$$\therefore \qquad u = \left[1 + \frac{1}{(2n+1)(2n+3)}\right]\left[1 + \frac{1}{(2n+3)(2n+5)}\right] \dots \dots .$$

$$\therefore \qquad \log u = \frac{1}{(2n+1)(2n+3)} + \frac{1}{(2n+3)(2n+5)} + \dots \dots$$

$$- \tfrac{1}{2}.\left[\frac{1}{(2n+1)(2n+3)}\right]^2 - \tfrac{1}{2}\left[\frac{1}{(2n+3)(2n+5)}\right]^2 - .$$

Now $$\frac{1}{(2n+1)(2n+3)} = \tfrac{1}{2}\left[\frac{1}{2n+1} - \frac{1}{2n+3}\right].$$

Hence, neglecting small quantities of a higher order than $1/n$, we may write

$$\log u = 1/2(2n+1)$$

$$\therefore \qquad u = e^{1/2(2n+1)}$$

$$= 1 + 1/2(2n+1) + \dots \dots ;$$

which shows that the error made in the above approximation to π is of the order stated.

[Another elementary proof of Stirling's theorem, by Mr J. W. L. Glaisher, is given in the *Quarterly Journal of Mathematics*, vol. xv., p. 57.]

A Device for the Analysis of Intervals and Chords in Music.

By A. Y. Fraser, M.A., F.R.S.E.

§ 1. The device here described has been found to simplify greatly the "somewhat laborious discussion" of the different musical intervals as given, say, in Sedley Taylor's *Sound and Music* (chap. viii.) or in Helmholtz's *Sensations of Tone*. It has been found particularly helpful in giving an account, necessarily rapid, of the nature of harmony to classes studying sound as a part of physics, from whom much familiarity with musical terms and notation is not to be expected.

§ 2. The leading idea of the device is that the octave being what may be called a periodic phenomenon, should be represented on a circular or spiral curve, and not, as is usual, on a straight vertical line. Such a representation of four octaves is given in figure 14. (To read this and the other figures, begin at the extreme right and follow the spiral round counter-clock-wise.) The compactness of this representation as compared with that on a straight line is at once obvious. The different stretches of intervals are represented by the different sizes of angles (which are, as shown, proportional to the numbers 45, 40, and 24). The various observations that have to be made regarding intervals, for example, that the fourth is the inversion of the fifth, that the interval D to *m* is equal to the interval *s* to *t*, and so on, can be got from this figure, at least as well as from the usual vertical scale. It remains to show how this diagram can be developed to exhibit readily the nature of harmonies, that is, of two or more notes sounded together; but before this can be shown, some further preliminary explanation is necessary.

§ 3. When two *simple* tones are sounded together, the effect is agreeable* or the reverse, according to the interval between them. The effect is found to be unpleasant if the interval be less than a minor third; the interval of a tone being less disagreeable than the interval of a semi-tone, and any disagreeable interval becoming less disagreeable if both notes are taken at a higher pitch. This unpleasantness has been shown to be due to rapid waxings and wanings of sound, called *beats*, as the vibrations of the two notes assist or oppose each other. The interval of a minor third is regarded, in fact, as the smallest smooth interval. Such an interval can be readily distinguished on our diagram, for it is represented by what is almost exactly a right angle.

§ 4. So much for *simple* tones; but, if we except the tuning fork, which, in certain circumstances does so, no musical instrument gives out simple tones; every tone is *compound*—is, in fact, a congeries of simple tones. Such a compound tone is made up as follows: There is (A) a loud sounding simple tone and (B) a collection of simple tones higher in pitch than (A), differing in relative intensity, and all in general much weaker than the fundamental tone (A). These

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* I use the words agreeable and disagreeable to save circumlocution, knowing all the time that musicians object *in toto* to the use of the words. A "disagreeable" interval may be used in a musical composition like a "disagreeable" incident in any other work of art, with the most happy general effect. Absolute smoothness in music, like absolutely pure water, is neither quite attainable, nor at all pleasing to the taste if attained.

component simple tones are called *partial tones*, the fundamental tone (A) being called the *prime partial*, and the rest (B) the *upper partials.* If the frequency of vibration of the prime partial be 1, then the series of upper partials have frequencies 2, 3, 4, 5, 6, &c., and thus if C represent the pitch of the prime partial tone, the whole series is (Fig. 15)

1,	2,	3,	4,	5,	6,	&c.
C	C′	G′	C″	E″	G″,	&c.

As a rule, beyond the last note given above, the upper partials are so weak as to be practically non-existent. The *quality* of any tone is determined by the number and relative intensities of the partial tones composing it, while the *pitch* of a tone is that of its prime partial.

§ 5. The last two sections are, of course, an epitome of facts, due for the most part to Helmholtz, and quite well known, but necessary for the understanding of the contrivance now to be described. In our examples we shall suppose the tones to be each made up of the six partials given above. They are the most important*, and the analysis of an interval with any other combination of partial tones is in principle quite the same.

§ 6. Figure 15 represents a diagram drawn on a large sheet of cardboard. Behind the cardboard is glued a strip of wood, which affords a support to a pin (say two inches of an ordinary pen holder), which projects from the centre of the spiral at right angles to the surface of the cardboard. This diagram, in which the six arrowheads represent six partials, will be taken to stand for all the partial tones of the lower of two notes whose combined effect is to be analysed.

Figure 16 represents a moveable arrangement to be used in connexion with figure 15. In the middle is a circle of wood with a hole in its centre, so that it can be made to rotate stiffly about the pin in fig. 15. This circle of wood carries radially three rods (steel knitting wires), and on each rod there slides a strip of cardboard cut out to represent one, two, and three arrowheads. The size of the arrangement is such that the whole can be made to cover exactly the corres-

* "The sounds of most musical instruments practically contain only the first six partial tones."—Sedley Taylor, *Sound and Music*, 2nd edit., p. 167.

ponding parts of fig. 15. Figure 16 is to be employed to represent the higher of two notes that are sounded together.

§ 7. Suppose, now, the framework of fig. 16 to be superposed on fig. 15 so as to coincide with it exactly. This will represent *unison*, that is, the sounding together of two notes of the same pitch and quality, with, therefore, complete coincidence in all the partial tones. Turn now the upper part a degree or two in either direction, and this will, of course, put every arrowhead of the one a degree or two from the corresponding arrowhead of the other. The interpretation of the diagram now is that there are six beating pairs of partials, which shows the *definiteness* of the unison combination—a chord being definite when it is bounded on each side by sharp discords.

§ 8. Again, let us look at the nature of the *major third* interval. Turn* round the moving part of our diagram till its "C" coincide with the "E" of the diagram on fig. 15. The dotted lines on fig. 17 show the position taken up by the moving part.

An examination of the diagram (fig. 17) shows that there is beating between the 4th partial of C and the 3rd of E, which two are a semi-tone apart, and also between the 6th of C and the 5th of E, also a semi-tone apart, but at a much higher pitch, and on that account, as well as from the fact that they will in general be weaker, they are a less rough pair. So much for the beating, due to partials in the major third.

If, now, the upper of the two notes be raised an octave (or the lower depressed an octave)—that is, if a male and a female voice sing the notes—the above beating intervals are cut out, and thus, so far as the first six partials are concerned, the *major tenth* (for such the interval now is) is a smooth interval. This bears out the known fact that a "major third" is smoother when taken by a male and a female voice than when taken by two females.

§ 9. We need not go into further detail in this direction. By

* As the moving part turns round to the left, its arrowheads project beyond the lines of the spiral to which they belong, and have either to be pushed back by hand or to be pulled back by strings passing round the fixed pin in the centre and shortening by being wound up. A small piece of india-rubber cord joining the point of each of the three cardboard strips of the outer extremity of its rod, will keep the string tight.

turning round the movable part (fig. 16, that is to say) every combination of two notes can be exhibited; angles less than a right angle—that is, intervals less than a minor third—at once catch the eye, and the numbers written on the arrowheads tell without trouble what partials are concerned. The roughness of any interval being thus ascertained, we can examine into the definiteness (due really to coinciding partials) by turning the upper part a degree or two out of position.

§ 10. One or two miscellaneous suggestions may be useful.

By using two moving parts like fig. 16 in conjunction with fig. 15, combinations of three notes can be studied. This requires no further illustration here.

A diagram such as fig. 14, with a spiral making four or five circuits, can be used as a repository of a good deal of information. One circuit may have the notes named as D, *r*, *m*, &c., while the next bears the letters C, D, E, &c. A third circle might carry the vibration ratios 1, $\frac{9}{8}$, $\frac{5}{4}$, &c., while the fourth might give a standard series of vibration numbers. The outermost circuit of a fair sized spiral would afford abundance of room for particulars concerning the various sharpened and flattened notes.

Our analysis has left combinational tones out of account, but the insertion of vibration numbers, as has just been suggested—even if a special diagram were devoted to this alone—would make the placing of the combinational tones an easy enough affair.

The method of using the diagram as described above is not the only way of turning the idea to account. A private student may use in connection with fig. 15 semi-transparent ground glass, which can be written on, or common glass with radial paper lines gummed on, or even a copy of fig. 16 cut out in cardboard; while for demonstration to a large audience the diagrams done on glass can be projected in the usual way by the lantern.