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Alexander Gorodnik and Amos Nevo

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Abstract

In [Gorodnik and Nevo, *Counting lattice points*, J. Reine Angew. Math. **663** (2012), 127–176] an effective solution of the lattice point counting problem in general domains in semisimple S-algebraic groups and affine symmetric varieties was established. The method relies on the mean ergodic theorem for the action of G on G/Γ, and implies uniformity in counting over families of lattice subgroups admitting a uniform spectral gap. In the present paper we extend some methods developed in [Nevo and Sarnak, *Prime and almost prime integral points on principal homogeneous spaces*, Acta Math. **205** (2010), 361–402] and use them to establish several useful consequences of this property, including:

(1) effective upper bounds on lifting for solutions of congruences in affine homogeneous varieties;

(2) effective upper bounds on the number of integral points on general subvarieties of semisimple group varieties;

(3) effective lower bounds on the number of almost prime points on symmetric varieties;

(4) effective upper bounds on almost prime solutions of congruences in homogeneous varieties.

1. Introduction and statement of results

Throughout the paper, F denotes a number field, and V_F denotes the set of absolute values of F extending the standard normalised absolute values of the rational numbers; F_v, $v \in V_F$, will denote the corresponding local fields.

We introduce local and global heights. For Archimedean $v \in V_F$, and for $x = (x_1, \dots, x_d) \in F_v^d$, we set

$$\mathrm{H}_v(x) = (|x_1|_v^2 + \cdots + |x_d|_v^2)^{1/2},$$

and for non-Archimedean v we set

$$\mathrm{H}_v(x) = \max\{|x_1|_v, \dots, |x_d|_v\}.$$

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For $x = (x_1, \dots, x_d) \in F^d$, we set

$$\mathrm{H}(x) = \prod_{v \in V_F} \mathrm{H}_v(x).$$

1.1 Effective lifting of solutions of congruences

Let S be a finite subset of V_F containing all Archimedean absolute values, and

$$O_S = \{x \in F : |x|_v \leqslant 1 \text{ for } v \notin S\}$$

be the ring of S-integers in F. We consider a system X of polynomial equations with coefficients in O_S. Given an ideal $\mathfrak{a}$ of O_S, we denote by $\mathsf{X}^{(\mathfrak{a})}$ the system of polynomial equations over the factor-ring $O_S/\mathfrak{a}$ obtained by reducing X modulo $\mathfrak{a}$. There is a natural reduction map

$$\pi_{\mathfrak{a}} \colon \mathsf{X}(O_S) \to \mathsf{X}^{(\mathfrak{a})}(O_S/\mathfrak{a}).$$

The question of whether a solution in $\mathsf{X}^{(\mathfrak{a})}(O_S/\mathfrak{a})$ can be lifted to an integral solution in $\mathsf{X}(O_S)$ is of fundamental importance in number theory. It is closely related to the strong approximation property for algebraic varieties (see [PR94, § 7.1]). For instance, if G is a connected F-simple simply connected algebraic group which is isotropic over S, then G satisfies the strong approximation property (see [PR94, § 7.4]) and, in particular, the map $\pi_{\mathfrak{a}}$ is surjective in this case. For more general homogeneous varieties, the map $\pi_{\mathfrak{a}}$ need not be surjective, but the image $\pi_{\mathfrak{a}}(\mathsf{X}(O_S))$ can be described using the Brauer–Manin obstructions (see [BD09, CX09, Har08]).

In this paper, we consider the problem of whether a solution in $\mathsf{X}^{(\mathfrak{a})}(O_S/\mathfrak{a})$ can be lifted to an integral solution in $\mathsf{X}(O_S)$ *effectively*: given $\bar{x} \in \mathsf{X}^{(\mathfrak{a})}(O_S/\mathfrak{a})$, can one find $x \in \mathsf{X}(O_S)$, with $\mathrm{H}(x)$ bounded in terms of $|O_S/\mathfrak{a}|$, such that $\pi_{\mathfrak{a}}(x) = \bar{x}$?

We give a positive answer to this question for affine homogeneous varieties of F-simple algebraic groups. Let X be an affine variety defined over F and equipped with a transitive action (defined over F) of a connected simply connected F-simple algebraic group $\mathsf{G} \subset \mathrm{GL}_m$.

Let S be a finite subset of V_F, containing all Archimedean absolute values, such that both G and X have models defined over O_S (i.e., schemes $\mathcal{G}$ and $\mathcal{X}$ over O_S with generic fibers isomorphic to G and X respectively), the action of G on X extends to the models, and $\mathrm{Lie}(\mathsf{G}) \cap \mathrm{M}_m(O_S)$ has a basis over O_S as an O_S-module. We note that every sufficiently large subset S of V_F satisfies the above assumptions. Moreover, the last assumption on S is satisfied when O_S is a principal ideal domain. In particular, the last assumption always holds when the field F has class number one. Here, and below, the notation $\mathsf{G}(O_S)$ and $\mathsf{X}(O_S)$ means $\mathcal{G}(O_S)$ and $\mathcal{X}(O_S)$ respectively.

Theorem 1.1. *There exist q_0 and $\sigma > 0$ such that for every ideal $\mathfrak{a}$ of O_S satisfying $|O_S/\mathfrak{a}| \geqslant q_0$ and every $\bar{x} \in \pi_{\mathfrak{a}}(\mathsf{X}(O_S))$ there exists $x \in \mathsf{X}(O_S)$ such that*

$$\pi_{\mathfrak{a}}(x) = \bar{x} \quad \textit{and} \quad \mathrm{H}(x) \leqslant |O_S/\mathfrak{a}|^{\sigma}. \tag{1.1}$$

The parameter σ in (1.1) can be explicitly computed. For instance, for group varieties, an explicit value of σ is given in Theorem 2.1 below. The parameter q_0 is computable too (see Remark 2.3 below).

Remark 1.2. The finiteness of the exponent σ for the case of S-integral points in the group variety follows from the fact that each Cayley graph $\mathsf{G}^{(\mathfrak{a})}(O_S/\mathfrak{a})$ has a logarithmic diameter. The bound provided by this approach depends on a choice of generating set of $\mathsf{G}(O_S)$, and when measured in terms of the height H it is of lesser quality than the estimate on σ which is developed below explicitly in terms of geometric and representation-theoretic data for G. We thank Peter Sarnak for this remark.

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Let us now consider the case of a connected F-simple simply connected algebraic group $\mathsf{G} \subset \mathrm{GL}_m$ which is isotropic over S. Then it is known to satisfy the strong approximation property, and our method gives an asymptotic formula for the number of solutions of (1.1).

THEOREM 1.3. *For every $\sigma > \sigma_0$ (as in (2.1) below), every ideal $\mathfrak{a}$ in O_S and every $\bar{x} \in \mathsf{G}^{(\mathfrak{a})}(O_S/\mathfrak{a})$,*

$$\begin{aligned}&|\{x \in \mathsf{G}(O_S);\, \pi_{\mathfrak{a}}(x) = \bar{x}, \mathrm{H}(x) \leqslant |O_S/\mathfrak{a}|^{\sigma}\}| \\ &\quad = \frac{1}{|\mathsf{G}^{(\mathfrak{a})}(O_S/\mathfrak{a})|} \cdot |\{x \in \mathsf{G}(O_S);\, \mathrm{H}(x) \leqslant |O_S/\mathfrak{a}|^{\sigma}\}|(1 + O_{\epsilon}(|O_S/\mathfrak{a}|^{\dim(\mathsf{G})(1-\sigma_0^{-1}\sigma)+\epsilon}))\end{aligned}$$

for every $\epsilon > 0$.

This result indicates that the properties $\pi_{\mathfrak{a}}(x) = \bar{x}$ and $\mathrm{H}(x) \leqslant |O_S/\mathfrak{a}|^{\sigma}$ are asymptotically independent.

We illustrate our results on a classical example; the problem of representing a quadratic form by another quadratic form (see, for instance, [O'Me00]).

Example 1.4. Let A be an integral non-degenerate symmetric $(n \times n)$-matrix and B be an integral non-degenerate symmetric $(m \times m)$-matrix with $n \leqslant m$. The variety

$$\mathsf{X} = \{x \in \mathrm{M}_{m\times n}(\mathbb{C}) : {}^t xBx = A\} \tag{1.2}$$

parametrises all possible representations of the quadratic form corresponding to A by the quadratic form corresponding to B. For simplicity, we assume that $m - n \geqslant 3$ and A is isotropic over $\mathbb{R}$. Then if the equation ${}^t xBx = A$ has a solution over $\mathbb{R}$ and over $\mathbb{Z}_p$ for every p, then it has an integral solution, and the reduction map $\mathsf{X}(\mathbb{Z}) \to \mathsf{X}(\mathbb{Z}/q)$ is surjective for every $q \geqslant 1$ (see [O'Me00, ch. X]). Our results implies that under the same assumptions, for every $q \geqslant 1$ and $\bar{x} \in \mathrm{M}_{m\times n}(\mathbb{Z}/q)$ satisfying

$${}^t\bar{x}B\bar{x} = A \mod q,$$

there exists $x \in \mathrm{M}_{m\times n}(\mathbb{Z})$ such that

$${}^t xBx = A, \quad x = \bar{x} \mod q, \quad \mathrm{H}_{\infty}(x) \ll q^{\sigma}, \tag{1.3}$$

where $\sigma > 0$ is a computable constant. For instance, when B has the signature $(\lfloor m/2 \rfloor, m - \lfloor m/2 \rfloor)$, this estimate holds for

$$\sigma > \sigma_m = \begin{cases} \dfrac{4m(m^2 - m + 1)n_e}{m - 1} & \text{when } m \text{ is odd,} \\ \dfrac{4(m-1)(m^2 - m + 1)n_e}{m + 2} & \text{when } m \text{ is even,} \end{cases} \tag{1.4}$$

where n_e denotes the least even integer greater than or equal to $\lfloor m/2 \rfloor$. We will provide details of this computation in § 2.

The following example demonstrates that the polynomial bound established in Theorem 1.1 does not hold for other homogeneous varieties.

Example 1.5. Let F be a number field of degree d with an infinite group of units and $\{\xi_1, \ldots, \xi_d\}$ be a basis of the ring O of integers of F. We consider the integral polynomial

$$f(x_1, \ldots, x_d) = N_{K/\mathbb{Q}}(x_1\xi_1 + \cdots + x_d\xi_d)$$

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and the variety $\mathsf{X} = \{f = 1\}$. Note that the set of integral points on X is exactly the group U of units in the number field F. We note that

$$|\{x \in \mathsf{X}(\mathbb{Z}) : \mathrm{H}_\infty(x) \leqslant T\}| \ll (\log T)^{r+s-1} \tag{1.5}$$

where r and s denote the number of real and complex absolute values of F respectively. This claim can be checked by representing the group of units as a lattice in $\mathbb{R}^{r+s-1}$, similarly to the proof of Dirichlet's theorem.

We also note that there are infinitely many primes $\mathfrak{p}$ in the ring of integers O of F such that

$$(O/\mathfrak{p})^\times = U \mod \mathfrak{p}. \tag{1.6}$$

It was proved in [CW75] that the set of such primes has positive density if one assumes the generalised Riemann hypothesis, and in [Nar88] that in most cases (for instance, when $[F : \mathbb{Q}] > 3$) there are infinitely many such primes unconditionally. Now it follows from (1.6) that

$$|\pi_p(\mathsf{X}(\mathbb{Z}))| \geqslant p - 1 \tag{1.7}$$

for infinitely many prime numbers p. Comparing (1.5) and (1.7), we conclude that the polynomial bound as in Corollary 1.1 is impossible in this case.

1.2 Integral points on subvarieties

We now turn to consider the problem of bounding the number of integral points on algebraic varieties. This has been an active field of research in recent years, and we refer the reader to the survey [Hea06] and the book [Bro09] for overviews of results and conjectures concerning upper estimates on the number of integral points. We will concentrate on homogeneous varieties, and our methods and results are motivated by those developed in [NS10, § 4.3].

Given affine varieties $\mathsf{Y} \subset \mathsf{X}$ defined over a number field F, we fix models for $\mathsf{Y} \subset \mathsf{X}$ defined over a ring O_S of S-integers in F, and set

$$N_T(\mathsf{Y}(O_S)) = |\{y \in \mathsf{Y}(O_S) : \mathrm{H}(y) \leqslant T\}|.$$

The problem we will focus on is establishing an upper estimate on $N_T(\mathsf{Y}(O_S))$ for arbitrary proper affine subvariety Y of X. We will prove a *non-concentration phenomenon* for the collection of proper subvarieties of a semisimple group variety G, namely that the number of S-integral points on Y has strictly lower rate of growth than G. We remark that this important property does not hold for general irreducible varieties X. Indeed a bound of the form

$$N_T(\mathsf{Y}(O_S)) \ll_{\mathsf{X},\deg(\mathsf{Y})} N_T(\mathsf{X}(O_S))^{1-\sigma_\mathsf{Y}}$$

with $\sigma_\mathsf{Y} > 0$, where we write $\deg(\mathsf{Y})$ for the degree of the projective closure of Y, is false in general. This can be demonstrated by the variety $x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0$, where most of the rational points lie on lines (see [Hea97]). However, in the case of group varieties we have the following theorem.

Theorem 1.6. *Let G be a connected F-simple simply connected algebraic group and Y an absolutely irreducible proper affine subvariety of G defined over a number field F. Let $S \subset V_F$ be a finite subset containing all Archimedean absolute values such that G is isotropic over S, and $\mathsf{Y} \subset \mathsf{G}$ have models defined over O_S. Then there exists $\sigma = \sigma(\mathsf{G}, S, \dim(\mathsf{Y})) \in (0, 1)$ such that*

$$N_T(\mathsf{Y}(O_S)) \ll_{\mathsf{G},\deg(\mathsf{Y})} N_T(\mathsf{G}(O_S))^{1-\sigma}.$$

An explicit formula for the exponent σ is given in Theorem 3.1 below, demonstrating that σ depends only on $\dim \mathsf{Y}$, and increases monotonically with the codimension of Y.

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To demonstrate Theorem 3.1 let us consider the case of integral points on subvarieties of the special linear group SL_n, $n \geqslant 2$.

Example 1.7. For every absolutely irreducible proper affine subvariety Y of SL_n defined over $\mathbb{Z}$, we have

$$N_T(\mathsf{Y}(\mathbb{Z})) \ll_{n,\deg(\mathsf{Y}),\epsilon} T^{n^2-n-((n^2-1-\dim(\mathsf{Y}))/(n^2+n)2n_e)+\epsilon}, \quad \epsilon > 0, \tag{1.8}$$

as $T \to \infty$, where n_e is the least even integer greater than or equal to $n-1$. This improves the trivial estimate $N_T(\mathsf{Y}(\mathbb{Z})) \ll T^{n^2-n}$. Details of this computation will be given in § 3.

We note that the assumption of absolute irreducibility is not crucial for the conclusion of Theorem 1.6. Another version of Theorem 1.6 which can be proved using the argument of [NS10, Lemma 4.2] is as follows.

Theorem 1.8. *With notation as in Theorem 1.6, for every proper affine subvariety* Y *of* G*, we have*

$$N_T(\mathsf{Y}(O_S)) \ll_{\mathsf{G},\mathsf{Y}} N_T(\mathsf{G}(O_S))^{1-\sigma}.$$

Let now Y_i, $1 \leqslant i \leqslant k$, be a collection of k hypersurfaces in G. Since the number of lattice points in each hypersurface has a lower rate of growth than the number of lattice points in G, the same holds for their union. Thus, the rate of growth of the number of lattice points in the complement of these hypersurfaces is the same as the rate of growth of all lattice points. This observation gives rise to a host of results asserting that the set of integral points which are generic (i.e. avoid the union of the hypersurfaces) has the maximal possible rate of growth. Let us illustrate this principle concretely by the following example.

Example 1.9. Denote by N_T the number of unimodular integral $(n \times n)$-matrices $(n \geqslant 3)$ with norm bounded by T, and by N'_T the number of such matrices satisfying the following.

- All the matrix entries are non-zero.
- All the principal minors do not vanish.
- All the eigenvalues are distinct.
- All the singular values (eigenvalues of A^tA) are distinct.

Then

$$N'_T = N_T \cdot (1 + O_\epsilon(T^{-(1/2n_e n(n+1))+\epsilon})), \quad \epsilon > 0,$$

where n_e is the least even integer greater than or equal to $n-1$.

1.3 Almost prime points on varieties and orbits

We now turn to the question of how often a polynomial map $f : \mathbb{Z}^n \to \mathbb{Z}$ admits prime (or, more realistically, almost prime) values. This problem has long been studied using sieve methods (see, for instance, [HR74]). Recently, substantial progress has been achieved in the papers [BGS10, LS10, NS10], establishing results on the abundance of almost prime values for polynomials defined on homogeneous varieties and orbits of linear groups. The goal of this section is to generalise one of the main results of [NS10] to the setting of symmetric varieties.

Let G be a connected $\mathbb{Q}$-simple simply connected algebraic group isotropic over $\mathbb{Q}$ and $\mathsf{G} \to \mathrm{GL}_n$ a representation of G which is also defined over $\mathbb{Q}$. Fix $v \in \mathbb{Z}^n$. We assume that $\mathsf{X} = \mathsf{G}v$ is Zariski closed, and $\mathsf{L} = \mathrm{Stab}_{\mathsf{G}}(v)$ is connected and has no non-trivial characters. Then the coordinate ring $\mathbb{C}[\mathsf{X}]$ is a unique factorisation domain (see Lemma 4.3 below). Let f be a regular

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function on X defined over $\mathbb{Q}$ such that it has a decomposition into irreducible factors $f = f_1 \cdots f_t$ where all the f_i are distinct and defined over $\mathbb{Q}$. Let $\mathcal{O} = \Gamma v$ be the orbit of $\Gamma = \mathsf{G}(\mathbb{Z})$, defined with respect to an integral model of G. We assume that f takes integral values on $\mathcal{O}$ and is weakly primitive (that is, $\gcd(f(x) : x \in \mathcal{O}) = 1$). The saturation number $r_0(\mathcal{O}, f)$ of the pair $(\mathcal{O}, f)$ is the least r such that the set of $x \in \mathcal{O}$ for which $f(x)$ has at most r prime factors is Zariski dense in X, which is the Zariski closure of $\mathcal{O}$ by the Borel density theorem. It is natural to ask whether the saturation number $r_0(\mathcal{O}, f)$ is finite and to establish quantitative estimates on the set $\{x \in \mathcal{O} : f(x)$ has at most r prime factors$\}$.

We fix a norm on $\mathbb{R}^n$ and set $\mathcal{O}(T) = \{w \in \mathcal{O} : \|w\| \leqslant T\}$. It was shown in [NS10] that when $\mathsf{X} \simeq \mathsf{G}$ is a group variety, the saturation number is finite and there exists explicit $r \geqslant 1$ such that

$$|\{x \in \mathcal{O}(T) : f(x) \text{ has at most } r \text{ prime factors}\}| \gg \frac{|\mathcal{O}(T)|}{(\log T)^{t(f)}} \tag{1.9}$$

as $T \to \infty$. As remarked in [BGS10, NS10], the assumption that $\mathsf{X} \simeq \mathsf{G}$ is not crucial if only finiteness of the saturation number is concerned, and $r_0(\mathcal{O}, f)$ is finite for general orbits. However, the effective lower estimate (1.9) is much more demanding, and so far it has only been established for two-dimensional quadratic surfaces [LS10] and for group varieties [NS10]. Our goal here is to prove (1.9) for general symmetric varieties.

Theorem 1.10. *Let $\mathcal{O}$ and f be as above and assume in addition that $\mathsf{L} = \mathrm{Stab}_{\mathsf{G}}(v)$ is symmetric (that is, L is the set of fixed points of an involution of G). Then there exists $r \geqslant 1$ such that*

$$|\{x \in \mathcal{O}(T) : f(x) \textit{ has at most } r \textit{ prime factors}\}| \gg \frac{|\mathcal{O}(T)|}{(\log T)^{t(f)}}$$

as $T \to \infty$.

An explicit value of the number r is given in Theorem 4.2 below.

We illustrate Theorem 1.10 by three examples.

Example 1.11. Let Q be a non-degenerate integral quadratic form in n variables, which is indefinite over $\mathbb{R}$. Let $v \in \mathbb{Z}^n$, $\Gamma = \mathrm{Spin}(Q)(\mathbb{Z})$, and $\mathcal{O} = \Gamma v$. If we assume that $Q(v) \neq 0$, then the stabiliser of v in $\mathrm{Spin}(Q)$ is a symmetric subgroup of $\mathrm{Spin}(Q)$. Moreover, we assume that $n \geqslant 4$, which implies that this stabiliser is connected and has no non-trivial characters. Then Theorem 1.10 applies and (1.9) holds. An explicit estimate for the number r of prime factors is as follows. If Q has signature $(1, n-1)$ over $\mathbb{R}$, then (1.9) holds with r the least integer satisfying

$$r > \frac{9(n^2 - n + 2)(3n^2 - 3n + 2)}{2n - 4} \cdot n_e \cdot t(f) \deg(f),$$

where n_e is the least even integer greater than or equal to $9(n-1)/7$. On the other hand, if Q has signature $(\lfloor n/2 \rfloor, n - \lfloor n/2 \rfloor)$ over $\mathbb{R}$, then (1.9) holds with r as above where n_e is the least even integer greater than or equal to $\lfloor n/2 \rfloor$. We will explain these computations in § 4.

Example 1.12. Let A be a non-degenerate integral symmetric matrix of dimension n. We say that another matrix B is integrally equivalent of A if there exists $\gamma \in \mathrm{SL}_n(\mathbb{Z})$ such that $B = {}^t\gamma A \gamma$, and write $B \sim_{\mathbb{Z}} A$. Let

$$\mathcal{O} = \{B \in \mathrm{M}_n(\mathbb{Z}) : B \sim_{\mathbb{Z}} A\}.$$

If $n \geqslant 3$, then Theorem 1.10 implies estimate (1.9) with

$$r > \frac{36n(3n^2 - 2)}{n - 1} \cdot n_e \cdot t(f) \deg(f),$$

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where n_e is the least even integer greater than or equal to $n-1$. This will be explained in detail in § 4.

1.4 Almost prime solutions of congruences on varieties and orbits

Our next aim is to discuss an analogue of the Linnik theorem [Lin44a, Lin44b] on the least prime in an arithmetic progression, which states that there exists $c, \sigma > 0$ such that for every coprime $b, q \in \mathbb{N}$ one can find a prime number p such that

$$p = b \mod q \quad \text{and} \quad p \leqslant cq^{\sigma}. \tag{1.10}$$

It is a very challenging goal to establish such a result in the setting of the previous section, so to keep things more realistic we settle for the existence of solutions of polynomially bounded size which are almost primes.

Let $\mathsf{G} \subset \mathrm{GL}_n$ be a connected $\mathbb{Q}$-simple simply connected algebraic group defined over $\mathbb{Q}$. We fix $v \in \mathbb{Z}^n$ and consider the orbit $\mathcal{O} = \Gamma v$ of $\Gamma = \mathsf{G}(\mathbb{Z})$, defined with respect an integral model of G. Let $f : \mathcal{O} \to \mathbb{Z}$ be a polynomial map. We assume that f is weakly primitive, and the regular function $\tilde{f} : \mathsf{G} \to \mathbb{C}$ defined by $\tilde{f}(g) = f(gv)$ decomposes as a product of t irreducible factors which are distinct and defined over $\mathbb{Q}$.

THEOREM 1.13. *There exist q_0, r and $\sigma > 0$ (as in Theorem 5.1 below) such that, for every coprime $b, q \in \mathbb{N}$ satisfying $q \geqslant q_0$ and $b \in f(\mathcal{O}) \bmod q$, one can find $x \in \mathcal{O}$ satisfying:*

(i) *$f(x)$ is a product of at most r prime factors;*

(ii) *$f(x) = b \bmod q$ and $\|x\| \leqslant q^{\sigma}$.*

The explicit values of r and σ are given in Theorem 5.1 below, and q_0 could be computed, in principle, as well.

Coming back to Example 1.11, we conclude that for a polynomial function f on $\mathrm{M}_{m\times n}(\mathbb{C})$ satisfying the above conditions, the system of equations,

$$^t xBx = A, \quad f(x) = b \mod q, \quad \mathrm{H}_{\infty}(x) \ll q^{\sigma},$$

has a solution $x \in \mathrm{M}_{m\times n}(\mathbb{Z})$ such that $f(x)$ is a product of at most r prime factors, provided that

$$^t xBx = A, \quad f(x) = b$$

has a solution modulo q, and q is sufficiently large. For instance, when B has the signature $(\lfloor m/2 \rfloor, m - \lfloor m/2 \rfloor)$, this holds for $\sigma > \sigma_m$ as in (1.4), and

$$r > \frac{9\alpha_m\, \sigma}{\alpha_m\, \sigma - m(m-1)/2} \cdot \sigma_m \cdot t(f) \deg(f),$$

where $\alpha_m = (m-1)^2/4$ for odd m and $\alpha_m = m(m+2)/4$ for even m.

We remark that in Theorem 1.13 we do not assume that the stabiliser of v in G is symmetric. Under this assumption, our method implies a result on the number of solutions.

THEOREM 1.14. *Under the additional assumption that $\mathrm{Stab}_{\mathsf{G}}(v)$ is symmetric, for every $\sigma > \sigma_0$ (as in (4.7)), r (as in (5.12)), and coprime $b, q \in \mathbb{N}$ satisfying $b \in f(\mathcal{O}) \bmod q$,*

$$\left| \left\{ x \in \mathcal{O}(q^{\sigma}) : \begin{array}{l} f(x) \text{ has at most } r \text{ prime factors,} \\ f(x) = b \bmod q \end{array} \right\} \right| \gg_{\sigma} \frac{1}{|f(\mathcal{O}) \bmod q|} \cdot \frac{|\mathcal{O}(q^{\sigma})|}{(\log q)^{t(f)}}$$

for sufficiently large q.

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1.5 The fundamental lattice point counting result

We now state the uniform solution given in [GN12, Theorem 5.1] to the lattice point counting problem, which underlies the results in the present paper. Let G be a connected F-simple simply connected algebraic group defined over a number field F, S a finite subset of V_F, and O_S the ring of S-integers. We fix a model of G defined over O_S. Let $G = \prod_{v\in S} \mathsf{G}(F_v)$, $\Gamma = \mathsf{G}(O_S)$, and

$$\Gamma(\mathfrak{a}) = \{\gamma \in \Gamma : \gamma = I \bmod \mathfrak{a}\} \quad \text{for an ideal } \mathfrak{a} \text{ of } O_S.$$

We shall use the following notation throughout the paper:

$$\begin{aligned}
p_S &= \text{the least number such that the representations of } G \text{ on } L^2_0(G/\Gamma(\mathfrak{a})) \\
&\qquad \text{are } L^{p_S^+}\text{-integrable for all ideals } \mathfrak{a} \text{ (see [GN10, Definition 5.2])}, \\
n_e(p) &= \text{the least even integer greater than or equal to } p/2, \text{ if } p > 2, \text{ and } 1, \text{ if } p = 2, \\
d_S &= \textstyle\sum_{v\in V_\infty} \dim \mathsf{G}(F_v), \\
B_T &= \{g \in G : \mathrm{H}(g) \leqslant T\}, \\
a_S &= \text{the Hölder exponent of the family of sets } B_{e^t} \text{ (see [GN10, Definition 3.12])}.
\end{aligned}$$

We note that the finiteness of the integrability exponent p_S is a manifestation of property (τ), established in full generality by Clozel [Clo03] (see [Clo03, Theorem 3.1]). We note that [Clo03, Theorem 3.1] is stated in terms of the isolation of representations of $\mathsf{G}(F_v)$, $v \in S$, appearing in $L^2_0(G/\Gamma(\mathfrak{a}))$ from the trivial representation, but this implies that all these representations are integrable, and, in fact, the argument in [Clo03] gives an explicit estimate on p_S. We refer to [Sar05] for a comprehensive discussion of property (τ). Hölder-admissibility of the sets B_{e^t} was established in [GN10, Theorem 7.19] and [BO07]. In many cases, one can take $a_S = 1$ (see [GN10, ch. 7]). For instance, this is the case when $F = \mathbb{Q}$ and $S = \{\infty\}$.

We can now state the following theorem.

Theorem 1.15 [GN12, Theorem 5.1]. *For every $\gamma_0 \in \Gamma$ and all ideals $\mathfrak{a}$ of O_S,*

$$|\gamma_0\Gamma(\mathfrak{a}) \cap B_T| = \frac{\mathrm{vol}(B_T)}{[\Gamma : \Gamma(\mathfrak{a})]} + O_\epsilon(\mathrm{vol}(B_T)^{1-(2n_e(p_S))^{-1}a_S/(a_S+d_S)+\epsilon}), \quad \epsilon > 0,$$

where the Haar measure on G is normalised so that $\mathrm{vol}(G/\Gamma) = 1$.

We set

$$\alpha_S(G) = \limsup_{T\to\infty} \frac{\log |\{\gamma \in \Gamma : \mathrm{H}(\gamma) \leqslant T\}|}{\log T} = \limsup_{T\to\infty} \frac{\log \mathrm{vol}(B_T)}{\log T}. \tag{1.11}$$

We note that $\alpha_S(G) > 0$ provided that G is isotropic over S (see [GW07, § 7], [Mau07], [GOS09, § 6]).

We will also have occasion below to consider the volume growth in a homogeneous space G/H, in which case we will denote the exponent by $\alpha(G/H)$.

Although the asymptotic formula for $|\{\gamma \in \Gamma : \mathrm{H}(\gamma) \leqslant T\}|$ is also known, it will not be needed in our argument.

2. Effective lifting of solutions of congruences

We first establish a version of Theorem 1.1 in the case of group varieties, and Theorem 1.1 will be deduced from the following result.

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Theorem 2.1. *Let $\mathsf{G} \subset \mathrm{GL}_m$ be a connected simply connected F-simple algebraic group, and let S be a finite subset of V_F, containing all Archimedean absolute values, such that G is isotropic over S and has a model defined over O_S, and $\mathrm{Lie}(\mathsf{G}) \cap \mathrm{M}_m(O_S)$ has a basis over O_S as an O_S-module. Let*

$$\sigma > \sigma_0 := \alpha_S(G)^{-1} \dim(\mathsf{G}) \frac{a_S + d_S}{a_S} 2n_e(p_S). \tag{2.1}$$

Then there exists $q_0 > 0$ such that for every ideal $\mathfrak{a}$ of O_S satisfying $|O_S/\mathfrak{a}| \geqslant q_0$ and every $\bar{x} \in \mathsf{G}^{(\mathfrak{a})}(O_S/\mathfrak{a})$ there exists $x \in \mathsf{G}(O_S)$ such that

$$\pi_{\mathfrak{a}}(x) = \bar{x} \quad \text{and} \quad \mathrm{H}(x) \leqslant |O_S/\mathfrak{a}|^{\sigma}. \tag{2.2}$$

We note that the exponent σ can be further improved, for instance, by considering a smooth density on the sets $\{\mathrm{H} \leqslant T\}$, and when $p_S = 2$, this leads to essentially optimal bound on σ. However, we do not pursue this direction in the present paper and rely only on the counting estimate of Theorem 1.15.

Proof. Since G is isotropic over S, it satisfies the strong approximation property with respect to S (see [PR94, § 7.4]). Then it follows that the map $\pi_{\mathfrak{a}}$ is surjective, and there exists $\gamma_0 \in \Gamma$ such that $\bar{x} = \pi_{\mathfrak{a}}(\gamma_0)$ for some $\gamma_0 \in \Gamma$. Moreover, we have $\bar{x} = \pi_{\mathfrak{a}}(\gamma_0 \Gamma(\mathfrak{a}))$.

By Theorem 1.15, for every $\delta < \delta_0 = (2n_e(p_S))^{-1} a_S/(a_S + d_S)$ and $c_\delta > 0$, we have

$$\left| |\gamma_0 \Gamma(\mathfrak{a}) \cap B_T| - \frac{\mathrm{vol}(B_T)}{|\Gamma : \Gamma(\mathfrak{a})|} \right| \leqslant c_\delta \ \mathrm{vol}(B_T)^{1-\delta}. \tag{2.3}$$

It is important to emphasise here that this estimate is uniform over all $\gamma_0 \in \Gamma$ and all ideals $\mathfrak{a}$ of O_S. It follows from (2.3) that for T satisfying

$$\mathrm{vol}(B_T) > (c_\delta \, |\Gamma : \Gamma(\mathfrak{a})|)^{1/\delta} \tag{2.4}$$

there exists $x \in \gamma_0 \Gamma(\mathfrak{a}) \cap B_T$. Then we have $\pi_{\mathfrak{a}}(x) = \bar{x}$ and $\mathrm{H}(x) \leqslant T$.

Now it remains to analyse for which values of T inequality (2.4) holds. By Lemma 2.2 below,

$$|\Gamma : \Gamma(\mathfrak{a})| \ll |O_S/\mathfrak{a}|^{\dim(\mathsf{G})}. \tag{2.5}$$

By (1.11), for every $\alpha < \alpha_S$ and $T > T(\alpha)$,

$$\mathrm{vol}(B_T) \geqslant T^{\alpha}. \tag{2.6}$$

Therefore, we conclude that (2.4) holds for $T = |O_S/\mathfrak{a}|^{\sigma}$ with $\sigma > \dim(\mathsf{G})/(\alpha\delta)$ and sufficiently large $|O_S/\mathfrak{a}|$. Since this is the case for every $\alpha < \alpha_S(G)$ and $\delta < \delta_0$, this concludes the proof. □

To complete the proof of Theorem 2.1, we therefore only have to establish the following lemma.

Lemma 2.2. *Let $\mathsf{G} \subset \mathrm{GL}_n$ be a connected algebraic group, and let S be a finite subset of V_F, containing all Archimedean absolute values, such that G has a model defined over O_S, and $\mathrm{Lie}(\mathsf{G}) \cap \mathrm{M}_m(O_S)$ has a basis over O_S. Then*

$$|\Gamma : \Gamma(\mathfrak{a})| \asymp |O_S/\mathfrak{a}|^{\dim(\mathsf{G})}$$

uniformly over ideals $\mathfrak{a}$ of O_S.

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Proof. Let O_v denote the local ring with the prime ideal $\mathfrak{p}_v$ corresponding to non-Archimedean $v \in V_F$. It follows from the formulas for local Tamagawa measures (see, for instance, [Vos98, 14.2 and 14.3]) and the Lang–Weil estimates [LW54] that, for all valuations v outside S and all $n \geqslant 0$,

$$|\mathsf{G}^{(\mathfrak{p}_v^n)}(O_v/\mathfrak{p}_v^n)| \asymp |O_v/\mathfrak{p}_v|^{n\dim(\mathsf{G})} = |O_v/\mathfrak{p}_v^n|^{\dim(\mathsf{G})},$$

and it follows from the Chinese reminder theorem that, for every ideal $\mathfrak{a}$ of O_S,

$$|\mathsf{G}^{(\mathfrak{a})}(O_S/\mathfrak{a})| \asymp |O_S/\mathfrak{a}|^{\dim(\mathsf{G})}.$$

Since the kernel of the reduction map $\pi_\mathfrak{a}\colon \Gamma \to \mathsf{G}^{(\mathfrak{a})}(O_S/\mathfrak{a})$ is equal to $\Gamma(\mathfrak{a})$, this implies the claim of the lemma. □

Remark 2.3. The constant q_0 in our results can be computed in principle. It depends on the implicit constant in Theorem 1.15, which is given explicitly in [GN12], and on $T(\alpha)$ in (2.6). An explicit value of $T(\alpha)$ can derived from the asymptotic formula for $\mathrm{vol}(B_T)$ (see [GN10, ch. 7]).

Proof of Theorem 1.1. By the Borel–Harish-Chandra theorem [BH62], the set $\mathsf{X}(O_S)$ is a union of finitely many orbits of $\Gamma = \mathsf{G}(O_S)$. Hence, it suffices to prove the claim for every $\bar{x} \in \mathsf{X}^{(\mathfrak{a})}(O_S/\mathfrak{a})$ that lifts to a point $x \in \mathsf{X}(O_S)$ contained in a Γ-orbit Γx_0 for some fixed $x_0 \in \mathsf{X}(O_S)$. If G is anisotropic over every $v \in S$, then Γ is finite, and the claim is trivial. Hence, we may assume that G is isotropic for some $v \in S$. Then Theorem 2.1 applies. We have

$$\bar{x} = \pi_\mathfrak{a}(\gamma \cdot x_0) = \pi_\mathfrak{a}(\gamma) \cdot \pi_\mathfrak{a}(x_0)$$

for some $\gamma \in \Gamma$. By Theorem 2.1, there exists $\gamma' \in \Gamma$ such that

$$\pi_\mathfrak{a}(\gamma') = \pi_\mathfrak{a}(\gamma) \quad \text{and} \quad \mathrm{H}(\gamma') \leqslant |O_S/\mathfrak{a}|^\sigma$$

where σ is as in Theorem 2.1. Since $\bar{x} = \pi_\mathfrak{a}(\gamma') \cdot \pi_\mathfrak{a}(x_0) = \pi_\mathfrak{a}(\gamma' \cdot x_0)$, it remains to observe that

$$\mathrm{H}(\gamma' \cdot x_0) \ll \mathrm{H}(\gamma')^N \tag{2.7}$$

for some uniform $N > 0$ determined by the action. □

Proof of Theorem 1.3. Let $\gamma_0 \in \Gamma$ be such that $\pi_\mathfrak{a}(\gamma_0) = \bar{x}$. By Theorem 1.15,

$$|\gamma_0\Gamma(\mathfrak{a}) \cap B_T| = \frac{\mathrm{vol}(B_T)}{|\Gamma : \Gamma(\mathfrak{a})|}\left(1 + O_\delta\left(\frac{|\Gamma : \Gamma(\mathfrak{a})|}{\mathrm{vol}(B_T)^\delta}\right)\right)$$

for every $\delta < \delta_0 = (2n_e(p_S))^{-1}a_S/(a_S + d_S)$. Hence, it follows from (2.5) and (2.6) that, for every $\alpha < \alpha_S(G)$ and $T > T(\alpha)$, we have

$$|\gamma_0\Gamma(\mathfrak{a}) \cap B_T| = \frac{\mathrm{vol}(B_T)}{|\Gamma : \Gamma(\mathfrak{a})|}(1 + O_\delta(|O_S/\mathfrak{a}|^{\dim(\mathsf{G})}T^{-\alpha\delta})).$$

Hence, if we pick $T = |O_S/\mathfrak{a}|^\sigma$ with $\sigma > \sigma_0$ as in (2.1) and sufficiently large $|O_S/\mathfrak{a}|$, then

$$\begin{aligned}|\gamma_0\Gamma(\mathfrak{a}) \cap B_T| &= \frac{\mathrm{vol}(B_T)}{|\Gamma : \Gamma(\mathfrak{a})|}(1 + O_{\alpha,\delta}(|O_S/\mathfrak{a}|^{\dim(\mathsf{G})-\sigma\alpha\delta})) \\ &= \frac{\mathrm{vol}(B_T)}{|\Gamma : \Gamma(\mathfrak{a})|}(1 + O_\epsilon(|O_S/\mathfrak{a}|^{\dim(\mathsf{G})-\dim(\mathsf{G})\sigma_0^{-1}\sigma+\epsilon}))\end{aligned}$$

for every $\epsilon > 0$, where we have used the fact that $\sigma_0 = \dim(\mathsf{G})/(\alpha_S\delta_0)$. Finally, to complete the proof, we note that

$$|\Gamma : \Gamma(\mathfrak{a})| = |\mathsf{G}^{(\mathfrak{a})}(O_S/\mathfrak{a})|$$

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and, by Theorem 1.15,

$$\mathrm{vol}(B_T) = |\{x \in \mathsf{G}(O_S); \mathrm{H}(x) \leqslant |O_S/\mathfrak{a}|^{\sigma}\}|(1 + O_{\alpha,\delta}(|O_S/\mathfrak{a}|^{-\sigma\alpha\delta})). \qquad \square$$

Coming back to Example 1.4, we note that the variety X defined in (1.2) is a homogeneous space of the spinor group $\mathsf{G} = \mathrm{Spin}(B)$. We have $\dim(\mathsf{G}) = m(m-1)/2$. The Hölder exponent is $a_S = 1$ by [GN10, Proposition 7.3]. Now we assume that G has maximal $\mathbb{R}$-rank (i.e., the signature of B is $(\lfloor m/2 \rfloor, m - \lfloor m/2 \rfloor)$). Then the integrability exponent is $p_S = m - 1$ for odd m and $p_S = m$ for even m by [Li95, Oh02]. By [DRS93, EM93], the growth rate $\alpha_S(G)$ of integral points in $\mathsf{G}(\mathbb{Z})$ can be estimated in terms of volume growth of the norm balls which is computable in terms of the root data of G (see [GOS09, Mau07]). This gives $\alpha_S(G) = (m-1)^2/4$ for odd m and $\alpha_S(G) = m(m+2)/4$ for even m. Hence, Theorem 2.1 holds with

$$\sigma > \begin{cases} \dfrac{2m(m^2-m+1)n_e}{m-1} & \text{when } m \text{ is odd,} \\ \dfrac{2(m-1)(m^2-m+1)n_e}{m+2} & \text{when } m \text{ is even,} \end{cases}$$

where n_e denotes the least even integer greater than or equal to $\lfloor m/2 \rfloor$. We note that the action of $\mathrm{Spin}(B)$ on X can be given by the standard Clifford algebra construction (see [Die63, ch. II, § 7]) which implies that (2.7) holds with $N = 2$. This explains (1.3).

3. Integral points on subvarieties

The following result is a precise version of Theorem 1.6. In the statement we use notation introduced in Theorem 1.15.

Theorem 3.1. *Let G be a connected F-simple simply connected algebraic group and Y an absolutely irreducible proper affine subvariety of G defined over a number field F. Let $S \subset V_F$ be a finite subset containing all Archimedean absolute values such that G is isotropic over S, and $\mathsf{Y} \subset \mathsf{G}$ have models defined over O_S. Then*

$$N_T(\mathsf{Y}(O_S)) \ll_{\mathsf{G},\deg(\mathsf{X}),\epsilon} N_T(\mathsf{G}(O_S))^{1-(a_S(\dim(\mathsf{G})-\dim(\mathsf{Y}))/\dim(\mathsf{G})(a_S+d_S)2n_e(p_S))+\epsilon}, \quad \epsilon > 0,$$

as $T \to \infty$.

Coming back to Example 1.7, we note that, in this case, $\dim(\mathrm{SL}_n(\mathbb{R})) = n^2 - 1$, and so $N_T(\mathrm{SL}_n(\mathbb{Z})) \sim c_n T^{n^2-n}$ with $c_n > 0$. Furthermore $p_S = 2(n-1)$ (see [DRS93]), and $a_S = 1$ (see [GN10, Proposition 7.3]). Hence, the estimate (1.8) is a special case of Theorem 3.1.

Proof of Theorem 3.1. For non-Archimedean $v \in V_F$, we denote by f_v the corresponding residue field and by $\mathfrak{p}_v$ the corresponding prime ideal.

We consider the reduction $\mathsf{Y}^{(v)}$ of the variety Y modulo a valuation v. Then by Noether's theorem, $\mathsf{Y}^{(v)}$ is absolutely irreducible for almost all v. Moreover, $\dim(\mathsf{Y}^{(v)}) = \dim(\mathsf{Y})$ and $\deg(\mathsf{Y}^{(v)}) = \deg(\mathsf{Y})$ for almost all v (see [Odo79, § 1]). Therefore, by [GL02, Proposition 12.1], we have the following estimate:

$$|\mathsf{Y}^{(v)}(f_v)| \ll_{\deg(\mathsf{Y})} |f_v|^{\dim(\mathsf{Y})}, \tag{3.1}$$

valid for almost all v. We observe that each fiber of the reduction map $\mathsf{Y}(O_S) \to \mathsf{Y}^{(v)}(f_v)$ is contained in a coset of the subgroup $\Gamma_v = \{\gamma \in \Gamma : \gamma = I \bmod \mathfrak{p}_v\}$ of $\Gamma = \mathsf{G}(O_S)$. Hence, it follows that $\mathsf{Y}(O_S)$ is contained in a union of at most $O_{\deg(\mathsf{Y})}\left(|f_v|^{\dim(\mathsf{Y})}\right)$ cosets $\gamma\Gamma_v$ with $\gamma \in \Gamma$.

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The crucial ingredient of the proof is Theorem 1.15, which gives an estimate of the number of points in the cosets $\gamma\Gamma_v$ uniformly over $\gamma \in \Gamma$. More precisely, by Theorem 1.15, for all v,

$$|\gamma\Gamma_v \cap B_T| = \frac{\mathrm{vol}(B_T)}{|\Gamma : \Gamma_v|} + O_\epsilon(\mathrm{vol}(B_T)^{1-(a_S/(a_S+d_S)2n_e(p_S))+\epsilon}), \quad \epsilon > 0. \tag{3.2}$$

For almost all v, the reduction $\mathsf{G}^{(v)}$ is a smooth geometrically irreducible variety of dimension $\dim \mathsf{G}$. Therefore, we have the Lang–Weil estimate (see [LW54]),

$$|\mathsf{G}^{(v)}(f_v)| = |f_v|^{\dim(\mathsf{G})} + O_{\mathsf{G}}(|f_v|^{\dim(\mathsf{G})-1/2}).$$

Since G is simply connected F-simple and isotropic over S, it follows from the strong approximation property (see [PR94, Theorem 7.12]) that the reduction map $\Gamma \to \mathsf{G}^{(v)}(f_v)$ is surjective for all $v \notin S$. This implies the estimate

$$|\Gamma : \Gamma_v| = |\mathsf{G}^{(v)}(f_v)| \gg |f_v|^{\dim(\mathsf{G})}$$

for almost all v.

Finally, we conclude from (3.1) and (3.2) that, for all v,

$$|\mathsf{Y}(O_S) \cap B_T| \ll_{\mathsf{G},\deg(\mathsf{Y}),\epsilon} |f_v|^{\dim(\mathsf{Y})} \left(\frac{\mathrm{vol}(B_T)}{|f_v|^{\dim(\mathsf{G})}} + \mathrm{vol}(B_T)^{1-(a_S/(a_S+d_S)2n_e(p_S))+\epsilon} \right), \quad \epsilon > 0.$$

To optimise this estimate, we take v such that

$$\mathrm{vol}(B_T)^{a_S/(a_S+d_S)2n_e(p_S)} \leqslant |f_v|^{\dim(\mathsf{G})} \leqslant 2\,\mathrm{vol}(B_T)^{a_S/(a_S+d_S)2n_e(p_S)}.$$

For sufficiently large T, such v exists by the prime number theorem for the ring of integers O in F. This gives the estimate

$$\begin{aligned} N_T(\mathsf{Y}(O_S)) &= |\mathsf{Y}(O_S) \cap B_T| \\ &\ll_{\mathsf{G},\deg(\mathsf{Y}),\epsilon} \mathrm{vol}(B_T)^{1-(a_S(\dim(\mathsf{G})-\dim(\mathsf{Y}))/\dim(\mathsf{G})(a_S+d_S)2n_e(p_S))+\epsilon}, \quad \epsilon > 0, \end{aligned}$$

as $T \to \infty$. Since $N_T(\mathsf{G}(O_S)) \sim \mathrm{vol}(B_T)$ by Theorem 1.15, this completes the proof. □

4. Almost prime points on varieties and orbits

We now turn to the problem of establishing the existence of almost prime points on symmetric varieties. We shall use the notation from § 1.3. In particular, G is a connected $\mathbb{Q}$-simple simply connected algebraic group defined over $\mathbb{Q}$ and L is a symmetric $\mathbb{Q}$-subgroup. Let $G = \mathsf{G}(\mathbb{R})$ and $L = \mathsf{L}(\mathbb{R})$. Then G is a connected semisimple Lie group with finite center and L is a closed symmetric subgroup of G. We shall use the structure theory of affine symmetric spaces (see [HS94, Part II]). Fix a maximal compact subgroup K of G compatible with L and a Cartan subgroup A for the pair (K, L). Then the Cartan decomposition

$$G = KA^+L$$

holds, where A^+ denotes a closed positive Weyl chamber in A. Let M denote the centraliser of A in $K \cap L$. We fix a bounded subset Ψ of $M\backslash L$ with non-empty interior which we assume to be Lipschitz well-rounded (in the sense of [GN12, § 7]). We also denote by $\dot{A}^+$ the interior of the Weyl chamber A^+ and set

$$S_T = \{g \in K\dot{A}^+\Psi : \|gv\| \leqslant T\}.$$

We note that it was shown in [GN12, Proposition 8.4] that the sets S_{e^t} are Hölder well-rounded with exponent 1/3.

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Our main tool is the following result on counting of lattice points in S_T for the congruence subgroups $\Gamma(q) = \{\gamma \in \Gamma : \gamma = I \bmod q\}$ of $\Gamma = \mathsf{G}(\mathbb{Z})$. We note that representations $L^2_0(G/\Gamma(q))$ are all L^{p+} with uniform $p > 0$ by [Clo03], so that the following theorem is a special case of [GN12, Theorem 8.1].

THEOREM 4.1 [GN12]. *For every $\gamma_0 \in \Gamma$ and $q \geqslant 1$,*

$$|\gamma_0\Gamma(q) \cap S_T| = \frac{\mathrm{vol}(S_T)}{[\Gamma : \Gamma(q)]} + O_\epsilon(\mathrm{vol}(S_T)^{1-(2n_e(p))^{-1}(1+3\dim G)^{-1}+\epsilon}), \quad \epsilon > 0,$$

where the Haar measure is normalised so that $\mathrm{vol}(G/\Gamma) = 1$.

We note that by [DRS93, EM93]

$$|\mathcal{O}(T)| \sim \frac{\mathrm{vol}(L/(L \cap \Gamma))}{\mathrm{vol}(G/\Gamma)} \cdot \mathrm{vol}(S_T v) \quad \text{as } T \to \infty, \tag{4.1}$$

where vol denote G-invariant measures on the corresponding spaces. It was shown in [GOS09, § 6] that

$$\mathrm{vol}(S_T v) \sim v_0 T^{\alpha(G/H)} (\log T)^\beta \quad \text{as } T \to \infty, \tag{4.2}$$

for some $v_0 > 0$, $\alpha(G/H) \in \mathbb{Q}^+$, and $\beta \in \mathbb{Z}^+$. Also, it is clear that

$$\mathrm{vol}(S_T) = \mathrm{vol}(S_T v) \cdot \mathrm{vol}(\Psi). \tag{4.3}$$

Now we prove the following theorem, which is a more explicit version of Theorem 1.10 stated in § 1.3 (we refer there for the notation used below).

THEOREM 4.2. *With the notation above, let r be the least integer satisfying*

$$r > 9\alpha(G/H)^{-1}\,(1 + \dim(G))(1 + 3\dim(G))2n_e(p) \cdot t(f)\deg(f).$$

Then

$$|\{x \in \mathcal{O}(T) : f(x) \text{ has at most } r \text{ prime factors}\}| \gg \frac{|\mathcal{O}(T)|}{(\log T)^{t(f)}}$$

as $T \to \infty$.

In the case of Example 1.11, we have $\dim(\mathrm{Spin}(Q)) = n(n-1)/2$ and $\alpha(G/H) = n - 2$. When Q has signature $(n, 1)$, one can take $p = 9(n-1)/7$ (see [BS91]). For other signatures, the group $\mathrm{Spin}(Q)(\mathbb{R})$ has $\mathbb{R}$-rank at least 2 and we can utilise the estimates on integrability exponents obtained in [Li95, Oh02]. In particular, when Q has signature $(\lfloor n/2 \rfloor, n - \lfloor n/2 \rfloor)$ (i.e., when $\mathrm{Spin}(Q)$ is split over $\mathbb{R}$), we have $p = n - 1$ for odd n and $p = n$ for even n.

In the case of Example 1.12, we have $\dim(\mathrm{SL}_n) = n^2 - 1$, $\alpha = (n^2 - n)/2$ (see [GOS09, § 2.3]), and $p = 2(n-1)$ (see [DRS93]).

Before we start the proof of Theorem 4.2, we show that the decomposition $f = f_1 \cdots f_t$ into irreducible factors is well defined.

LEMMA 4.3. *Let G be a connected semisimple simply connected algebraic group and L a closed connected subgroup with no non-trivial characters. Then the coordinate ring $\mathbb{C}[\mathsf{G}/\mathsf{L}]$ is a unique factorisation domain.*

Proof. We refer to [FI73, KKV89, Pop74] for computation of Picard groups of homogeneous spaces. There is an exact sequence

$$\mathcal{X}(\mathsf{G}) \to \mathcal{X}(\mathsf{L}) \to \mathrm{Pic}(\mathsf{G}/\mathsf{L}) \to \mathrm{Pic}(\mathsf{G}),$$

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where $\mathcal{X}(\mathsf{G})$ and $\mathcal{X}(\mathsf{L})$ denote the character groups. Since G is simply connected, $\mathrm{Pic}(\mathsf{G}) = 1$. Hence, it follows from the exact sequence that $\mathrm{Pic}(\mathsf{G}/\mathsf{L}) = 1$, and $\mathbb{C}[\mathsf{G}/\mathsf{L}]$ is a unique factorisation domain by [Har77, Proposition 6.2]. □

Proof of Theorem 4.2. Using the dominant map $\mathsf{G} \to \mathsf{X}$, every element $f \in \mathbb{C}[\mathsf{X}]$ lifts to an element $\tilde{f} \in \mathbb{C}[\mathsf{G}]$. Since G is simply connected, the ring $\mathbb{C}[\mathsf{G}]$ is a unique factorisation domain. We claim that the decomposition of $\tilde{f}$ into irreducible factors in $\mathbb{C}[\mathsf{G}]$ is of the form $\tilde{f} = \tilde{f}_1 \cdots \tilde{f}_t$, where $f = f_1 \cdots f_t$ is the decomposition in $\mathbb{C}[\mathsf{X}]$. Indeed, suppose that $\tilde{f}_i = g_1 \cdots g_s$ for $g_1, \ldots, g_s \in \mathbb{C}[\mathsf{G}]$ is the decomposition into irreducibles. We consider the right action of L on $\mathbb{C}[\mathsf{G}]$. Since $\tilde{f}_i$ is L-invariant and L is connected, it follows from uniqueness of the decomposition that each g_i is also L-invariant and descends to a function on $\mathbb{C}[\mathsf{X}]$, which implies that this decomposition must be trivial. Hence, the $\tilde{f}_i$ are irreducible.

Now we apply the argument of [NS10] to the polynomial function $\tilde{f} : \Gamma \to \mathbb{Z}$ and the sets $\Gamma \cap S_T$ (instead of sets $\{\gamma \in \Gamma : \|\gamma\| < T\}$). It follows from Theorem 4.1 that, for every $q \geqslant 1$ and $\gamma_0 \in \Gamma$,

$$\frac{|\gamma_0\Gamma(q) \cap S_T|}{\mathrm{vol}(S_T)} = \frac{1}{[\Gamma : \Gamma(q)]} + O_\epsilon(\mathrm{vol}(S_T)^{-(2n_e(p))^{-1}(1+3\dim G)^{-1}+\epsilon}), \quad \epsilon > 0. \tag{4.4}$$

Therefore, by (4.2) and (4.3),

$$\frac{|\gamma_0\Gamma(q) \cap S_T|}{\mathrm{vol}(S_T)} = \frac{1}{[\Gamma : \Gamma(q)]} + O_\epsilon(T^{-(\theta/(1+3\dim(G)))+\epsilon}), \quad \epsilon > 0,$$

where $\theta = \alpha(G/H)/2n_e(p)$. This estimate is a substitute for [NS10, Theorem 3.2]. Given the estimate above for a family of sets S_T, the argument in [NS10] for norm balls can be carried out without change, and we conclude that, for sufficiently large T,

$$\sum_{\gamma \in \Gamma \cap S_T : \gcd(\tilde{f}(\gamma), P_z) = 1} 1 \gg \frac{|\Gamma \cap S_T|}{(\log |\Gamma \cap S_T|)^{t(f)}}, \tag{4.5}$$

where

$$P_z = \prod_{p \leqslant z} p, \quad z = |\Gamma \cap S_T|^\kappa, \quad \kappa = (9t(f)(1 + \dim(G))(1 + 3\dim(G))2n_e(p))^{-1}.$$

For every $\gamma \in \Gamma \cap S_T$, we have

$$|\tilde{f}(\gamma)| = |f(\gamma v)| \ll T^{\deg(f)}.$$

On the other hand, if $\gcd(\tilde{f}(\gamma), P_z) = 1$, then every prime factor of $\tilde{f}(\gamma)$ is at least z, and $z \gg T^{\alpha(G/H)\kappa}$ by (4.2)–(4.4). Therefore, for every term in the sum (4.5), the number of prime factors of $\tilde{f}(\gamma)$ is bounded above by

$$\frac{\deg(f)}{\alpha(G/H)\kappa} = 9\alpha(G/H)^{-1}(1 + \dim(G))(1 + 3\dim(G))2n_e(p)t(f)\deg(f)$$

provided T is sufficiently large. We conclude that

$$|\{\gamma \in \Gamma \cap S_T : f(\gamma v) \text{ has at most } r \text{ prime factors}\}| \gg \frac{|\Gamma \cap S_T|}{(\log |\Gamma \cap S_T|)^{t(f)}}. \tag{4.6}$$

To finish the proof, we consider the projection map

$$\pi : \Gamma \cap S_T \to \mathcal{O}(T) : \gamma \mapsto \gamma v.$$

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It follows from the uniqueness properties of the Cartan decomposition (see [HS94, p. 108]) that if $\gamma_0, \gamma \in \Gamma \cap S_T$ satisfy $\gamma_0 v = \gamma v$, then their KA^+-components are equal modulo M, and $\gamma_0^{-1}\gamma \in \Psi^{-1}\Psi$. Hence,

$$\pi^{-1}(\gamma_0 v) \subset \gamma_0 \Psi^{-1}\Psi \cap \Gamma,$$

and the cardinality of every fiber of π is bounded by $|\Psi^{-1}\Psi \cap \Gamma|$. It follows from (4.6) that

$$|\{w \in \mathcal{O}(T) : f(w) \text{ has at most } r \text{ prime factors}\}| \gg \frac{|\Gamma \cap S_T|}{(\log |\Gamma \cap S_T|)^{t(f)}}$$

as $T \to \infty$. Since $|\Gamma \cap S_T| \asymp \mathrm{vol}(S_T)$, the claim of the theorem now follows from (4.1)–(4.3).

We also establish a quantitative version of Theorem 1.1 for lifting solutions of congruences in $\mathcal{O}$, which will be used to prove Theorem 1.14 in § 5.

THEOREM 4.4. *For every*

$$\sigma > \sigma_0 := \alpha(G/H)^{-1} \dim(G)(1 + 3\dim(G))2n_e(p), \tag{4.7}$$

sufficiently large q, and $b \in \mathcal{O} \bmod q$,

$$|\{x \in \mathcal{O}(q^\sigma); x = b \bmod q\}| \gg_\sigma |\mathcal{O}(q^\sigma)| \cdot \frac{1}{|\mathcal{O} \bmod q|}.$$

Proof. Using Theorem 4.1 and arguing exactly as in the proof of Theorem 1.3, we get the estimate

$$|\gamma\Gamma(q) \cap S_T| \gg_\sigma \frac{1}{|\Gamma : \Gamma(q)|} \cdot |\Gamma \cap S_T|,$$

for $T = q^\sigma$ with sufficiently large q and every $\gamma \in \Gamma$. This implies that

$$\begin{aligned} |\{\gamma \in \Gamma \cap S_T; \gamma v = b \bmod q\}| &= \sum_{\gamma \in \Gamma/\Gamma(q) : \gamma v = b \bmod q} |\gamma\Gamma(q) \cap S_T| \\ &\gg_\sigma \frac{|\mathrm{Stab}_\Gamma(b \bmod q) : \Gamma(q)|}{|\Gamma : \Gamma(q)|} \cdot |\Gamma \cap S_T| = \frac{1}{|\mathcal{O} \bmod q|} \cdot |\Gamma \cap S_T|. \end{aligned}$$

Recall from the previous proof that the cardinality of the fibers of the map

$$\pi : \Gamma \cap S_T \to \mathcal{O}(T) : \gamma \mapsto \gamma v.$$

is uniformly bounded. Therefore,

$$|\{x \in \mathcal{O}(T); x = b \bmod q\}| \gg |\{\gamma \in \Gamma \cap S_T; \gamma v = b \bmod q\}|.$$

Since $|\Gamma \cap S_T| \asymp |\mathcal{O}(T)|$ by (4.1), this completes the proof. □

5. Almost prime solutions of congruences on varieties and orbits

We start by proving Theorem 1.13 for group varieties. Let $\mathsf{G} \subset \mathrm{GL}_n$ be a connected $\mathbb{Q}$-simple simply connected algebraic group defined over $\mathbb{Q}$. We assume that G is isotropic over $\mathbb{R}$, and denote by $\alpha = \alpha(G) > 0$ the volume growth exponent of $\mathsf{G}(\mathbb{R})$, defined as in (1.11). Let f be a regular function on G defined over $\mathbb{Q}$ that decomposes into a product of $t = t(f)$ absolutely irreducible factors defined over $\mathbb{Q}$. We assume that $f(\mathsf{G}(\mathbb{Z})) \subset \mathbb{Z}$ and f is weakly primitive.

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Theorem 5.1. *Let*

$$\sigma > \sigma_0 := \alpha^{-1} \dim(\mathsf{G})(1+\dim(\mathsf{G}))2n_e(p),$$
$$r > \frac{9\alpha\sigma}{\alpha\sigma - \dim(\mathsf{G})} \cdot \sigma_0 \cdot t(f) \deg(f).$$

Then there exists $q_0 > 0$ such that, for every coprime $b, q \in \mathbb{N}$ satisfying $q \geqslant q_0$ and $b \in f(\mathsf{G}(\mathbb{Z})) \bmod q$, one can find $x \in \mathsf{G}(\mathbb{Z})$ such that:

(i) *$f(x)$ is a product of at most r prime factors;*

(ii) *$f(x) = b \bmod q$ and $\|x\| \leqslant q^\sigma$.*

Proof. We write $f(x) = (1/N)g(x)$ where $g(x)$ is a polynomial with integral coefficients and $N \in \mathbb{N}$. Since f is weakly primitive,

$$\gcd(g(\gamma) : \gamma \in \Gamma) = N. \tag{5.1}$$

Let $N = N_1 N_2$, where N_1 is the product of all prime factors of N with multiplicities which are coprime to q. Then the condition $f(\gamma) = b \bmod q$ is equivalent to $g(\gamma) = bN \bmod qN$. Moreover, because of (5.1), it is equivalent to $g(\gamma) = bN \bmod qN_2$.

According to our assumptions, there exists $\gamma_0 \in \mathsf{G}(\mathbb{Z})$ such that $f(\gamma_0) = b \bmod q$. We set

$$\Gamma = \mathsf{G}(\mathbb{Z}),$$
$$\Gamma_q = \Gamma(qN_2) = \{\gamma \in \Gamma : \gamma = \mathrm{id} \bmod qN_2\},$$
$$\mathcal{O}_q(T) = \{\gamma \in \gamma_0\Gamma_q : \|\gamma\| \leqslant T\}.$$

Note that every $\gamma \in \gamma_0\Gamma_q$ satisfies $f(\gamma) = b \bmod q$.

Let $\mathcal{P}_{q,z}$ be the set of prime numbers which are coprime to q and bounded by z. Our aim is to estimate from below the cardinality of points $\gamma \in \mathcal{O}_q(T)$ such that $f(\gamma)$ is coprime to $\mathcal{P}_{q,z}$, which we denote by $S(T, q, z)$. This will achieved by applying the combinatorial sieve as in [HR74, Theorem 7.4] and [NS10, § 2]. Let

$$a_k = |\{\gamma \in \mathcal{O}_q(T) : f(\gamma) = k\}| \quad \text{and} \quad X = |\mathcal{O}_q(T)| = \sum_{k \geqslant 0} a_k.$$

In order to apply the combinatorial sieve, we need to verify the following conditions.

(A_0) For every square-free d in $\mathcal{P}_{q,z}$,

$$\sum_{k=0 \bmod d} a_k = \frac{\rho(d)}{d} X + R_d, \tag{5.2}$$

where $\rho(d)$ is a non-negative multiplicative function such that, for primes $p \in \mathcal{P}_{q,z}$, we have

$$\frac{\rho(p)}{p} \leqslant c_1 \tag{5.3}$$

for some $c_1 < 1$.

(A_1) Summing over square-free d in $\mathcal{P}_{q,z}$,

$$\sum_{d \leqslant X^\tau}{}' |R_d| \leqslant c_2 X^{1-\zeta}$$

for some $c_2, \tau, \zeta > 0$.

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(A_2) For every $2 \leqslant w \leqslant z$,

$$-l \leqslant \sum_{p \in \mathcal{P}_{q,z} : w \leqslant p < z} \frac{\rho(p) \log p}{p} - t \log \frac{z}{w} \leqslant c_3 \tag{5.4}$$

for some $c_3, l, t > 0$.

Assuming (A_0)–(A_2), [HR74, Theorem 7.4] (combined with the properties of the function η_κ (see [HR74, § 7.4])) implies that, for $z = X^{\tau/s}$ with $s > 9t$, the following estimate holds:

$$S(T, q, z) \geqslant XW(z)\left(C_1 - C_2 l \frac{(\log\log 3X)^{3t+2}}{\log X}\right), \tag{5.5}$$

where

$$W(z) = \prod_{p \in \mathcal{P}_{q,z} : p \leqslant z} \left(1 - \frac{\rho(p)}{p}\right),$$

and the constants $C_1, C_2 > 0$ are determined by c_1, c_2, c_3, τ, ζ, t. The reader should be aware that there is a typographical error in [HR74, (6.2)] and the inequality should be in the opposite direction. We also note that an estimate of the form (5.5) could be, in principle, produced by other lower-bound sieves (for instance, by the β-sieve treated in [FI10]).

We deduce (A_0) and (A_1) from the estimates on the cardinality of lattice points given by Theorem 1.15. Let $\pi_{dN_1} : \Gamma \to \Gamma/\Gamma(dN_1)$ denote the factor map. It follows from the strong approximation property that the image of $\gamma_0\Gamma_q$ under π_{dN_1} is the whole of $\Gamma/\Gamma(dN_1)$. We set $B_T = \{h \in \mathsf{G}(\mathbb{R}) : \|h\| \leqslant T\}$. By Theorem 1.15, for every d coprime to q and $\delta \in \Gamma/\Gamma_q(dN_1)$, we have $\Gamma_q(dN_1) = \Gamma(qdN)$ and

$$\begin{aligned}
|\delta\Gamma_q(dN_1) \cap B_T| &= \frac{\mathrm{vol}(B_T)}{[\Gamma : \Gamma(qdN)]} + O_\epsilon(\mathrm{vol}(B_T)^{1-(2n_e(p))^{-1}(1+\dim(\mathsf{G}))^{-1}+\epsilon}) \\
&= \frac{\mathrm{vol}(B_T)}{[\Gamma : \Gamma(dN_1)] \cdot [\Gamma : \Gamma_q]} + O_\epsilon(\mathrm{vol}(B_T)^{1-(2n_e(p))^{-1}(1+\dim(\mathsf{G}))^{-1}+\epsilon}) \\
&= \frac{|\gamma_0\Gamma_q \cap B_T|}{[\Gamma : \Gamma(dN_1)]} + O_\epsilon(\mathrm{vol}(B_T)^{1-(2n_e(p))^{-1}(1+\dim(\mathsf{G}))^{-1}+\epsilon}) \\
&= \frac{X}{|\mathsf{G}(\mathbb{Z}/(dN_1))|} + O_\epsilon(X^{1-(2n_e(p))^{-1}(1+\dim(\mathsf{G}))^{-1}+\epsilon})
\end{aligned}$$

for every $\epsilon > 0$. We note that, for d coprime to q, we have $f(\gamma) = 0 \bmod d$ if and only if $g(\gamma) = 0 \bmod dN_1$. Restricting the sums below to d coprime to q, we have

$$\begin{aligned}
\sum_{k = 0 \bmod d} a_k &= |\{\gamma \in \gamma_0\Gamma_q \cap B_T;\, f(\gamma) = 0 \bmod d\}| \\
&= \sum_{\delta \in \pi_{dN_1}(\gamma_0\Gamma_q) : g(\delta) = 0 \bmod dN_1} |\delta\Gamma_q(dN_1) \cap B_T| \\
&= |\mathsf{G}(\mathbb{Z}/(dN_1)) \cap \{g = 0\}| \cdot \left(\frac{X}{|\mathsf{G}(\mathbb{Z}/(dN_1))|} + O_\epsilon(X^{1-(2n_e(p))^{-1}(1+\dim(\mathsf{G}))^{-1}+\epsilon})\right) \\
&= \frac{\rho(d)}{d} X + O_\epsilon(|\mathsf{G}(\mathbb{Z}/(dN_1)) \cap \{g = 0\}| X^{1-(2n_e(p))^{-1}(1+\dim(\mathsf{G}))^{-1}+\epsilon}),
\end{aligned}$$

where

$$\rho(d) = \frac{d|\mathsf{G}(\mathbb{Z}/(dN_1)) \cap \{g = 0\}|}{|\mathsf{G}(\mathbb{Z}/(dN_1))|}.$$

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As in [NS10, § 4.1], we deduce that ρ is a multiplicative function, (5.3) holds, and

$$\rho(p) = t(f) + O_f(p^{-1/2}). \tag{5.6}$$

Using the fact that

$$|\mathsf{G}(\mathbb{Z}/(dN_1)) \cap \{g = 0\}| \ll d^{\dim(\mathsf{G})-1},$$

we obtain

$$\begin{aligned}\sum_{d \leqslant X^\tau}{}' |R_d| &\ll_\epsilon \sum_{d \leqslant X^\tau} d^{\dim(\mathsf{G})-1} X^{1-(2n_e(p))^{-1}(1+\dim(\mathsf{G}))^{-1}+\epsilon} \\ &\ll (X^\tau)^{\dim(\mathsf{G})} X^{1-(2n_e(p))^{-1}(1+\dim(\mathsf{G}))^{-1}+\epsilon} \ll X^{1-\zeta}\end{aligned}$$

for some $\zeta > 0$, provided that $\tau < \tau_0 = (2n_e(p))^{-1} \dim(\mathsf{G})^{-1}(1 + \dim(\mathsf{G}))^{-1}$. This concludes the proof of (A_0) and (A_1).

To prove (A_2), we observe that from (5.6) (see, for example, [MV07, Theorem 2.7(b)]) it follows that

$$\sum_{z \leqslant p \leqslant w} \frac{\rho(p) \log p}{p} - t(f) \log \frac{z}{w} \leqslant c_3$$

for some $c_3 > 0$. This implies the upper estimate in (5.4). The lower estimate with $l = O(\log \log q)$ follows from Lemma 5.2 below.

Now it follows from (5.5) that

$$S(T, q, X^{\tau/s}) \gg \frac{X}{(\log X)^{t(f)}} \left(C_1 - C_2' (\log \log q) \frac{(\log \log X)^{3t(f)+2}}{\log X} \right). \tag{5.7}$$

Here we used the fact that $W(z) \gg (\log z)^{-t(f)}$, which follows from (5.6).

We apply (5.7) with $T = q^\sigma$ with $\sigma > \sigma_0$ and sufficiently large q. Then by Theorem 1.3, Lemma 2.2, and (2.6),

$$X = |\gamma_0 \Gamma_q \cap B_T| \gg_\sigma \frac{\mathrm{vol}(B_T)}{|\Gamma : \Gamma_q|} \gg_{\alpha'} q^{\alpha'\sigma - \dim(\mathsf{G})} \tag{5.8}$$

with $\alpha' < \alpha$. Hence, for sufficiently large q,

$$S(T, q, X^{\tau/s}) \gg_{\sigma,\alpha'} \frac{X}{(\log X)^{t(f)}}. \tag{5.9}$$

We note that every point γ which is counted in $S(T, q, X^{\tau/s})$ satisfies conclusion (ii) of the theorem, and

$$|f(\gamma)| \ll T^{\deg(f)} = q^{\sigma \deg(f)},$$

and every prime p which is coprime to q and divides $f(\gamma)$ must satisfy

$$p > X^{\tau/s} \gg_{\sigma,\alpha'} q^{(\alpha'\sigma - \dim(\mathsf{G}))\tau s^{-1}}.$$

Hence, the number of such prime factors is bounded from above by

$$\frac{\sigma \deg(f)}{(\alpha'\sigma - \dim(\mathsf{G}))\tau s^{-1}}$$

provided that q is sufficiently large. Moreover, since b and q are coprime, $f(\gamma)$ is not divisible by any prime which divides q. Hence, the number of prime factors of $f(\gamma)$ (with multiplicities)

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is bounded by

$$r > \frac{\sigma \deg(f)}{(\alpha\sigma - \dim(\mathsf{G}))\tau_0(9t(f))^{-1}} = \frac{9\alpha\sigma}{\alpha\sigma - \dim(\mathsf{G})} \cdot \sigma_0 \cdot t(f) \deg(f).$$

Hence, every γ counted in $S(T, q, X^{\tau/s})$ satisfies conclusion (i) of the theorem as well.

We have shown that every γ counted in $S(T, q, X^{\tau/s})$ satisfies (i) and (ii). Since it follows from (5.8) and (5.9) that $S(T, q, X^{\tau/s}) \geqslant 1$ for sufficiently large q, this completes the proof. □

In order to complete the proof Theorem 5.1, we need the following quite standard estimate.

Lemma 5.2. *The following estimate holds:* $\sum_{p|q} \log p/p = O(\log\log q)$.

Proof. It is sufficient to prove the claim for square-free q. Moreover, since the function $p \mapsto (\log p)/p - c_1 \log(p + c_2)$, $c_1, c_2 > 0$, is decreasing for $p \geqslant 3$, it remains to verify the estimate when q is the product of all consecutive primes less than z. In this case,

$$\sum_{p|q} \frac{\log p}{p} = O(\log z)$$

(see, for instance, [MV07, Theorem 2.7(b)]), and, by the prime number theorem,

$$\log q = \sum_{p \leqslant z} \log p \sim z,$$

which implies the claim. □

We note that the proof of Theorem 5.1 not only implies the existence of solutions for congruences, but also gives the following quantitative estimate.

Theorem 5.3. *With the notation of Theorem 5.1,*

$$\left| \left\{ x \in \mathsf{G}(\mathbb{Z}) : \begin{array}{l} f(x) \text{ has at most } r \text{ prime factors,} \\ f(x) = b \bmod q \text{ and } \|x\| \leqslant q^\sigma \end{array} \right\} \right| \gg_\sigma \frac{1}{|f(\mathsf{G}(\mathbb{Z})) \bmod q|} \cdot \frac{N_{q^\sigma}(\mathsf{G}(\mathbb{Z}))}{(\log q)^{t(f)}}$$

for sufficiently large q.

Proof. Since by (5.8) and Theorem 1.15, for every $\gamma_0 \in \Gamma_q$ and sufficiently large q,

$$X = |\gamma_0\Gamma_q \cap B_{q^\sigma}| \gg_\sigma \frac{N_{q^\sigma}(\mathsf{G}(\mathbb{Z}))}{|\Gamma : \Gamma_q|},$$

the claim of the theorem follows from (5.9) by summing over $\gamma_0 \in \Gamma/\Gamma_q$ such that $f(\gamma) = b \bmod q$. □

Proof of Theorem 1.13. If G is anisotropic over $\mathbb{R}$, then Γ is finite. Hence, we may assume that G is isotropic. We apply Theorem 5.1 with the function $\tilde{f} : \mathsf{G} \to \mathbb{C}$ given by $\tilde{f}(g) = f(gv)$. Since $\|\gamma v\| \ll \|\gamma\|$, the claim of Theorem 1.13 follows. □

Proof of Theorem 1.14. We apply the argument of the proof of Theorem 5.1 with the sets $S_T \subset \mathsf{G}(\mathbb{R})$ introduced in § 4 (in place of the sets B_T) and the polynomial function $\tilde{f}$ on G defined by $\tilde{f}(x) = f(xv)$. Using the estimate on $|\delta\Gamma_q(dN_1) \cap S_T|$ provided by Theorem 4.1, this argument can be carried out with no changes. Let $T = q^\sigma$ with σ as in Theorem 4.4. We conclude from (4.2) that, for sufficiently large q,

$$X = |\gamma_0\Gamma_q \cap S_T| \gg_\sigma \frac{\mathrm{vol}(S_T)}{|\Gamma : \Gamma_q|} \gg q^{\alpha\sigma - \dim(\mathsf{G})}, \tag{5.10}$$

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where α is as in (4.2), and

$$S(T, q, X^{\tau/s}) \gg_\sigma \frac{X}{(\log X)^{t(f)}}, \tag{5.11}$$

where $\tau < \tau_0 = (2n_e(p))^{-1} \dim(\mathsf{G})^{-1}(1 + 3\dim(\mathsf{G}))^{-1}$. As in the proof of Theorem 5.1, we conclude that every γ counted in $S(T, q, X^{\tau/s})$ is a product of at most r factors, where

$$\begin{aligned} r &> \frac{\sigma \deg(f)}{(\alpha\sigma - \dim(\mathsf{G}))\tau_0(9t(f))^{-1}} \\ &= \frac{\sigma}{\alpha\sigma - \dim(\mathsf{G})} 9t(f)\deg(f)\dim(\mathsf{G})(1 + 3\dim(\mathsf{G}))2n_e(p). \end{aligned} \tag{5.12}$$

Now, using (5.10) and (5.11), for every $\gamma_0 \in \Gamma$ such that $f(\gamma_0 v) = b \bmod q$, we have the estimate

$$\begin{aligned} &|\{\gamma \in \gamma_0\Gamma_q \cap S_T : f(x) \text{ has at most } r \text{ prime factors}\}| \\ &\gg_\sigma \frac{X}{(\log X)^{t(f)}} \gg_\sigma \frac{1}{|\Gamma : \Gamma_q|} \cdot \frac{|\Gamma \cap S_T|}{(\log q)^{t(f)}}. \end{aligned}$$

Since every $\gamma \in \gamma_0\Gamma_q$ satisfies $f(\gamma v) = b \bmod q$, we conclude that

$$\begin{aligned} &\left|\left\{\gamma \in \Gamma \cap S_T : \begin{array}{l} f(\gamma v) \text{ has at most } r \text{ prime factors}, \\ f(\gamma v) = b \bmod q \end{array}\right\}\right| \\ &\gg_\sigma \frac{|\{\gamma \in \Gamma/\Gamma_q : f(\gamma v) = b \bmod q\}|}{|\Gamma : \Gamma_q|} \cdot \frac{|\Gamma \cap S_T|}{(\log q)^{t(f)}} \\ &= \frac{1}{|f(\mathcal{O}) \bmod q|} \cdot \frac{|\Gamma \cap S_T|}{(\log q)^{t(f)}}. \end{aligned}$$

Since the cardinality of the fibers of the map $\pi : \Gamma \cap S_T \to \mathcal{O}(T) : \gamma \mapsto \gamma v$ is uniformly bounded, we conclude that

$$\left|\left\{x \in \mathcal{O}(T) : \begin{array}{l} f(x) \text{ has at most } r \text{ prime factors}, \\ f(x) = b \mod a \end{array}\right\}\right| \gg_\sigma \frac{1}{|f(\mathcal{O}) \bmod q|} \cdot \frac{|\Gamma \cap S_T|}{(\log q)^{t(f)}},$$

which implies the theorem because of (4.1) and (4.3). □

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Alexander Gorodnik a.gorodnik@bristol.ac.uk
School of Mathematics and Statistics, University of Bristol, Bristol BS8 1TW, UK

Amos Nevo anevo@tx.technion.ac.il
Department of Mathematics, Technion-Israel Institute of Technology, 32000 Haifa, Israel