \cup \{\infty, -\infty\}$ such that $f_{-1}(\pm\infty)$ is nowhere dense.

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