REVISIONISM REVISITED

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Abstract. This paper offers a substantial improvement in the revision-theoretic approach to conditionals in theories of transparent truth. The main modifications are (i) a new limit rule; (ii) a modification of the extension to the continuum-valued case; and (iii) the suggestion of a variation on how universal quantification is handled, leading to more satisfactory laws of restricted quantification.

§1. Background and preview. It seems central to the notion of truth that there is a kind of equivalence between the attribution of truth to a proposition and the proposition itself. Not merely that '$True(\lambda p)$' and 'p' are co-assertable, but that one can be substituted for the other (except inside quotation marks, attitude contexts and the like) without affecting assertability. Call this *the transparency of truth*. But there seem to be propositions that directly or indirectly attribute untruth to themselves.[1] Given transparency, their truth is equivalent to their untruth, which is inconsistent in classical logic. Given that there are such propositions, we must decide between restricting transparency and restricting classical logic. To make an informed choice, we must explore both classical theories that restrict transparency, and non-classical theories that keep it.[2] The present paper, like a great deal of recent literature, is part of an exploration of the latter option.

Instead of speaking of propositions, we can speak of *sentences as used on a given occasion*.[3] I think this changes little in the theory of truth. (A small advantage is that the

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[1] This is sometimes challenged, but for a rebuttal to the challenges see [10]. That paper also elaborates some of the points in notes 3 and 5 below.

[2] Poincare once argued that Euclidean geometry is so much simpler than alternative geometries that it would never make sense to take seriously a physical theory based on an alternative geometry. There are many today who have the attitude toward classical logic that Poincare had toward Euclidean geometry: keep it no matter what the cost elsewhere. Maybe there are also some who are "Poincarean" in the opposite direction: keep the naive laws of truth and/or property-instantiation come what may. But I think the proper attitude, here as with geometry, is to choose the simplest *overall* theory (logic and truth rules together). To do that, we must know what the simplest overall theory is, which requires that we develop each one as best we can.

[3] A sentence as used on a given occasion can be indeterminate, but instead of regarding that as involving its expressing multiple propositions, we can regard it as expressing an indeterminate

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slightly contentious assumption of *propositions* that attribute untruth to themselves is replaced by the harder to deny assumption of *sentences that on a given occasion* attribute untruth to *themselves as used on that occasion*. Another advantage is that it enables us to put aside questions of propositional identity, and of how to extend transparency to accommodate it.)[4] If we focus on sentences rather than propositions, it's natural to consider toy languages in which there are no ambiguous or indexical sentences, so that we can take sentence-types of the language as bearers of truth. I will follow most of the literature on paradoxes of truth by pretending that sentence-types are the bearers of truth, but what I say could easily be rewritten in terms of propositional truth.[5]

Far and away the most influential paper in the literature on non-classical approaches to truth is [13] (despite Kripke's own protests that his paper has nothing to do with non-classical logic). In particular, his fixed point constructions using Strong Kleene semantics have served as the basis for transparent truth in a variety of non-classical logics. (I will assume that the reader knows these constructions. They start from a ground language, without '*True*', in which a theory of syntax (supplying the bearers of truth) can be developed, and classical ground models for this language that are standard for syntax. We extend these models to 3-valued models by adding a transparent truth predicate; different fixed points extend them in different ways.[6] Sentences true in a ground model M_0 get value 1 in its 3-valued extensions; sentences false in M_0 get 0; and value $\frac{1}{2}$ is used for certain sentences involving '*True*', such as Liar sentences that can't coherently be assigned 0 or 1.) Here I'll restrict attention to the Strong Kleene quantificational logic K_3 (extended to include transparent truth by the construction). Call an inference "TK_3-valid" if for all fixed point models, if the premises get value 1 in the model, so does the conclusion. (One gets a slightly stronger logic if one restricts to *minimal* fixed point models.)

The 3-valued Kleene models treat, $\neg$, $\wedge$, $\vee$, $\forall$ and $\exists$ in very natural ways: the value of $\neg A$ is 1 minus the value of A; $|A \wedge B|$ is $min\{|A|, |B|\}$ and $|\forall x A(x)|$ is $min\{|A(o)| : o$ in the domain$\}$; and $\vee$ and $\exists$ are similar but with *max* instead of *min*. But there is no connective that serves as a reasonable conditional. (I'm assuming that a reasonable conditional should license the assertion of all instances of $A \to A$, $A \wedge B \to A$, and the like, but if $\to$ were defined from $\neg$ and $\vee$ as in classical logic, some instances of these will get the undesignated value $\frac{1}{2}$.) So it is natural to want to extend Kripke's

proposition, and taking the attribution of truth to an indeterminate proposition p to be indeterminate in just the way that p is.

4 But in the sentential case, the notion of equivalence involved in the definition of transparency requires a bit more explanation: it is something like equivalence *modulo the existence of the utterance*. (In the propositional case, it's also modulo their existence, but that might be thought unimportant since propositions "exist necessarily".)

5 Talk of propositions is sometimes regarded as merely a convenient paraphrase for quantification into sentential position. The paradoxes of truth arise equally in this framework. (If quantification both in and out of intentional contexts is allowed, one can use sentences like $\exists p$(The person in Room 202 is now saying that p, but not p), said by the person on Room 202. Even if that sort of sentential quantification is disallowed, one can mimic Gödel–Tarski diagonalization; at least, one can if quantification into predicate position as well as sentential position is allowed, and the latter is needed anyway to deal with related issues about properties.) I think that what follows could also be rewritten more laboriously in terms of sentential quantification.

6 Since the extended models merely add a new predicate, they too are standard, which means that a rule form of induction is guaranteed to hold even for predicates that contain '*True*'.

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theory to a language involving a reasonable conditional $\rightarrow$ in addition to the Kleene connectives and quantifiers plus '*True*'. But this requires one or another substantial alteration of Kripke's methods.[7]

One such extension was proposed in [3]. Brady first noted that from an arbitrary assignment h of values in $\{0, \frac{1}{2}, 1\}$ to conditional sentences $A \rightarrow B$ in the language with '*True*', we can give what I'll call a "Kripkean microconstruction" to generate an assignment v_h of values to all sentences in that language. (More accurately, h and v_h need to assign values to conditional formulas *relative to* assignments of objects in the domain to their free variables. For brevity's sake, I'll be sloppy about this in my formulations; if you like, you can take my 'sentence' to mean 'quasi-sentence', defined as a pair of a formula and a function assigning objects to its free variables.)[8] In this construction, conditionals are treated as atomic, with their values given by h, and otherwise the construction is just like Kripke's minimal fixed point construction. v_h will assign the same values to conditionals as h does, and as long as h is transparent in its assignments to conditionals, v_h will be transparent in its assignments to all sentences.

So now the question is, how do we choose h? For this, Brady proposed a fixed point "macro-construction": we start out with an h_0 that assigns 1 to all conditionals; then at each subsequent α, we let h_α assign to $A \rightarrow B$ the value 1 iff $(\forall\beta < \alpha)(v_{h_\beta}(A) \leq v_{h_\beta}(B))$, 0 iff $(\exists\beta < \alpha)(v_{h_\beta}(A) = 1 \wedge v_{h_\beta}(B) = 0)$, and $\frac{1}{2}$ otherwise.[9] Because for every conditional, the value assigned to it by h_α never increases as α increases, we eventually reach a fixed point h_Ψ. The "final value" of a sentence in the Brady construction is its value in the Kripkean fixed point micro-construction over h_Ψ.

A Brady conditional has some nice properties, but also a very odd one: conditionals invalidated at early stages (early micro-constructions like that from h_0, where the assignment to conditionals is obviously bad) can never recover. For instance, if $\top$ is $\forall x(x = x)$ and $\bot$ is its negation, h_0 assigns 1 to $\top \rightarrow \bot$, so h_1 assigns value 0 to $(\top \rightarrow \bot) \rightarrow \bot$; and because at every subsequent stage h_α looks back at *all* earlier stages, every h_α assigns 0 to this sentence.

Maybe there are purposes for which we might want a conditional where this isn't so bad. But one role for conditionals is to restrict universal quantification: to define "All A are B" as "$\forall x(Ax \rightarrow Bx)$". For this purpose, the Brady conditional seems plainly inadequate: it invalidates the inference from "There are no A" to "All A are B". More generally, it fails because the $\rightarrow$ fails to reduce to the ordinary $\supset$ in classical contexts, and in classical contexts, restricted universal quantification goes by $\supset$. *It's the use of conditionals for defining restricted universal quantification that I'll be considering in*

7 [16] improves on Kripke's construction using basically Kripkean methods (a multi-stage monotonic construction), but its conditional is not "reasonable" in the sense of this paragraph: some instances of $A \rightarrow A$ come out undesignated. Maybe for some purposes that doesn't make it unreasonable, but as I'll soon make explicit, my interest in this paper is with conditionals that can be used to restrict universal quantification, and I don't think it acceptable not to be able to assert in full generality such laws as "All A are A", "Everything that's A and B is A", and so forth.

8 In fact, I'll take '*True*' to apply to quasi-sentences as well as ordinary sentences, so that the theory to be discussed really treats of satisfaction rather than just truth as normally understood (though truth as normally understood is a special case).

9 If I (following Brady) hadn't restricted to minimal fixed points, then each h wouldn't generate a single v_h but a large set V_h of them; then presumably $h_\alpha(A \rightarrow B)$ would be 1 iff for all $\beta < \alpha$ and all v in V_{h_β}, $v(A) \leq v(B)$.

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this paper.[10] (If you think that the restricted universal quantifier should be taken as primitive rather than defined in terms of unrestricted quantification and a conditional, you can easily rewrite what follows in terms of extending Kripke by adding a restricted universal quantifier to the language.)

An obvious diagnosis of why the Brady construction has this odd feature is that in defining h_α, it looks back at *all* previous v_{h_β}; what if instead we just look at the *recent* ones? One way of developing this thought is to use a revision construction.[11] In the simplest version of this, when α is a successor ordinal, we look only at values generated by the previous stage:

$$h_{\beta+1}(A \to B) = 1 \text{ iff } v_{h_\beta}(A) \leq v_{h_\beta}(B).$$

There's more than one possibility for the 0 clause, but I think the best is the revision-theoretic variation of [3] (which models Łukasiewicz 3-valued logic as well as a revision theory can):

$$h_{\beta+1}(A \to B) = 0 \text{ iff } v_{h_\beta}(A) = 1 \wedge v_{h_\beta}(B) = 0;$$
$$\text{so } \frac{1}{2} \text{ iff } v_{h_\beta}(A) - v_{h_\beta}(B) = \frac{1}{2}.$$

(In what follows, I'll use the notation $|A|_\beta$ in place of $v_{h_\beta}(A)$.)

But what about 0 and limit ordinals? The starting assignment h_0 matters only slightly in a 3-valued revision construction: we want it to be transparent, but other than that the values there *largely* wash out in the end. But the treatment of h_λ when λ is a limit ordinal matters a great deal. And *in all my prior writings on this, I made an ill-advised choice*.

In my previous writings, I used a rule that like Kripke's gives a kind of default status to value $\frac{1}{2}$: in the 3-valued case, it gave value $\frac{1}{2}$ to a conditional at a limit ordinal unless either the limit inferior of prior values was 1 or the limit superior was 0. That is, for limit λ,

$$(\mathbf{SYM_3})h_\lambda(A \to B) = \begin{cases} 1 & \text{iff } (\exists\beta < \alpha)(\forall\gamma \in [\beta,\alpha))(h_\gamma(A \to B) = 1) \\ 0 & \text{iff } (\exists\beta < \alpha)(\forall\gamma \in [\beta,\alpha))(h_\gamma(A \to B) = 0) \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

This has some attractive mathematical properties, with which I was much taken: in particular, what I called "the Fundamental Theorem". But I now think that a different limit rule that does not yield a "Fundamental Theorem" actually leads to a much better theory. The preferred rule (in the 3-valued case, for limit λ) is

$$(\mathbf{LimInf_3})h_\lambda(A \to B) = \begin{cases} 1 & \text{iff } (\exists\beta < \alpha)(\forall\gamma \in [\beta,\alpha))(h_\gamma(A \to B) = 1) \\ 0 & \text{iff } (\forall\beta < \alpha)(\exists\gamma \in [\beta,\alpha))(h_\gamma(A \to B) = 0) \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

In other words: for $r > 0$, $h_\lambda(A \to B) \geq r$ iff $(\exists\beta < \alpha)(\forall\gamma \in [\beta,\alpha))(h_\gamma(A \to B) \geq r)$.

[10] Restricted existential quantification is of course $\exists x(Ax \wedge Bx)$, so the two restricted quantifiers aren't interdefinable except in classical settings.

[11] [11] is a well-known discussion of revision theories. But their focus is on revision theories for truth directly, as an *alternative* to Kripke (and also, revision theories for circular definitions). The application of revision theory to conditionals, using a Kripkean background, has quite a different flavor.

The successor and limit cases can be combined: for any $\alpha > 0$ and $r > 0$, $h_\alpha(A \to B) \geq r$ iff $(\exists\beta < \alpha)(\forall\gamma \in [\beta, \alpha))(|A|_\gamma - |B|_\gamma \leq 1 - r)$. The equivalence to the official definition is trivial in the successor case, and proved by an easy induction in the limit case. (The combined formulation works even for $\alpha = 0$ if the starting hypothesis h_0 assigns 0 to all conditionals. 'Hypothesis' here and later will mean 'assignment of values to conditionals'.)

In §2, I'll explore *the sentential logic* that results from adopting (LimInf_3), and say a bit more about the choice of it over (SYM_3). In §3, I'll discuss the advantages of *moving to a continuum-valued context* (the advantages are huge, and the discussion of §2 is little affected), and how the revision theory with a generalization of (LimInf_3) should be developed in such a context. (I discussed continuum-valued revision theory with a generalization of (SYM_3) in [8], but in addition to the new limit rule, I will make another substantial change.) In §4, I'll explore *the quantifier logic*. In §5, I'll discuss *restricted quantification* and raise *an issue about the interpretation of the English word 'all'*. §6 concludes, with comparisons to earlier work combining conditionals with transparent truth. The Appendices contain some proofs.

§2. The sentential logic. A revision construction doesn't lead to a fixed point. However, it's easy to see on cardinality grounds that there are *recurrent* valuations of conditionals, i.e., valuations h that occur arbitrarily late: $\forall\beta(\exists\lambda > \beta)(h_\gamma = h)$. Indeed there comes a point α_0 (the "critical ordinal") such that for every $\alpha \geq \alpha_0$, h_α is recurrent.[12] A valuation v *of sentences generally* is called recurrent if it's v_h for some recurrent h. (Or more generally if one doesn't restrict to the minimal Kripke fixed point: if it is a member of V_h (see n. 9) for some recurrent h.) Which valuations are recurrent obviously depends on the ground model. An inference is taken as valid iff for all ground models, it preserves value 1 in all recurrent valuations based on the ground model.

Putting it another way, let the value space $\mathbf{V}_M$ for a ground model M be the set of functions that assign a value in $\{0, \frac{1}{2}, 1\}$ to every recurrent hypothesis based on M. Then a revision construction based on ground model M assigns to every sentence A a value $||A||$ in $\mathbf{V}_M$. The sole designated value of $\mathbf{V}_M$ is the constant function $\mathbf{1}$, assigning 1 to every recurrent hypothesis based on M. Validity is the preservation of this value $\mathbf{1}$, on all ground models. Equivalence of two sentences is their having the same function as value, on all ground models; in other words, validity of the biconditional between them.

Another feature of revision constructions (at least, those with fixed rules at limits, like (SYM) or (LimInf)) is the existence of *strong reflection ordinals*: ordinals Δ so complicated that for every $\beta < \Delta$, every recurrent hypothesis occurs in the interval $[\beta, \Delta)$. It's clear that on both (SYM) and (LimInf), the value of a conditional is 1 at a strong reflection ordinal Δ iff it is 1 for all recurrent h_α.[13] That is, $||A \to B|| = \mathbf{1}$ iff $|A \to B|_\Delta = 1$. That's a nice feature.

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12 These facts, and the existence of reflection ordinals as stated in the paragraph after next, are standard results in revision theory (see, for instance, [11]).

13 A *weak reflection ordinal* is any ordinal that satisfies this biconditional. The easiest way to prove the existence of weak reflection ordinals is to prove the existence of strong ones.

With (SYM), the value of a conditional is 0 at Δ iff it is 0 for all recurrent h_α; so the value of the negation of a conditional is 1 at Δ iff it is 1 for all recurrent h_α. So the nice feature for conditionals extends to negations of conditionals. In fact, it can be shown that it extends to every sentence of the language: for every sentence, not just for conditionals, $||A|| = \mathbf{1}$ iff $|A|_\Delta = 1$. That's what in earlier various papers, I called the Fundamental Theorem. But it depends on (SYM): indeed, with (LimInf), the value of a conditional is 0 at Δ iff it is 0 for *some* recurrent h_α; so the value of the negation of a conditional is 1 whenever it is 1 for *some* recurrent h_α. So for (LimInf), the Fundamental Theorem fails very badly.

And that does have a philosophical cost. Typical classical theories based on supervaluationist and revision-theoretic semantics have a feature sometimes thought odd: a disjunction can be valid even though both disjuncts lead to inconsistency. Indeed, in such classical theories, the disjunction of a Liar sentence and its negation is an example. In revision theories based on (SYM), the Fundamental Theorem rules out there being any such examples. When we move to (LimInf), on the other hand, there are such examples, though in the theory ultimately to be recommended, it happens only with rather arcane disjunctions, like $R \vee \neg R$, where R is the Restall sentence discussed in §3.

Despite this apparent disadvantage of LimInf, it leads to a much nicer logic, one that more closely approximates classical logic and hence is easier to use in formal reasoning involving the restricted quantifier conditional. Here's a simple example, but it's just the tip of the iceberg. Consider the schema

Adjunction: $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$.

This is not valid on (SYM), but is on (LimInf). (The rule form $(A \to B) \wedge (A \to C) \vdash A \to B \wedge C$ is valid on both.) Indeed on (SYM), even the special case

$$(\top \to B) \wedge (\top \to \neg B) \to (\top \to B \wedge \neg B)$$

fails. For suppose $|B|_\alpha$ is 1 for odd α, 0 when α is an even successor, and $\frac{1}{2}$ when α is a limit.[14] Then (SYM) gives $\top \to B$ and $\top \to \neg B$ value $\frac{1}{2}$ at limits, but gives $\top \to B \wedge \neg B$ 0 at limits of form $\lambda + \omega$, so the displayed conditional has value $\frac{1}{2}$ at any ordinal of form $\lambda + \omega + 1$ and hence at any multiple of ω^2. (With (LimInf) that counterexample fails since $\top \to B$ and $\top \to \neg B$ have value 0 at all limit ordinals.) The soundness of Adjunction for (LimInf) is easily verified: I'll give a proof in a more general setting in Appendix A.

A difference between (SYM) and (LimInf) that has big ramifications is that with (SYM) we have the biconditional

$$(\top \to \neg B) \leftrightarrow \neg(\top \to B),$$

whereas with (LimInf) we have only the left-to-right direction. (Though even with (LimInf) we have the rule form of the right to left: $\neg(\top \to B) \vdash \top \to \neg B$.) Equivalently, writing $A \sqcap B$ for $\neg(A \to \neg B)$, (LimInf) yields $(\top \to B) \to (\top \sqcap B)$ but not the converse (since at limit ordinals, $|\top \to B|_\lambda$ is the *liminf* of prior values of B, whereas $|\top \sqcap B|_\lambda$ is the *limsup* of those values). (SYM) leads to both directions,

[14] A sentence B which has this feature on (SYM_3) is the sentence Q_2 that says $True(\langle Q_2 \rangle) \to \neg True(\langle Q_2 \rangle)$. (A continuum-valued theory would give Q_2 a constant value, viz., **2/3**, but there are other sentences that would have this valuation pattern there.)

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which might at first seem like an advantage to (SYM); but in fact having the distinction between $\top \to B$ and $\top \sqcap B$ is crucial to getting approximations to many important classical laws.

For instance, one such law is

Classical Permutation: $[A \to (B \to C)] \to [B \to (A \to C)]$.

There's no way one can get this on any revision construction that doesn't lead to a fixed point, basically due to the fact that A differs in its depth of conditional embedding on the two sides of the main conditional, and so does B. The obvious idea of how to fix this is to "depth-level it", replacing the occurrence of A on the left by $\top \to A$ or $\top \sqcap A$, and analogously for the occurrence of B on the right. It turns out that on (LimInf), you need $\sqcap$ on the left and $\to$ on the right:

Depth-Leveled Permutation: $[(\top \sqcap A) \to (B \to C)] \to [(\top \to B) \to (A \to C)]$.

That this is sound with (LimInf) will be shown in Appendix A. With (SYM) on the other hand, there is no distinction between $\top \sqcap B$ and $\top \to B$, and as a result, no depth-leveled version of Permutation is available. (For a counterexample in the 3-valued construction, let A be the Q_2 of n. 14, C be $\neg A$, and B be $\top$.)

I won't say a lot more about (SYM). With (LimInf), the following derivation system for the sentential logic with '*True*' is sound in the given semantics (and also in the continuum-based semantics to be given later).

S1 (Identity). $A \to A$.

S2 ($\wedge$-Elim). $A \wedge B \to A$, and $A \wedge B \to B$.

S3 (Distribution). $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$.

S4 (Double Negation). $\neg\neg A \to A$.

S5 (Contraposition). $(A \to \neg B) \to (B \to \neg A)$.

S6 (Adjunction). $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$.

S7a (Depth-Leveled Suffixing). $[\top \to (A \to B)] \to [(B \to C) \to (A \to C)]$.

S7b (Depth-Leveled Prefixing). $[\top \to (A \to B)] \to [(C \to A) \to (C \to B)]$.

S7c. $[\top \to (A \sqcap B)] \to [(B \to C) \to (A \sqcap C)]$.

S8 (Depth-Leveled Weakening). $[\top \to (\neg A \vee B)] \to [A \to B]$.

S9 (Depth-Leveled Permutation). $[(\top \sqcap A) \to (B \to C)] \to [(\top \to B) \to (A \to C)]$.

S10. $(\top \to \neg A) \to \neg(\top \to A)$.

S11 (Depth-Leveled special Łukasiewicz law). $[(A \to B) \to (\top \to B)] \to [\top \to ((\top \sqcap A) \vee (\top \sqcap B))]$.

T-Schema. $True(\langle A \rangle) \leftrightarrow A$ [When we add quantifiers we'll want to strengthen this and add composition laws.]

R1 (Modus Ponens). $A, A \to B \vdash B$.

R2 ($\wedge$-Introd). $A, B \vdash A \wedge B$.

R3a (Positive Weakening Rule). $B \vdash \top \to B$.

R3b (Negative Weakening Rule). $\neg(\top \to B) \vdash \neg B$.

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R4a (No decreases). $(\top \to B) \to B \vdash B \to (\top \to B)$.
R4b (No increases). $B \to (\top \sqcap B) \vdash (\top \sqcap B) \to B$.

Mostly these require little comment beyond what I've given already. R4a and R4b reflect the fact that if a sentence doesn't reach a fixed point in a revision construction, it must increase at some final ordinals (ordinals greater than the critical ordinal) and decrease at others: if, e.g., it never decreased from α on, but did increase, then the valuation at α couldn't be recurrent. It easily follows from R4a and b (with S10 and the Prefixing and Suffixing rules below) that their reverse directions are valid and that $(\top \to B) \leftrightarrow B \vdash (\top \sqcap B) \leftrightarrow B$ and conversely. S11 is depth-leveled versions of the L to R of the Łukasiewicz (and classical) equivalence $[(A \to B) \to B] \leftrightarrow [A \vee B]$. (The system proves depth-level versions of the R to L as well: $[\top \to (\top \to A)] \to [(A \to B) \to (\top \to B)]$ and $[\top \to (\top \to B)] \to [(A \to B) \to (\top \to B)]$.[15]

What makes the logic more complicated than the Łukasiewicz is the need to depth-level, due to the lack of a general equivalence between $\top \to B$ and B. (If you were to add that general equivalence as an additional axiom schema, the Łukasiewicz continuum-valued laws as given in [14, pp. 228–229], would be immediate consequences: all those laws are obtained from those here by deleting some occurrences of '$\top \to$' and '$\top\sqcap$'.) Any revision construction (not leading to a fixed point) will have failures of this equivalence, e.g., when A is the Restall sentence discussed below.

What we might hope, though, is that for "most" sentences, even "most" paradoxical ones, the "regularity assumption" $(\top \to B) \leftrightarrow B$ is valid. (As noted, that's equivalent to the validity of $(\top \sqcap B) \leftrightarrow B$, so that could equally serve as a regularity assumption; so could either direction of either of these biconditionals.)[16] With a revision theory based on 3-valued logic, a great many paradoxical sentence are irregular. But if we move to an analog that replaces $\{0, \frac{1}{2}, 1\}$ with the unit interval [0, 1], regularity can be assumed for all but the very arcane sentences (ones that employ quantification in a very specific way). In particular, in the continuum-valued semantics, regularity and hence Łukasiewicz continuum-valued laws can be assumed for "quantifier-independent" sentences, in a sense I'll define in the next section. And since the axioms of [14] are complete for Łukasiewicz semantics (for inferences involving only finitely many premises), this means that if we supplement the system above with a schema stating the regularity of quantifier-independent sentences, the system so supplemented is complete for (finite premise) inferences involving only such sentences.[17]

For now, I just note that the unsupplemented system is powerful enough to derive lots of desirable consequences. Here are a few obvious ones:

Suffixing Rule: $A \to B \vdash (B \to C) \to (A \to C)$.

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15 The second is immediate by the Suffixing Rule below; the first is a special case of Depth-Leveled Suffixing.

The Łukasiewicz (and classical) equivalence in the text entails $[(A \to B) \to B] \to [(B \to A) \to A]$. Here we get an approximation: $[(A \to B) \to (\top \to B)] \to [(B \to A) \to (\top \sqcap A)]$. This is almost immediate from S11 and S7c.

16 Note that the formulas themselves are not equivalent; only their validity is.

17 The complexity of the revision-theoretic model theory precludes any sound axiomatic system satisfying a more general form of completeness for finite-premise sentential inferences: see [20] (for an earlier version of the revision-theoretic semantics, but the point still applies).

Proof. $A \to B \vdash \top \to (A \to B)$ by the Positive Weakening rule, so with Depth-Leveled Suffixing and Modus Ponens, we get the result. □

Similarly for the analogous *Prefixing Rule*; and of course from either one, we get the

Transitivity Rule: $A \to B, B \to C \vdash A \to C$.

Using Contraposition, we get negative versions of Suffixing and Prefixing, e.g., $A \to B \vdash \neg(A \to C) \to \neg(B \to C)$. Using this, we can get

Intersubstitution Law: For any $n \geq 0$: $(\top \to^n (A \to B)) \to [X(A) \to X(B)]$, where A occurs only positively and of depth n in $X(...)$; and similarly with the consequent replaced by $X(B) \to X(A)$, if A occurs only negatively and of depth n. $(\top \to^n (A \to B))$ means the result of prefixing $A \to B$ with '$\top \to$' n times. This can be explicitly defined for variable n, by recursion.) If the antecedent is strengthened to $(\top \to^n (A \leftrightarrow B))$ we only need to assume that substitutions are of depth n.

Various laws relating $\wedge$ and $\to$, such as (i) $[(A \to B) \wedge (C \to D)] \to [A \wedge C \to B \wedge D]$ and (ii) $(A \to B) \to [(A \wedge C) \to (B \wedge C)]$.

These would not be valid under (SYM); their proofs rest on Adjunction.

Proof of (i): Using $A \wedge C \to A$ and $A \wedge C \to C$, the antecedent is easily shown to strongly entail $(A \wedge C \to B) \wedge (A \wedge C \to D)$; we get $A \wedge C \to B \wedge D$ by Adjunction. □

Proof of (ii): Take D to be C in (i). □

Reverse Adjunction: $(A \to C) \wedge (B \to C) \to (A \vee B \to C)$.

(This follows from Adjunction using contraposition.) Given this, obvious disjunctive analogs of the laws relating $\wedge$ and $\to$ are derivable in an analogous fashion.

Note that depth-level prefixing with $C = \top$ is

$$\vdash [\top \to (A \to B)] \to [(\top \to A) \to (\top \to B)],$$

which makes the prefix $\top \to$ "necessity-like", with a necessity that in this system doesn't obey the T axiom corresponding to a reflexive accessibility operation. Of course this "necessity" has nothing to do with necessity as normally understood, and for sentences that can be assumed regular, it is totally vacuous.

If we define $\$A$ as $(\top \to A) \wedge A$, \$ does of course obey the T axiom, and it too is necessity-like under (LimInf):

\$-Introduction: $A \vdash \$A$, and
K-law for \$: $\vdash \$(A \to B) \to (\$A \to \$B)$.

(\$-Introduction is immediate from R3a and R2, and the K-law for \$ follows from the K-law for "$\top \to$", together with law (i) relating $\wedge$ and $\to$. As the dependence on the law relating $\wedge$ and $\to$ suggests, it isn't sound under (SYM).) $\$(A \wedge B) \leftrightarrow \$A \wedge \$B$ is clearly derivable. Of course when A can be assumed regular, the "necessity" operators $\top \to$ and \$ are vacuous.

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Strengthened Positive Weakening Rule: $\neg A \vee B \vdash A \rightarrow B$.

Proof. Use R3a and either Depth-Leveled Suffixing or Depth-Leveled Weakening. □

Strengthened Negative Weakening Rule: $\neg(A \rightarrow B) \vdash A \wedge \neg B$.

Proof. By Depth-Leveled Weakening, Contraposition, and transitivity rule, $\vdash \neg(A \rightarrow B) \rightarrow \neg[\top \rightarrow (\neg A \vee B)]$; so $\neg(A \rightarrow B) \vdash \neg[\top \rightarrow (\neg A \vee B)]$ and so by R3b, $\neg(A \rightarrow B) \vdash \neg(\neg A \vee B)$, which yields the result. □

The Strengthened Positive Weakening rule also yields explosion (it entails $\neg A \vdash A \rightarrow B$, which yields $A, \neg A \vdash B$ by modus ponens).

The absence of a full weakening axiom $(\neg A \vee B) \rightarrow (A \rightarrow B)$ might seem as if it would make trouble for restricted quantification. However, when we add quantifiers we'll be able to define an operator $\$^{\omega}$, where $\$^{\omega} A$ behaves something like an infinite conjunction of A, $\top \rightarrow A$, $\top \rightarrow (\top \rightarrow A)$, and so on. (See third paragraph of §4.) And we'll be able to derive the law

(#) $\$^{\omega}(\neg A \vee B) \rightarrow \$^{\omega}(A \rightarrow B)$.

This is the secret behind the treatment of restricted quantification that I'll suggest.

I've noted that on sufficient regularity assumptions, all the laws of Łukasiewicz [0,1]-valued sentential logic hold. This enables us to immediately see that we must be able to prove such things as

$$Reg(A), Reg(B) \vdash (A \rightarrow B) \vee (B \rightarrow A),$$

with $Reg(A)$ defined as $(\top \rightarrow A) \leftrightarrow A$. (One could verify this directly by taking a standard Łukasiewicz derivation of it[18] and using the regularity assumptions as needed.)

Of course at the moment I'm working in a 3-valued revisionist framework, where relatively few such regularity assumptions are valid; so it may be unclear why I've appealed to the Łukasiewicz [0,1]-valued logic. The answer is that nothing in the derivation system exploits the limitation to three values, as we'll soon see;[19] and there are strong reasons to avoid such a limitation.

§3. Values in [0,1].

In the 3-valued revision theory, there are a great many rather simple sentences B that don't get constant values: equivalently, neither $(\top \rightarrow B) \rightarrow B$ nor $B \rightarrow (\top \rightarrow B)$ get value $\mathbf{1}$. This leads to complexities in reasoning with them. One example is the sentence Q_2 that says of itself that if it's true it isn't true. On the LimInf rule in 3-valued revision theory, it eventually goes in ω-cycles of form $< 0, 1, 0, 1, 0, 1, \ldots >$. But it's natural to try to avoid the cycling. We could do this by going to a 7-valued revision theory with values $\frac{k}{6}$ where $0 \leq k \leq 6$: then we could (with

18 For example, [17, pp. 419–421].

19 Łukasiewicz 3-valued logic is Łukasiewicz continuum-valued plus one extra (and rather unnatural) axiom that limits the values to three; and I haven't included any analog of that axiom in my derivational system.

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a suitable starting hypothesis) give Q_2 the value $\frac{2}{3}$ at each stage.[20] But there are other simple sentences that would still cycle, e.g., a sentence that asserts that if it's true then so is the Liar sentence: to avoid cycling, we'd need a value space with the value $\frac{3}{4}$.

An obvious idea for avoiding cycling to the extent possible is to do a revision theory over the unit interval [0,1]. This requires that we generalize the Kripkean micro-constructions, to allow them to take arbitrary values in [0,1], but doing so is easy. If we use minimal Kripke fixed points then at the initial Kripke stage we still assign each $True(t)$ value $\frac{1}{2}$ (relative to an assignment to its free variables) when (relative to that assignment) t denotes (the Gödel number of) a sentence A; at a successor stage $\sigma + 1$, we assign $True(t)$ the value that A gets (via the obvious generalization of the Kleene rules to [0,1]) at stage σ; and at a limit ordinal λ, we assign to $True(t)$ the limit of the values at prior stages. The limit exists, because it's easily shown that any change of values in the construction is always away from $\frac{1}{2}$ (in the same direction).[21]

We adapt this to the revision theory by allowing *the assignments h to conditionals* to take on arbitrary values in [0,1]; then the Kripke construction over h yields a [0,1]-valued assignment v_h to all sentences in the language.

We now need to generalize the revision-theoretic macro-construction. Let's defer the issue of the starting hypothesis for the moment. The obvious generalization of the successor stages is:

(Suc) $h_{\beta+1}(A \to B) = L_\beta(A, B)$,

where $L_\beta(A, B)$ is 1 if $|A|_\beta \leq |B|_\beta$, $1 - (|A|_\beta - |B|_\beta)$ otherwise. (This is the obvious adaptation of Łukasiewicz continuum-valued semantics to revision theory.) And the obvious generalization of the LimInf rule for limit stages is that $h_\lambda(A \to B)$ is the limit inferior of the $h_\gamma(A \to B)$ as γ approaches λ. In other words,

(**LimInf** $[0,1]$) For limit λ, $h_\lambda(A \to B) \geq r$ iff $(\forall \varepsilon > 0)(\exists \beta < \lambda)(\forall \gamma \in [\beta, \lambda))(h_\gamma(A \to B) \geq r - \varepsilon)$.[22]

I tentatively propose that we use (Suc) and ($\text{LimInf}_{[0,1]}$), which taken together say in effect that for any $\alpha > 0$, $h_\alpha(A \to B)$ is $liminf_\alpha\{L_\gamma(A, B)\}$ (where that means the liminf of the sequence of L_γ values prior to α). On this proposal, the derivational system in §2 is still sound, as shown in Appendix A.

But for this proposal to serve its desired purpose of eliminating unnecessary cycling, we need care in selecting the starting valuation h_0. For instance, if h_0 assigns every conditional a value in $\{0, \frac{1}{2}, 1\}$, then no sentence will ever get a value outside of $\{0, \frac{1}{2}, 1\}$. If it assigns every conditional the value $\frac{p}{q}$ for positive integers p and q with $p < q$, then no sentence will ever get a value that isn't a multiple of $\frac{1}{q}$ if q is even, or $\frac{1}{2q}$

[20] The 4-valued logic with values $\{0, \frac{1}{3}, \frac{2}{3}, 1\}$ wouldn't work, because we need the value $\frac{1}{2}$ to accommodate Liar sentences.

[21] For a more careful formulation, see [8].

[22] The simpler

(?) $h_\lambda(A \to B) \geq r$ iff $(\exists \beta < \lambda)(\forall \gamma \in [\beta, \lambda))(h_\gamma(A \to B) \geq r)$

is incoherent in the standard real numbers: for instance, if $h_\gamma(A \to B)$ are forever increasing prior to λ, then if r is their limit, (?) would say that the value at λ is $\geq$ each real less than r, but not $\geq r$.

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if q is odd. This makes it unevident what to choose as the starting valuation, especially given that we need the starting valuation to be transparent.

For this reason, [8] proposed a modification of (Suc), that eliminated cycling for a huge variety of sentences even with simple h_0 like those that assign 0 (or $\frac{1}{2}$) to all conditionals. (That paper used a generalization ($\text{SYM}_{[0,1]}$) of (SYM_3), rather than ($\text{LimInf}_{[0,1]}$), but the modification would work as well with ($\text{LimInf}_{[0,1]}$).) The modification was called "slow corrections". Instead of taking $h_{\beta+1}(A \to B)$ to be $L_\beta(A, B)$, as (Suc) does, the "slow correction" revision rule takes it to be the average of this and its previous value:

(SC) $h_{\beta+1}(A \to B) = \frac{1}{2}[h_\beta(A \to B) + L_\beta(A, B)]$.

Under (SC) (and with either (SYM) or (LimInf)), a great many sentences like Q_2 reach fixed point values other than 0, $\frac{1}{2}$ and 1 (and do so even by stage ω).

As we'll see, there are some rather arcane sentences that can't reach fixed points in any revision construction. (A typical example is the Restall sentence R, constructed by diagonalization to be equivalent to the claim $\exists n(True(\langle R \to^n \bot \rangle))$, where $R \to^n \bot$ is the result of prefixing $\bot$ with '$R \to$' n times. I think all other examples will involve a similar quantification over depth of embedding in the scope of $\to$'s, combined with truth, though I'm not clear just how to make this precise.) In dealing with such arcane sentences, (SC) is much less convenient to work with than (Suc), and leads to somewhat messier laws. So *it would be better if we could stick to (Suc) by using a more complicated* h_0.

One way to do that would be to use a "pre-construction" based on (SC), for the sole purpose of arriving at a starting valuation for the "real construction" based on (Suc). The starting valuation for the preconstruction could be a simple one, e.g., assigning 0 to all conditionals; we could use the assignment of values to conditionals at a reflection ordinal of the pre-construction as the starting value of the real construction with the simpler rule for successors.

It would be nice if this procedure led, at the very least, to all "quantifier-independent" sentences having constant value. (The importance of doing so is discussed at the end of this Section.) More precisely, call a sentence quantifier-*dependent* if it is in the smallest set X that contains all sentences with quantifiers and also contains all sentences with $True(t)$ where t denotes a member of X. Call it quantifier-independent otherwise.[23] If all quantifier-independent sentences have constant value in the slow-correction pre-construction, then of course they will retain that value in a real construction that uses that pre-construction for its starting valuation; but I don't know whether the antecedent is true.

In case it isn't true, there's an alternative way to specify a starting valuation for the revision construction based on (Suc), that does lead to constant values for all quantifier-independent sentences (provided that we make a slight alteration in the Kripkean micro-constructions). It uses the Brouwer fixed point theorem. That theorem has previously been used to show[24] that *in a language without quantifiers* (but which

[23] Defining "Z-independent" for other notions Z is usually more complicated because of the need to exclude occurrences of '$True(x)$', where the range of 'x' includes sentences that are Z-dependent. But a *quantifier-free* sentence can't contain '$True(x)$' for any variable x.

[24] This is spelled out in [6, pp. 97–99], but I'm sure the point was well known long before. [18] used the theorem in a similar way in connection with the axiom of comprehension in set

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may have a way to construct self-referential sentences without them),[25] any ground model (that's standard for syntax) can be extended to a model for the language of '*True*' satisfying Łukasiewicz sentential logic.[26] We can adapt that result here, by temporarily treating as atomic all formulas whose main connective is a quantifier, and giving them uninteresting values—say assigning them all value 0 (relative to any assignment of objects to their free variables), or assigning all universals 0 and all existentials 1. Let g be any function assigning values in $[0, 1]$ to every sentence. Let M_g be the Łukasiewicz sentential model in which (i) values of atomic sentences with predicates other than '*True*' are given in the ground model; (ii) sentences whose main connectives are quantifiers are given their chosen uninteresting values; (iii) sentences of form '$True(t)$' where t doesn't denote the Gödel number of a sentence are assigned value 0; and (iv) for any sentence x, the value $g(x)$ is assigned to any sentence of form '$True(t)$' where t denotes the Gödel number of x. Let F be the function that, applied to any g, yields the Łukasiewicz valuation function on M_g: that is, yields the function that assigns to each sentence its value in M_g under those sentential rules (hence assigns the uninteresting values to quantified sentences). It's evident from those sentential rules that this function is continuous on the product space $[0, 1]^{SENT}$. By the Brouwer theorem, F has fixed points: there are assignments g such that $F(g) = g$. Pick any such g (there will be many),[27] and let h_0 assign to each conditional what this g does.

We'd like that all subsequent h_α will agree with h_0 on all quantifier-independent conditionals (or equivalently, that the valuations they generate will agree with that generated by h_0 on all quantifier-independent sentences). In that case, all such sentences clearly get constant value in the macro-construction. But to get such agreement, we need to move beyond *minimal* Kripke fixed points.[28] For instance, if the Brouwer fixed point function g assigned a value other than $\frac{1}{2}$ to a Truth-Teller sentence U (taken as quantifier-independent: see n. 25), we'd need a non-minimal fixed point that gives it the same value: otherwise h_0 and h_1 would disagree on $\top \to U$, and there's no reason to think that the macro-construction would eventually settle down on all quantifier-independent sentences. But the fix is obvious: in the initial stage of the micro-constructions, we still assign $True(t)$ value $\frac{1}{2}$ if t denotes a quantifier-dependent

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theory, and [5] extends his results, in ways that could doubtless be used to extend the result as stated in the text to *partial* results in the language *with* quantifiers, and thus generalize the claim about quantifier-independent sentences made later in this paragraph.

25 For instance using direct naming, or the diagonalization operator of [19].

26 There can be no such result for Łukasiewicz quantifier logic: [15] pointed out (using the aforementioned Restall sentence) that adding the truth schema to Łukasiewicz quantifier logic gives an ω-inconsistent theory; and [12] showed that this becomes a flat out inconsistency if we beef up the truth schema to include the compositional principle for conditionals, that is, $\forall x \forall y[True(cond(x, y)) \leftrightarrow (True(x) \to True(y))]$. In a revision construction with (Suc) and (LimInf) then (whatever the starting valuation) the Restall sentence eventually gets value 0 at all limit ordinals and 1 at all successor ordinals. (This pattern is impossible for conditionals, but not for existential quantifications of conditionals.)

27 Most obviously, a fixed point g can assign any value to a Truth-teller, and any value $\geq \frac{1}{2}$ to a sentence that says of itself that if it's true then so is the Truth-teller. For an example with a very different flavor, suppose A is equivalent to $B \to \neg A$, B to $A \to B$. Some fixed points have $|A| = 1$, $|B| = 0$; others have $|A| = \frac{1}{2}$, $|B| = 1$; these are the only possibilities. (The slow correction pre-constructions that starts from giving all conditionals the same value lead to the latter.)

28 I'm grateful to Quinton Wood for bringing this issue to my attention.

sentence; but if t denotes a quantifier-independent sentence x, $True(t)$ is assigned $g(x)$ for the chosen Brouwerian function g. If we do this, all stages of the macro-construction will agree with g on quantifier-independent sentences, and so they will get constant values.

Invoking the Brouwer theorem in this way gives little information as to what values a fixed point might assign. To the extent that the pre-construction approach works, it can be used to actually determine fixed point values for sentences; so the question of how widely it determines constant values is worth investigating.

But one way or another, we have a revision theory based on (Suc) and (LimInf) that leads to a wide range of constant values. So the [0,1]-based revision theory validates the soundness of not only the derivation system of §2, but also

Regularity Schema $(\top \to A) \leftrightarrow A$, for quantifier-independent A.

(This could doubtless be extended beyond the quantifier-independent, though I don't know how far beyond.) And to repeat, adding this to the derivation system is enough to derive Łukasiewicz continuum-valued logic as restricted to such sentences.

§4. Quantifiers. I will now expand the derivation system (that of §2 plus the Regularity Schema) by adding quantifier laws. I'll take $\exists$ as primitive. As we'll see, there's an issue about how best to represent the English 'all', so I'll define more than one candidate. I'll use $\forall^0$ for the familiar $\neg\exists\neg$, and define $\forall^1 xA$ as $\$\neg\exists x\neg Ax$. (Recall that $\$A$ is $A \wedge (\top \to A)$.) So $|\forall^1 xA|_\alpha \geq r$ iff $(\forall\varepsilon > 0)(\exists\beta < \alpha)(\forall\gamma \in [\beta, \alpha])(\forall c \in |M|)(|A(c)|_\gamma \geq r - \varepsilon)$. (Note the use of a fully closed interval $[\beta, \alpha]$.) Obviously this valuation rule is very much dependent on using the (LimInf) rule: under (SYM), \$ is a very unnatural operator, and so is $\forall^1$. But in the derivation system to be developed for (LimInf), $\forall^1$ will actually play a more salient role than $\forall^0$.

Even without the quantifier rules, we can see that $\vdash \forall^1 xA \to \forall^0 xAx$ (by definition and $\wedge$-elimination) and $\forall^0 xAx \vdash \forall^1 xAx$ (by \$-Introduction, based on R3a).

A third candidate for 'all' is a quantifier I'll call $\forall^\omega$. To define it, I first use $\forall^0$ plus the truth predicate to define $\$^\omega A$ as something like an infinite conjunction of A, $\top \to A$, $\top \to (\top \to A)$, etc. More exactly, $\$^\omega A =_{df} (\forall^0 n \in N)[True(\langle \dot{\top} \dot{\to}^n A\rangle)]$; equivalently, $(\forall^0 n \in N)[True(\langle \dot{\$}^n A\rangle)]$, where $\$^0 A$ is just A. (As remarked earlier, $\to^n$ can be explicitly defined by recursion as a function of variable n, so this definition of $\$^\omega$ makes sense.) Semantically, we have

$$|\$^\omega A|_\alpha \geq r \text{ iff } (\forall\varepsilon > 0)(\exists\beta)[\beta + \omega \leq \alpha \wedge (\forall\gamma \in [\beta, \alpha])(|A|_\gamma \geq r - \varepsilon)].$$

(Note again the use of a fully closed interval $[\beta, \alpha]$, due to the inclusion of n=0 in the definition.)

It's obvious from this semantics that the inference from A to $\$^\omega A$ is valid, as is the sentence $\$^\omega A \to \$^{\boldsymbol{n}} A$ for any specific n; though we'll need quantifier rules before we can discuss their derivability in a finitary system. Similarly for $\$^\omega A \to \$^\omega(\top \to A)$; its validity reflects the fact that $1 + \omega = \omega$. The fact that $\omega + 1 \neq \omega$ suggests, correctly, that $\$(\$^\omega A)$ is a genuine strengthening of $\$^\omega A$; it might be called $\$^{\omega+1} A$.[29]

29 Indeed, we could extend this to define a $\$^\alpha$ for every ordinal notation in the language. They'd be strictly increasing in strength; no limit of them is definable in the language.

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I'll take $\forall^{\omega} xA$ to be $\$^{\omega}\neg\exists x\neg Ax$. Like $\forall^1$ (and unlike $\forall^0$), it will play a salient role in the quantifier rules. In the semantics,

$|\forall^{\omega} xA|_\alpha \geq r$ iff $(\forall\varepsilon > 0)(\exists\beta[\beta + \omega \leq \alpha \wedge (\forall\gamma \in [\beta, \alpha])(\forall c \in |M|)(|A(c)|_\gamma \geq r - \varepsilon)]$.

Of course $\forall^{\omega}$, $\forall^1$ and $\forall^0$ are all equivalent when applied to "ordinary" formulas A, those that can be assumed to be regular.

If you think it odd to have unrestricted quantifiers $\forall$ inequivalent to $\neg\exists\neg$, it may help to observe that we already have such an inequivalence for restricted quantification: the restricted universal is defined via $\rightarrow$ and the restricted existential via $\wedge$. (Also, many other non-classical logics, like intuitionist, have $\forall$ stronger than $\neg\exists\neg$, as is the case with $\forall^1$ and $\forall^{\omega}$.)

We use three axioms and a meta-rule, and strengthen the truth rule:

Q1 ($\exists$-Intro) $B(y) \rightarrow \exists x B(x)$ when no free occurrence of x in $B(x)$ is in the scope of an $\exists y$.

Q2 (Central Quantifier Rule) $\forall^1 x(Bx \rightarrow C) \rightarrow (\exists x Bx \rightarrow C)$ when x not free in C.

Q3 (InfDist) $C \wedge \exists x Bx \rightarrow \exists x(C \wedge Bx)$.

Metarule (UnivGen): If $A_1, \ldots, A_n \vdash Bx$ and x isn't free in any of $A_1, \ldots, A_n$ then $A_1, \ldots, A_n \vdash \forall^{\omega} xBx$.

We generalize the truth schema, and add composition principles. In these cases, it won't matter which version of the universal quantifier to use (as we'll see), so I write them without a superscript.

Generalized Truth Schema $(\forall t_1, \ldots t_n)[TERM(t_1) \wedge \ldots \wedge TERM(t_n) \supset [True(\langle A(t_1, \ldots, t_n)\rangle) \leftrightarrow A(den(t_1), \ldots, den(t_n))]]$

Generalized Composition Schema for $\dot{\rightarrow}$: $(\forall x)(\forall y)[QSENT(x) \wedge QSENT(y) \supset [True(x \dot{\rightarrow} y) \leftrightarrow (True(x) \rightarrow True(y))]]$,

with analogous composition rules for other connectives. ('$QSENT$' means 'quasi-sentence': see the text to which n. 8 is attached.) Given the compositional rules, it's obviously enough to postulate the generalized truth schema for atomic A.

Here and in §2, I've written most of the axioms and rules schematically, but now that we have quantifiers and transparent truth, we should really use generalized versions. For instance, we should replace S1 with the generalization $\forall x[QSENT(x) \supset True(x \dot{\rightarrow} x)]$. (Again, it makes little difference which version of the quantifier we use.) From this, we can easily derive specific instances $A \rightarrow A$ using Q1 and the truth rules (and a bit more if we use $\forall^1$ or $\forall^{\omega}$). We can do this for each of the axiom schemas, except that we need a schema in the vicinity of Q1 as well as the generalizations so as to be able to derive the other schemas from their corresponding generalizations. We also should generalize the rules, e.g., replacing Modus Ponens by $True(x), True(x \dot{\rightarrow} y) \vdash True(y)$, and supplementing UnivGen as written by "If $True(y_1), \ldots, True(y_n) \vdash True(z)$ then for every variable x not free in any member of the y_i, $True((\dot{\forall}\dot{x})\dot{\$}^{\omega} z)$". If this is done, the proofs of the schematic theorems and schematic derived rules are easily transformed to proofs of the corresponding generalizations. For ease of reading, I will continue to write in schematic terms, except in one place in Appendix B, where the generalized form is needed.

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Q2 could be replaced by the following pair:

Q2a. $\forall^1 x(Bx \to Cx) \to (\exists x Bx \to \exists x Cx)$.
Q2b. $\exists x B \to B$ when x isn't free in B.

(Q1 entails the converse of Q2b.)

In Łukasiewicz logic, Q3 is derivable from other quantifier laws, but that seems unlikely here.[30]

Obviously $\vdash \$^\omega A \to A$, by definition of $\$^\omega$ plus the truth rules. More generally,

$\$^\omega$-Elim For any n, $\vdash \$^\omega A \to \$^{\boldsymbol{n}} A$.

(For a special case of the previous is $\vdash \$^\omega \$^{\boldsymbol{n}} A \to \$^{\boldsymbol{n}} A$, and it's not hard to prove the equivalence of $\$^\omega \$^{\boldsymbol{n}} A$ to $\$^\omega A$; see Appendix B.)

And using UnivGen, we get a rule form of the converse:

$\$^\omega$-Introduction $A \vdash \$^\omega A$.

Proof. $A \vdash A$, so by UnivGen $A \vdash \forall^\omega x A$ for any x not free in A, i.e., $A \vdash \$^\omega \forall^0 x A$; and using Q2b (plus contraposition, etc.) we get $A \vdash \$^\omega A$. □

In Q2, on the other hand, the use of $\forall^1$ is essential: the analog with $\forall^\omega$ is weaker than we want, and the analog with $\forall^0$ isn't sound. For suppose $B(n)$ is $R \to^n \bot$, where R is the Restall sentence. (More exactly: $B(n)$ is "either n isn't a natural number, or n is a natural number and the result of prefixing $\bot$ with '$R \to$' n times is true".) Then for each n, and each limit ordinal λ and natural numbers k, $|B(n)|_{\lambda+k}$ is 1 iff $n \geq k \geq 1$, 0 otherwise. Then $|\exists n B(n)|_{\lambda+k}$ is 1 for all $k \geq 1$ (since $|B(k)|_{\lambda+k}$ is 1 when $k \neq 0$), and so $|\exists n B(n) \to \bot|_{\lambda+k}$ is 0 for all $k \geq 1$, and so $|\exists n B(n) \to \bot|_{\lambda+\omega}$ is 0. But for each n, $|B(n) \to \bot|_{\lambda+k+1}$ is 1 if $|B(n) \to \bot|_{\lambda+k} = 0$, that is, if $k > n$; so for each n, $|B(n) \to \bot|_{\lambda+\omega}$ is 1, so $|\forall^0 n(B(n) \to \bot)|_{\lambda+\omega}$ is 1.

That Q2 as stated is sound in the semantics is proved in Appendix A. (It wouldn't be sound in a semantics based on (SYM), since it implies the K-law for \$.) The soundness of the others is obvious.

Of course we don't need $\forall^1$ for *the rule forms* of Q2 and Q2a: $\forall^0 x(Bx \to C) \vdash \exists x Bx \to C$ follows from Q2, because of \$-introduction, and similarly for Q2a.

A crucial feature of $\forall^\omega$ (not shared by $\forall^1$ or $\forall^0$) is

($\mathbf{RQ}_\to$) $\forall^\omega x(\neg Ax \vee Bx) \to \forall^\omega x(Ax \to Bx)$.

That this is not only valid, but provable in our system, is shown in Appendix B. I'll discuss its ramifications in §5.

It's also useful to introduce n-ary quantifiers, that are candidates for understanding "for all $x_1,\ldots,x_n$, $A(x_1,\ldots,x_n)$". I take $(\forall^0 x_1,\ldots,x_n)$ to be defined as a string of n unary $\forall^0$s: $(\forall^0 x_1,\ldots,x_n)A(x_1,\ldots,x_n)$ is just $(\forall^0 x_1)(\forall^0 x_2)\ldots(\forall^0 x_n)A(x_1,x_2,\ldots,x_n)$.

30 In Łukasiewicz, the contraposed form $\forall^0 x(C \vee Bx) \to C \vee \forall^0 x Bx$ is derived by deriving $\forall^0 x[(C \to Bx) \to Bx] \to [(C \to \forall^0 x Bx) \to \forall^0 x Bx]$, and using equivalence of each side to disjunction. But the equivalence of each side to disjunction fails here, in the absence of regularity assumptions.

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(Similarly for $(\exists x_1, \ldots, x_n)$.) But I take $(\forall^1 x_1, \ldots, x_n)$ to be defined as $\$(\forall^0 x_1, \ldots, x_n)$: that is, $(\forall^1 x_1)(\forall^0 x_2)\ldots(\forall^0 x_n)A(x_1, x_2, \ldots, x_n)$, which for $n > 1$ is weaker than a string of $\forall^1$s. Strings of more than one $\forall^1$s will turn out to be rather less useful. Similarly, $(\forall^\omega x_1, \ldots, x_n)$ is to be defined as $\$^\omega(\forall^0 x_1, \ldots, x_n)$: that is, $(\forall^\omega x_1)(\forall^0 x_2)\ldots(\forall^0 x_n)A(x_1, x_2, \ldots, x_n)$.

Here are some salient facts about what's derivable in the system, even without using the Regularity Schema. (More detail, with proofs, in Appendix B.) And about laws you might expect but that aren't sound.

Vacuous quantification: If x isn't free in C, then $\exists x C$ and $\forall^0 x C$ are each equivalent to C. But $\forall^1 x C$ is equivalent to $\$C$, and $\forall^\omega x C$ is equivalent to $\$^\omega C$. When $n > 1$ and x_1 isn't free in A, $(\forall^1 x_1, \ldots, x_n)A$ is equivalent to $(\forall^1 x_2, \ldots, x_n)A$ (and analogously for x_i when $i > 1$), since the one occurrence of \$ remains when the x_1 is deleted; analogously for $\forall^\omega$.

Change of bound variables: Works just as expected.

$\$\forall^0$ vs. $\forall^0\$$: We get the analog of the converse Barcan formula for $\forall^0$, that is, $\vdash \$\forall^0 x A x \to \forall^0 x \$ A x$; but only the rule form $\forall^0 x \$ A x \vdash \$\forall^0 x A x$ of the Barcan formula.[31]

$\$\forall^1$ vs. $\forall^1\$$, and $\$\forall^\omega$ vs. $\forall^\omega\$$: Analogously, the converse Barcan holds with $\forall^1$ and $\forall^\omega$, but the Barcan only in rule form.

Commutation of quantifiers: We have $\exists x \exists y A x y \to \exists y \exists x A x y$, and similarly when all the $\exists$'s are replaced by $\forall^0$. With $\forall^1$ and $\forall^\omega$ we have only the rule forms, which is part of the reason why the n-ary quantifiers $(\forall^1 x_1, \ldots, x_n)$ and $(\forall^\omega x_1, \ldots, x_n)$ are more useful than strings of the corresponding unary quantifiers. With the n-ary quantifiers, permuting the variables in the quantifier is unproblematic.

Universal analogs of Q2 (and Q2a): We have $\forall^1 x(C \to Bx) \to (C \to \forall^0 x Bx)$ (and $\forall^1 x(Ax \to Bx) \to (\forall^0 x Ax \to \forall^0 x Bx)$). (We can't replace $\forall^0$ in the consequent with $\forall^1$: see Appendix B.) Similarly $\forall^\omega x(C \to Bx) \to (C \to \forall^n x Bx)$, for any n (where $\forall^n$ is $\$^n\forall^0$); and $\$\forall^\omega x(C \to Bx) \to (C \to \forall^\omega x Bx)$.

$\exists$-Importation: If C doesn't contain x free, $\vdash \exists x(C \to Bx) \to (C \to \exists x Bx)$.

Failure of $\exists$-Exportation Rule: The rule $C \to \exists x Bx \vdash \exists x(C \to Bx)$ (for C not containing free x) is unsound in the semantics.

That $\exists$-Exportation fails is a good thing! For (as noted in [1]) its validity in Łukasiewicz logic is the source of the inconsistency mentioned in n. 26. (Taking C to be the Restall sentence R and $B(n)$ to be $R \to^n \bot$, $\exists$ –Exportation (together with transparency)

31 You can get counterexamples to the conditional form of the Barcan for $\forall^0$ (that is, of $\forall^0 x \$ A x \to \$\forall^0 x A x$) even with just values 1 and 0. E.g., with domain the natural numbers, suppose that $B(n)$ is $\neg(R \to^n \bot)$, with R the Restall sentence, so that for each n, $|B(n)|_{\lambda+k}$ is 1 iff either $k = 0$ or $k > n$, and 0 otherwise. Then $|B(n)|_{\lambda+\omega}$ is 1 also, so each $B(n)$ goes in cycles of length ω. $|\forall^0 n B(n)|_{\lambda+k}$ is 0 except when k=0, so $|\$\forall^0 n B(n)|_{\lambda+k}$ is 0 for all k. But $|\top \to B(n)|_\lambda$ $(= |\top \to B(n)|_{\lambda+\omega})$ is 1 since that's the limit of the$|\top \to B(n)|_{\lambda+k}$, and $|\top \to B(n)|_{\lambda+1}$ is 1 since $|B(n)|_\lambda$ is 1; so $|\$B(n)|_{\lambda+1}$ is 1 for all n, so $|\forall^0 n \$ B(n)|_{\lambda+1} = 1$. So at $\lambda + 1$, $\forall^0 n \$ B(n)$ has higher value than $\$\forall^0 n B(n)$.Variants of this also give counterexamples to Barcan for $\forall^1$ and for $\forall^\omega$. For $\forall^1$, let $B(n)$ be $\top \to \neg(R \to^n \bot)$; for $\forall^\omega$, the counterexamples are a bit more complicated, involving sentences that have cycles of length ω^2.

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immediately leads to the proof of R; and with that and the compositional rule for $\rightarrow$, $\neg R$ is easily proved as well.)[32]

These give enough of the flavor of the system to enable us to move to restricted quantification.

§5. Restricted quantification. I mentioned early on that my concern in this paper was with a conditional $\rightarrow$ adequate to defining universal restricted quantification: "All A are B" was to be understood as '$\forall x(Ax \rightarrow Bx)$'. But we've seen at least three candidates for '$\forall$', so three candidates for "All A are B": we could take it as $\forall^0 x(Ax \rightarrow Bx)$, as $\forall^1 x(Ax \rightarrow Bx)$, or as $\forall^\omega x(Ax \rightarrow Bx)$. Similarly for multi-place restricted: we could take "All $x_1 \ldots, x_n$ that stand in relation A stand in relation B" as $(\forall^0 x_1 \ldots, x_n)[A(x_1 \ldots, x_n) \rightarrow B(x_1 \ldots, x_n)]$, or as $(\forall^1 x_1 \ldots, x_n)[A(x_1 \ldots, x_n) \rightarrow B(x_1 \ldots, x_n)]$, or as $(\forall^\omega x_1 \ldots, x_n)[A(x_1 \ldots, x_n) \rightarrow B(x_1 \ldots, x_n)]$. I'll use the notations $\forall^0 x_{Ax} Bx$, $\forall^1 x_{Ax} Bx$ and $\forall^\omega x_{Ax} Bx$ for the unary restricted quantifiers (and analogously for the n-ary).

I think there's a big advantage to the $\forall^\omega$ readings for both the restricted and unrestricted quantifiers: together these validate the obviously desirable law "If everything is either not-A or B then all A are B":

(RQ) $\forall x(\neg Ax \vee Bx) \rightarrow \forall x_{Ax} Bx$.

(Using an ordinal bigger than ω for both unrestricted and restricted (see n. 29) would have the same advantage, but there seems little point to that.)

I don't of course mean this as a claim about ordinary English: ordinary speakers don't have coherent ways of dealing with truth-theoretic paradoxes, and certainly not of the arcane paradoxes involving Restall-like sentences; and only for such sentences is there any real distinction between $\forall^0$, $\forall^1$ and $\forall^\omega$. What I mean, rather, is that by and large, ordinary reasoning is best preserved even for arcanely paradoxical sentences if we understand both the unrestricted and the restricted 'all' as $\forall^\omega$.

If we were to understand $\forall$ in (RQ) as either $\forall^0$ or $\forall^1$ in both occurrences, (RQ) wouldn't be valid. We could get it by an *ad hoc* adjustment to the definition of restricted quantification, e.g., taking $\forall x_{Ax} Bx$ not as $\forall x(Ax \rightarrow Bx)$ but as $\forall x[(Ax \rightarrow Bx) \vee (\neg Ax \vee Bx)]$; but while that would deliver (RQ) it would fail to deliver related laws, such as

(RQ*) $\forall x(\neg Ax \vee Bx) \rightarrow \forall x_{Ax}(Ax \wedge Bx)$.

(For instance, suppose that for any object c, $|Ac|_\alpha$ is $\frac{1}{2}$ for all α and $|Bc|_\alpha$ is 1 for odd α and 0 for even α. On the proposed *ad hoc* reading, (RQ$*$) then amounts to $(\neg Ac \vee Bc) \rightarrow [(Ac \rightarrow Ac \wedge Bc) \vee \neg Ac \vee (Ac \wedge Bc)]$. But if $\alpha + 1$ is odd, then $|\neg Ac \vee Bc|_{\alpha+1}$ is 1 while $|Ac \rightarrow Ac \wedge Bc|_{\alpha+1}$ and $|\neg Ac|_{\alpha+1}$ are $\frac{1}{2}$ and $|Ac \wedge Bc|_{\alpha+1}$ is 0; so this instance of (RQ$*$) will get value $\frac{1}{2}$ at $\alpha + 2$.) To get laws (RQ) and (RQ$*$) without going to infinite iterations of \$, we need one more \$ on the left than on the right: e.g., $\forall^1$ on the left and $\forall^0$ on the right. If we want a single unrestricted '$\forall$',

[32] But given regularity assumptions for C and Bx, we get even the conditional form $(C \rightarrow \exists x Bx) \rightarrow \exists x(C \rightarrow Bx)$.

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and define restricted quantification in terms of it, we need to use $\forall^{\omega}$ (or $\forall^{\alpha}$ for some bigger α).

Might we do better by taking the restricted quantifier as primitive, and defining the unrestricted in terms of it ($\forall x Bx$ as $\forall x_{x=x} Bx$)? We'd need valuation rules for the restricted quantifier, but the obvious idea would mimic the revision construction here, evaluating $\forall x_{Ax} Bx$ at an ordinal directly in terms of the values of Ac and Bc at prior ordinals. But in that case, explaining $\forall x Bx$ as $\forall x_{x=x} Bx$ would be unacceptable: if the Bc's all have high values at ordinals prior to α, $\forall x Bx$ would automatically get a high value at α, even if some or all of its instances Bc have low values at α. This can be so even when α is past the critical ordinal; so universal instantiation would fail.

Of course we could get the equivalence of "everything is B" to "every self-identical thing is B" by reading the former as $\forall^1 x Bx$ and the latter as $\forall^0 x_{Ax} Bx$ (or more generally, the former as $\forall^{n+1} x Bx$ and the latter as $\forall^n x_{Ax} Bx$, for finite n). But that seems rather unnatural, which is why I prefer going to $\forall^{\omega}$.

§6. Final remarks. I expect that the derivation system presented in §2 and §4 could usefully be expanded (even beyond the added regularity principle suggested at the end of §3; and I expect that that principle too could usefully be extended to a wider class of sentences). A system that is fully complete on this semantics is not in the cards.[33] But it would certainly be desirable to strengthen the one here; this paper is only a first step.

But I think it is an important step in a number of ways. Probably most centrally, it shows that we can have a theory of transparent truth that has a conditional obeying natural laws and where in particular we can have the law

(RQ) $\forall x(\neg Ax \vee Bx) \rightarrow \forall x_{Ax} Bx$,

reading $\forall x_{Ax} Bx$ as "All A are B", and taking it to abbreviate $\forall x(Ax \rightarrow Bx)$. Admittedly, we get (RQ) only by treating the universal quantifier in an unusual way, not as $\neg\exists\neg$; but non-classical theories already pretty much required us to treat the universal *restricted* quantifier as different from $\neg\exists x_{Ax} \neg Bx$ (where $\exists x_{Ax} Cx$ is the restricted existential quantifier $\exists x(Ax \wedge Cx)$), so treating the universal *unrestricted* quantifier as different from $\neg\exists\neg$ doesn't seem totally outlandish.

Previously, it was reasonable to worry that a theory of transparent truth with (RQ) might be unobtainable (except in trivial ways, like adding '$\neg A \vee B$' as a disjunct, that don't deliver related laws like $\forall x(\neg Ax \vee Bx) \rightarrow \forall x(Ax \rightarrow Ax \wedge Bx)$.) Brady's pioneering papers on conditionals in transparent truth theories, and his book [4], don't validate even the Weakening *Rule*

(WR) $\forall x Bx \vdash \forall x(Ax \rightarrow Bx)$

(and though he didn't consider varying the semantics of the quantifier, it is pretty clear that there is no remotely acceptable treatment of quantifiers that would give this rule

33 (i) It's well known that the set of sentences valid in Łukasiewicz continuum-valued quantifier semantics is not recursively enumerable, and this system reduces to that on sufficient regularity assumptions. (ii) Even in the sentential case, the complexity is too great, as observed in n. 17.

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given his treatment of conditionals). In [2] Bacon modifies Brady's construction, and says that with his modification, we get not only that rule but a conditional form of it:

(WC) $\vdash \forall x Bx \rightarrow \forall x(Ax \rightarrow Bx)$.

But in the main body of his paper (all but the last section), Bacon gets (WC) only in a very limited language, with no proper negation operator—so for instance, it has no means to express ordinary Liar sentences.[34] The final section of his paper does contain interesting though quite undeveloped suggestions about how to extend his account to a language with a real negation, with a semantics that uses what is essentially the "Routley star" from relevance logic. But though this delivers (WC), it does not deliver (RQ) (which doesn't follow from (WC) in its logic since its conditional is non-contraposable). It doesn't deliver even the rule

(WCR) $\forall x \neg Ax \vdash \forall x(Ax \rightarrow Bx)$;

it takes it as consistent that there are no A's and yet not all A's are B's (e.g., in his preferred version, when Bx is $x \neq x$, and Ax is $x = x \wedge Q_2$, with Q_2 as in n. 14).

In earlier work, e.g., [7], I made a different sort of attempt to get a law "If everything is either not A or B then all A are B": I read it not as RQ but rather as

(RQ*) $\forall x(\neg Ax \vee Bx) \rhd \forall x(Ax \rightarrow Bx)$,

where $\rhd$ was a different sort of conditional from $\rightarrow$: $\rightarrow$ is a restricted quantifier conditional, $\rhd$ is an ordinary English conditional (perhaps something roughly like the epistemic conditionals in the tradition of Stalnaker 1968). But, while I don't reject the idea that some pre-theoretic laws about conditionals depend on reading some 'if...then's as ordinary English conditionals, the logic that I managed to get by these means was rather weak.

In this paper, I think I've achieved much more satisfactory results, and done so without need of a separate conditional $\rhd$. Part of this is due to my suggestion of a different reading for 'all'. But that wouldn't have led to useful results in a theory based on the revision-theoretic semantics in my earlier papers, because those papers made a poor choice for how the revision procedure behaved at limit ordinals. The earlier part of the present paper shows, I think, that even independent of quantification, the new limit rule (LimInf) is far preferable to the old one; and the treatment of quantifiers in later sections of this paper wouldn't have made any sense on the old limit rule.

I remarked at the beginning of the paper that we have a choice in how to deal with the paradoxes: we can keep the logic classical and complicate the truth theory, or keep the truth theory simple and complicate the logic. We can't know which is the better package of logic plus truth theory until we have developed both packages as best we can. The present paper is part of an ongoing effort to do this, focusing especially on the non-classical side. While just a first step, I think that by getting a closer approximation to classical laws, especially laws of restricted quantification, it

[34] Bacon notes that it does have a hierarchy of pseudo-negation operators $\neg_n A =_{df} A \rightarrow^n \perp$ for $n \geq 1$. But they turn out to be *very* unlike real negation: e.g., for any A, $\neg_n \neg_n A$ leads to contradiction, and for any A and B, we can prove $(\neg_n \neg_n A) \leftrightarrow (\neg_n \neg_n B)$.

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significantly improves the competitive advantages of the combination of transparent truth and non-classical logic over classical logic with restrictions on transparency. A further step in improving the competitive advantage would involve the addition of predicates "well-behaved" (guaranteeing classical behavior) and "minimally well-behaved" (guaranteeing regularity), as in [9].[35] But the improvement achieved by adding such predicates is somewhat orthogonal to the issues discussed in this paper.

Appendix A: Soundness. The soundness of many of the axioms and rules, with respect to the [0,1]-based revision theory with LimInf, is obvious. I'll give proofs for the less obvious ones: S6, S7a (S7b is analogous), S7c, S8 (though it too is pretty obvious), S9, S11, R4a (R4b is analogous) and Q2. The system is also sound with respect to the revision theory based on three values (and on $2n+1$ values more generally), and the proofs are simpler there. Of course the assumption of regularity for quantifier-independent sentences requires the continuum-based version.

In many of the proofs below, I'll tacitly rely on a fact noted in the main text, that the value of a conditional $A \to B$ at a limit ordinal can be characterized either as the liminf of *the values of that conditional* at prior ordinals, or as the liminf of *the values* $L_\gamma(A, B)$ at prior ordinals (where these are the values obtained from $|A|_\gamma$ and $|B|_\gamma$ by the Łukasiewicz evaluation rule). The former "continuity formulation" is what was built into the LimInf rule as I originally stated it; the latter "general rule" has the advantage of holding for successor ordinals as well as for limits. An easy transfinite induction shows their equivalence for limit ordinals. The general rule can be written more explicitly as

(GR) For any $\alpha > 0$ and $r < 1$, $h_\alpha(A \to B) \geq 1 - r$ iff $(\forall\varepsilon > 0)(\exists\beta < \lambda)(\forall\gamma \in [\beta, \lambda))(|A|_\gamma - |B|_\gamma \leq r + \varepsilon)$.

The equivalence of the two formulations leads immediately to the following lemma.

LEMMA A1. $\top \to (A \to B)$ *has the same value as* $A \to B$ at limit ordinals. (*So* $|\top \to (A \to B)|_{\lambda+1} = |\top \to (A \to B)|_\lambda$.)

Proof. By the "general rule", $|\top \to (A \to B)|_\lambda$ is $\mathit{liminf}_\lambda\{L_\gamma(\top, A \to B)\}$, which is $\mathit{liminf}_\lambda\{|A \to B|_\gamma\}$, which by "continuity formulation" is $|A \to B|_\lambda$. □

LEMMA A2. *For any* r: *if for some* β, $|A \to B|_\alpha \geq r$ *for all successor ordinals greater than* β, *then for all final ordinals (successor or limit)*, $|A \to B|_\alpha \geq r$ (*where a final ordinal is an ordinal greater than or equal to the critical ordinal*).

Proof. The premise guarantees that for all final ordinals β, $|A|_\beta \leq |B|_\beta$ and hence $L_\beta(A, B) = 1$; so by the "general rule" for evaluating conditionals, the conclusion follows. □

Soundness of S6: $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$.

Proof. By Lemma A2, we need only establish this for successor ordinals $\geq \alpha_0 + 1$ (where α_0 is the critical ordinal), and that amounts to showing that for all final α, and all reals $r < 1$, if $|A \to B|_\alpha \geq 1 - r$ and $|A \to C|_\alpha \geq 1 - r$ then

[35] A somewhat analogous "well-behaved" predicate improves the classical theory as well, but the two predicates together mark a bigger improvement in the non-classical case.

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$|A \to B \wedge C|_\alpha \geq 1 - r$. By (GR), the premises tell us that for any $\varepsilon > 0$, there are β_1 and β_2 such that $(\forall\gamma \in [\beta_1, \alpha))(|A|_\gamma - |B|_\gamma \leq r + \varepsilon)$ and $(\forall\gamma \in [\beta_2, \alpha))$ $(|A|_\gamma - |C|_\gamma \leq r + \varepsilon)$. Letting β be their maximum, we get that $(\forall\gamma \in [\beta, \alpha))(|A|_\gamma - min\{|B|_\gamma, |C|_\gamma\} \leq r + \varepsilon)$ (since $min\{|B|_\gamma, |C|_\gamma\}$ is either $|B|_\gamma$ or $|C|_\gamma$). So $(\forall\gamma \in [\beta, \alpha))(|A|_\gamma - |B \wedge C|_\gamma) \leq r + \varepsilon$, and so by (GR) again, $|A \to B \wedge C|_\lambda \geq 1 - r$. □

Lemma A3. *For any* $\alpha > 0$, $|A \to B|_\alpha + |B \to C|_\alpha \leq 1 + |A \to C|_\alpha$. (*So if* $|B \to C|_\alpha = 1$, $|A \to B|_\alpha \leq |A \to C|_\alpha$.)

[We obviously need the '1+': e.g., if A implies B and B implies C, both terms on left are 1.]

Proof. Using the well-known inequality $liminf\{s_n + t_n\} \geq liminf\{s_n\} + liminf\{t_n\}$, we get that the left-hand side is $\leq liminf_\alpha\{L_\gamma(A, B) + L_\gamma(B, C)\}$. And $L_\gamma(A, B) + L_\gamma(B, C)$ is easily seen to be $\leq 1 + L_\gamma(A, C)$ (they're equal when neither summand is 1). So the LHS is $\leq liminf_\alpha\{L_\gamma(A, C)\}$, and that's the RHS. □

Lemma A4. *For any* $\alpha > 0$, $|A \sqcap C|_\alpha + |A \to B|_\alpha \leq 1 + |B \sqcap C|_\alpha$.

Proof. Writing D for $\neg C$, Lemma A3 gives $|A \to B|_\alpha + |B \to D|_\alpha \leq 1 + |A \to D|_\alpha$. But $|A \sqcap C|_\alpha$ is $1 - |A \to D|_\alpha$, and similarly for $|B \sqcap C|_\alpha$, so $|A \to B|_\alpha - |B \sqcap C|_\alpha \leq 1 - |A \sqcap C|_\alpha$, which is equivalent to the corollary. □

Soundness of S7a: $[\top \to (A \to B)] \to [(B \to C) \to (A \to C)]$.

Proof. Obviously $|A \to B|_\alpha \leq 1$, and so by Lemma A3, when $\alpha > 0$, $|A \to B|_\alpha \leq min\{1, 1 - |B \to C|_\alpha + |A \to C|_\alpha\}$. So (for any $\alpha > 0$) $|\top \to (A \to B)|_{\alpha+1} \leq |(B \to C) \to (A \to C)|_{\alpha+1}$. So by Lemma A2, we get that for any final α, $|\top \to (A \to B)|_\alpha \leq |(B \to C) \to (A \to C)|_\alpha$. So for any final successor ordinal, $|[\top \to (A \to B)] \to [(B \to C) \to (A \to C)]|_\alpha = 1$. So by Lemma A2 again, $|[\top \to (A \to B)] \to [(B \to C) \to (A \to C)]|_\alpha = 1$ for any final ordinal. □

Soundness of S7c: Analogous to proof for S7a, but using Lemma A4 instead of A3.

Soundness of S8: $[\top \to (\neg A \vee B)] \to [A \to B]$.

Proof. For any α, $|\top \to (\neg A \vee B)|_{\alpha+1}$ is $max\{1 - |A|_\alpha, |B|_\alpha\}$, which is $\leq min\{1, 1 - |A|_\alpha + |B|_\alpha\}$, which is $|A \to B|_{\alpha+1}$. Using Lemma A2, we get the result. □

Soundness of S9: $[(\top \sqcap A) \to (B \to C)] \to [(\top \to B) \to (A \to C)]$.

Proof. We need that for all final α, $|(\top \sqcap A) \to (B \to C)|_\alpha \leq |(\top \to B) \to (A \to C)|_\alpha$. By Lemma A2, it suffices to show this when α is a successor $\beta + 1$. In that case, what we need is that

$$(*)\ |\top \to B|_\beta - |A \to C|_\beta \leq max\{0, |\top \sqcap A|_\beta - |B \to C|_\beta\}.$$

We prove $(*)$ by induction on $\beta > 0$.

When β is itself a successor $\delta + 1$, the LHS of $(*)$ is $B_\delta - min\{1, 1 - A_\delta + C_\delta\}$, i.e., $max\{B_\delta - 1, B_\delta - 1 + A_\delta - C_\delta\}$ (which may be negative).

And the RHS is $max\{0, A_\delta - min\{1, 1 - B_\delta + C_\delta\}\}$, i.e., $max\{0, A_\delta - 1, A_\delta - 1 + B_\delta - C_\delta\}$.

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If the LHS is $B_\delta - 1$, then it's ≤ 0, so $\leq$ RHS. And if it's $\leq B_\delta - 1 + A_\delta - C_\delta$ then its also $\leq$ RHS, since that's equivalent to the last term on right-hand side by rearrangement.

When β is a limit λ, we invoke induction hypothesis: for all $\gamma < \lambda$, $(*)$ holds, or rewriting it, $|A \to C|_\gamma \geq |\top \to B|_\gamma - max\{0, |\top \sqcap A|_\gamma - |B \to C|_\gamma\}$, i.e., $|A \to C|_\gamma \geq min\{|\top \to B|_\gamma, |\top \to B|_\gamma - |\top \sqcap A|_\gamma + |B \to C|_\gamma\}$.

Taking the $liminf_\lambda$ of both sides, we get

$$|A \to C|_\lambda \geq liminf_\lambda\{min\{|\top \to B|_\gamma, |\top \to B|_\gamma - |\top \sqcap A|_\gamma + |B \to C|_\gamma\}\};$$

and since the $liminf$ of the minimum of two sequences is the minimum of their $liminf$s,

$$|A \to C|_\lambda \geq min\{|\top \to B|_\lambda, liminf_\lambda\{|\top \to B|_\gamma - |\top \sqcap A|_\gamma + |B \to C|_\gamma\}.$$

But since $liminf(-y_\gamma) = - limsup(y_\gamma)$, we have

$$liminf(x_\gamma - y_\gamma + z_\gamma) \geq liminf(x_\gamma) - limsup(y_\gamma) + liminf(z_\gamma);$$

so the above yields

$$|A \to C|_\lambda \geq min\{|\top \to B|_\lambda, |\top \to B|_\lambda - |\top \sqcap A|_\lambda + |B \to C|_\lambda\},$$

which means

$$|A \to C|_\lambda - |\top \to B|_\lambda \geq min\{0, |B \to C|_\lambda - |\top \sqcap A|_\lambda\}.$$

Taking the negatives of both sides, we get $(*)$ for λ. □

Soundness of S11: $[(A \to B) \to (\top \to B)] \to [\top \to ((\top \sqcap A) \vee (\top \sqcap B))]$.

Proof. By Lemma A2, it suffices to show that at successor ordinals, the LHS has value $\leq$ the RHS; that is, that for all β,

$$1 - |A \to B|_\beta + |\top \to B|_\beta \leq |\top \sqcap (A \vee B)|_\beta.$$

(The fact that $|A \to B|_\beta \geq |\top \to B|_\beta$ avoids any need to minimize the LHS with 1. On the RHS, I've used $\top \sqcap (A \vee B)$ instead of the equivalent $(\top \sqcap A) \vee (\top \sqcap B)$, to simplify the proof of the limit stage of the induction below.)

If β is itself a successor $\delta + 1$, the LHS is $1 - min\{1, 1 - |A|_\delta + |B|_\delta\} + |B|_\delta$; that is, $max\{B_\delta, A_\delta\}$. That's the value of the RHS.

When β is a limit λ we use induction. We can rewrite the inequality as $1 - |\top \sqcap (A \vee B)|_\beta + |\top \to B|_\beta \leq |A \to B|_\beta$. Suppose this holds prior to λ. Taking $liminf$s (at λ), we have $liminf\{1 - |\top \sqcap (A \vee B)|_\beta + |\top \to B|_\beta\} \leq |A \to B|_\lambda$. But by the same liminf law used in proving S9, this yields $1 - limsup\{|\top \sqcap (A \vee B)|_\beta\} + liminf\{|\top \to B|_\beta\} \leq |A \to B|_\lambda$; that is, $1 - |\top \sqcap (A \vee B)|_\lambda + |\top \to B|_\lambda \leq |A \to B|_\lambda$, as desired. □

Soundness of R4a: $(\top \to B) \to B \vdash B \to (\top \to B)$.

Proof. If $|(\top \to B) \to B|_\alpha$ is 1 for all final α then for all final β, $|B|_\beta \geq |\top \to B|_\beta$. From this, it's easily seen by induction that the sequence $|B|_\beta$ is non-decreasing once the critical ordinal is reached. But then it must be constant: if it strictly increased then the valuations at earlier stages wouldn't be recurrent. So $|B \to (\top \to B)|_\alpha$ is 1 for all final α. □

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Soundness of Q2: $\forall^1 x(Bx \rightarrow C) \rightarrow (\exists x Bx \rightarrow C)$ when x not free in C.

Proof. It suffices to prove
(1): For successor $\alpha+1$, $|\forall^0 x(Bx \rightarrow C)|_{\alpha+1} \leq |\exists x Bx \rightarrow C|_{\alpha+1}$, and
(2): For limit λ, $|\top \rightarrow \forall^0 x(Bx \rightarrow C)|_\lambda \leq |\exists x Bx \rightarrow C|_\lambda$.
(1) is obvious: If $|\exists x Bx \rightarrow C|_{\alpha+1} < r$ then $|\exists x Bx|_\alpha - |C|_\alpha > 1 - r$, so there's a c such that $|Bc|_\alpha - |C|_\alpha > 1 - r$, and so for which $|Bc \rightarrow C|_{\alpha+1} < r$. So $|\forall^0 x(Bx \rightarrow C)|_{\alpha+1} < r$.
(2): If $|\top \rightarrow \forall^0 x(Bx \rightarrow C)|_\lambda \geq r$ then $(\forall \varepsilon > 0)(\exists \beta < \lambda)(\forall \gamma \in [\beta, \lambda))(\forall c \in |M|)(|Bx \rightarrow C|_\gamma \geq r - \varepsilon/2)$; this holds in particular at successor γ, so

$$(\forall \varepsilon > 0)(\forall \delta \in [\beta_\varepsilon, \lambda))(\forall c \in |M|)(1 - |Bc|_\delta + |C|_\delta \geq r - \varepsilon/2).$$

For any δ, there's a c such that

$$|Bc|_\delta \geq |\exists x Bx|_\delta - \varepsilon/2;$$

so $(\forall \varepsilon > 0)(\forall \delta \in [\beta_\varepsilon, \lambda))(1 - |\exists x Bx|_\delta + |C|_\delta \geq r - \varepsilon)$. So $|\exists x Bx \rightarrow C|_\lambda \geq r$. □

Appendix B: Quantifier derivations and invalidities. Proofs of most of the sentential laws mentioned in §2 were sketched there, so I'll just sketch derivations of some of the quantifier laws mentioned but not proved in §4, and a few others. I'll also elaborate on a few of the invalidities mentioned there.

K-law for $\$^\omega$: $\vdash \$^{\omega+1}(A \rightarrow B) \rightarrow (\$^\omega A \rightarrow \$^\omega B)$.

Proof. This is the one place where we need to use the generalized form of the laws, mentioned early in §4, rather than the schemas. That enables us to turn the schematic proof of the K-law for \$, given in §2, into a proof of
(A) $True(\dot{\$}(x \dot{\rightarrow} y)) \rightarrow [True(\dot{\$}(x)) \rightarrow True(\dot{\$}(y))]$.
The next step is to generalize this to
(B) $True(\dot{\$}^n(x \dot{\rightarrow} y)) \rightarrow [True(\dot{\$}^n(x)) \rightarrow True(\dot{\$}^n(y))]$.
Since we have induction in the theory (see n. 6), and the $n = 0$ case reduces to the composition law for $\rightarrow$, it suffices for (B) that we prove
(C) $[B(n) \rightarrow (C(n) \rightarrow D(n))] \rightarrow [B(n+1) \rightarrow (C(n+1) \rightarrow D(n+1))]$,
where $B(n)$ is $True(\dot{\$}^n(x \dot{\rightarrow} y))$, $C(n)$ is $True(\dot{\$}^n(x))$, and $D(n)$ is $True(\dot{\$}^n(y))$. Using contraposition, this amounts to $[B(n) \rightarrow (C(n) \rightarrow D(n))] \rightarrow [\$B(n) \rightarrow (\$C(n) \rightarrow \$D(n))]$. But using \$-introduction followed by three applications of (A), we get (C) and hence (B).
Using UnivGen on (B) (we only need the weaker form with $\forall^1$) and Q2a, we get
(D) $\forall^0 n True(\dot{\$}^n(x \dot{\rightarrow} y)) \rightarrow \forall^0 n[True(\dot{\$}^n(x)) \rightarrow True(\dot{\$}^n(y))]$.
By \$-Introduction and (A), this gives
(E) $\$\forall^0 n True(\dot{\$}^n(x \dot{\rightarrow} y)) \rightarrow \$\forall^0 n[True(\dot{\$}^n(x)) \rightarrow True(\dot{\$}^n(y))]$.
And then an application of Q2 to the consequent (together with the Suffixing Rule) yields
(F) $\$\forall^0 n True(\dot{\$}^n(x \dot{\rightarrow} y)) \rightarrow [\forall^0 n(True(\dot{\$}^n(x))) \rightarrow \forall^0 n(True(\dot{\$}^n(y)))]$.
In other words, $\$^{\omega+1}(A \rightarrow B)) \rightarrow (\$^\omega A \rightarrow \$^\omega B)$. □

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The K-laws obviously entail the following rules (given \$-Introduction and $\$^\omega$-Introduction, both proved in the text).

K-rules: $A \to B \vdash \$A \to \$B$ and $A \to B \vdash \$^\omega A \to \$^\omega B$.

N.B.: We can't strengthen the K-law for $\$^\omega$ to $\$^\omega(A \to B)) \to (\$^\omega A \to \$^\omega B)$. For take A to be $\top$ and B to be the Restall sentence R. Since $|R|_\alpha$ is 0 whenever α is a limit ordinal, $|\$^\omega R|_\alpha$ is 0 for all α; so $|\$^\omega \top \to \$^\omega R|_\alpha$ is as well. But since $|R|_{\beta+n}$ is 1 for any β and positive integer n, $|\top \to R|_{\beta+n}$ is 1 for any β and $n \geq 2$, so $\$^\omega(\top \to R)$ has value 1 at any $\beta + \omega$ (that is, at any limit ordinal that isn't a multiple of ω^2).

Crucial feature of $\$^\omega$ and $\forall^\omega$: (i) $\$^\omega B \to \$^\omega(\top \to B)$ and (ii) $\$^\omega\forall^0 xBx \to \$^\omega\forall^1 xBx$.

Proof. By definition, the consequent of (i) is $(\forall n)True(\top \to^n (\top \to B))$, which is obviously equivalent to $(\forall n > 0)True(\top \to^n B)$; which follows from $(\forall n)True(\top \to^n B)$, which is the antecedent. (ii) easily follows from (i). □

Vacuous quantification: If x isn't free in C, $\vdash \exists x \neg C \leftrightarrow \neg C$, by Q2b and Q1, from which we derive $\vdash \forall^0 xC \leftrightarrow C$ using contraposition. The K-rules for \$ and $\$^\omega$ then yield $\vdash \forall^1 xC \leftrightarrow \$C$ and $\vdash \forall^\omega xC \leftrightarrow \$^\omega C$.

Change of variables: $\vdash \exists xAx \to \exists yAy$ (when no y in Ay is in the scope of an $\exists x$), and analogously for $\forall^0$, and $\forall^1$ and $\forall^\omega$.

Proof. $\vdash Ax \to \exists yAy$, so by UnivGen and Q2, $\vdash \exists xAx \to \exists yAy$. (Recall that UnivGen has been stated in terms of $\forall^1$.) The result for $\forall^0$ follows from this using contraposition, and the results for $\forall^1$ and $\forall^\omega$ then follow by the K-rules for \$ and $\$^\omega$. □

Commutation of unary $\exists$ and $\forall^0$ quantifiers: $\vdash \exists x\exists yAxy \to \exists y\exists xAxy$, and similarly when all the $\exists$'s are replaced by $\forall^0$.

Proof. $\vdash Axy \to \exists xAxy$ by Q1. So $\vdash \exists yAxy \to \exists y\exists xAxy$ by UnivGen on y plus Q2a. So $\vdash \exists x\exists yAxy \to \exists y\exists xAxy$ by UnivGen on x plus Q2. The result for $\forall^0$ follows from this using contraposition. □

Commutation within an n-ary $\forall^1$ **quantifier or** $\forall^\omega$ **quantifier:** $\vdash (\forall^1 x_1, x_2, ...x_n)A \to (\forall^1 x_{\pi(1)}, x_{\pi(2)}, ...x_{\pi(n)})A$, where π is any permutation of $\{1, ..., n\}$. [Equivalently, $\forall^1 x_1\forall^0 x_2...\forall^0 x_n A \to \forall^1 x_{\pi(1)}\forall^0 x_{\pi(2)}, ..., \forall^0 x_{\pi(n)})A$.] Analogously for $\forall^\omega$.

Proof. An n-ary $\forall^0$quantifier is obviously equivalent to a string of n unary $\forall^0$ quantifiers, so commutation in the string yields commutation within the n-ary $\forall^0$. The result for the n-ary $\forall^1$ quantifier is then immediate by \$-introduction and the K-law for \$. □

Commutation *rule* for unary $\forall^1$ quantifiers or $\forall^\omega$ quantifiers: $\forall^1 x\forall^1 yAxy \vdash \forall^1 y\forall^1 xAxy$, and analogously for $\forall^\omega$.

Proof. $\forall^1 x\forall^1 yAxy \vdash Axy$. Apply UnivGen first to x, then to y. □

N.B.: We can't strengthen these to conditional form, e.g., to $\forall^1 x\forall^1 yA(x, y) \rightarrow \forall^1 y\forall^1 xA(x, y)$. For let Axy be of form $Bx \wedge y = y$. Clearly $\forall^0 yAxy$ is then equivalent to $\forall^0 yBx$, which is equivalent to Bx, so $\forall^1 yAxy$ is equivalent to $\$Bx$, so $\forall^1 x\forall^1 yA(x, y)$ is equivalent to $\forall^1 x\$Bx$. And clearly $\forall^1 xAxy$ is equivalent to $\forall^1 xBx$, so $\forall^1 y\forall^1 xA(x, y)$ is equivalent to $\$\forall^1 xBx$. So this instance of commutation for $\forall^1$ is equivalent to the Barcan conditional for $\forall^1$, ruled out in n. 31.

The one-way interchange laws $\forall^1\forall^0 \rightarrow \forall^0\forall^1$ and $\forall^\omega\forall^0 \rightarrow \forall^0\forall^\omega$ will be derived below.

Universal analogs of Q2a:

(1) $\vdash \forall^1 x(Ax \rightarrow Bx) \rightarrow (\forall^0 xAx \rightarrow \forall^0 xBx)$
(1*) $\forall^1 x(C \rightarrow Bx) \rightarrow (C \rightarrow \forall^0 xBx)$
(2) $\vdash \$\forall^1 x(Ax \rightarrow Bx) \rightarrow (\forall^1 xAx \rightarrow \forall^1 xBx)$
(2*) $\$\forall^1 x(C \rightarrow Bx) \rightarrow (\$C \rightarrow \forall^1 xBx)$
(3) $\$\forall^\omega x(Ax \rightarrow Bx) \rightarrow (\forall^\omega xAx \rightarrow \forall^\omega xBx)$
(3*) $\$\forall^\omega x(C \rightarrow Bx) \rightarrow (\$^\omega C \rightarrow \forall^\omega xBx)$.

Proofs. (1) is immediate from Q2a, by contraposition of its consequent and the prefixing rule. From (1), the K-rule for \$ yields $\vdash \$\forall^1 x(Ax \rightarrow Bx) \rightarrow \$(\forall^0 xAx \rightarrow \forall^0 xBx)$; the conditional form of K-law for \$ yields $\$(\forall^0 xAx \rightarrow \forall^0 xBx) \rightarrow (\forall^1 xAx \rightarrow \forall^1 xBx)$, and by Transitivity Rule, we get (2). For (3), we use the K-rule for $\$^\omega$ followed by that for \$ to get $\vdash \$^{\omega+1}\forall^1 x(Ax \rightarrow Bx) \rightarrow \$^{\omega+1}(\forall^0 xAx \rightarrow \forall^0 xBx)$; the "crucial feature" allows replacing the antecedent by $\forall^{\omega+1} x(Ax \rightarrow Bx)$, and the conditional form of K-law for $\$^\omega$ then allows replacing the consequent by $\forall^\omega xAx \rightarrow \forall^\omega xBx$. (1*)–(3∗) follow from (1) to (3) by vacuous quantifier rules. □

N.B.: We can't strengthen (2) to $\forall^1 x(Ax \rightarrow Bx) \rightarrow (\forall^1 xAx \rightarrow \forall^1 xBx)$. This is evident, since when Ax is $\top$ this is equivalent to $\forall^1 x(\top \rightarrow Bx) \rightarrow (\top \rightarrow \forall^1 xBx)$, and that implies the Barcan formula for $\forall^1$ which we've seen invalid (n. 31). [The antecedent of that Barcan formula, $\forall^1 x\$Bx$, is $\forall^1 x[Bx \wedge (\top \rightarrow Bx)]$, which implies $\forall^1 xBx \wedge \forall^1 x(\top \rightarrow Bx)$, which by the bad strengthening of (2) would imply $\forall^1 xBx \wedge (\top \rightarrow \forall^1 xBx)$, which is the consequent $\$\forall^1 xBx$ of that Barcan formula.] Similarly for (3).

Partial converse of Q2: $\vdash (\exists xBx \rightarrow C) \rightarrow \forall^0 x(Bx \rightarrow C)$, when x not free in C.

Proof. $\vdash Bx \rightarrow \exists xBx$, so by Suffixing Rule, $\vdash (\exists xBx \rightarrow C) \rightarrow (Bx \rightarrow C)$, so by UnivGen and (1*), $\vdash (\exists xBx \rightarrow C) \rightarrow \forall^0 x(Bx \rightarrow C)$. □

Corollary. $\exists xBx \rightarrow C \vdash \forall^1 x(C \rightarrow Bx)$.

Quantifiers with disjunction and conjunction:

(4) $\vdash \exists x(Ax \vee Bx) \leftrightarrow \exists xAx \vee \exists xBx$
(5) $\vdash (\forall^0 xAx \wedge \forall^0 xBx) \rightarrow \forall^0 x(Ax \wedge Bx)$
(6) $\vdash (\forall^1 xAx \wedge \forall^1 xBx) \rightarrow \forall^1 x(Ax \wedge Bx)$
(7) $\vdash \forall^0 x(C \vee Bx) \rightarrow (C \vee \forall^0 xBx)$, when x isn't free in C.

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Proofs. (4) The R to L is easy, using Reverse Adjunction. L to R: $\vdash Ax \to (\exists x Ax \vee \exists x Bx)$, using Q1, $\vee$-Introduction and transitivity; similarly $\vdash Bx \to (\exists x Ax \vee \exists x Bx)$. So by Reverse Adjunction, $\vdash (Ax \vee Bx) \to (\exists x Ax \vee \exists x Bx)$. So by UnivGen and Q2, $\vdash \exists x(Ax \vee Bx) \to (\exists x Ax \vee \exists x Bx)$.

(5): By (4) plus contraposition.

(6): From (5), use \$-Introduction plus K-law for \$ to get $\$(\forall^0 x Ax \wedge \forall^0 x Bx) \to \forall^1 x(Ax \wedge Bx)$. Using $\$C \wedge \$D \to \$(C \wedge D)$, conclusion follows.

(7) By Q3, $\vdash \neg C \wedge \exists x \neg Bx \to \exists x(\neg C \wedge \neg Bx)$. From this, we easily get $\neg(C \vee \neg\exists x \neg Bx) \to \exists x \neg(C \vee Bx)$, i.e., $\neg(C \vee \forall^0 x Bx) \to \exists x \neg(C \vee Bx)$. The result follows by contraposition (plus definition of $\forall^0$ and double negation laws). □

Three analogs of converse Barcan formula:

(8a) $\vdash (\top \to \forall^0 x Bx) \to \forall^0 x(\top \to Bx)$
(8b) $\vdash \$\forall^0 x Bx \to \forall^0 x \$Bx$ (that is, $\vdash \forall^1 x Bx \to \forall^0 x \$Bx$)
(8c) $\vdash \$^\omega \forall^0 x Bx \to \forall^0 x \$^\omega Bx$ (that is, $\vdash \forall^\omega x Bx \to \forall^0 x \$^\omega Bx$).

Proof of (8a): $\vdash \forall^0 x Bx \to Bx$, so by prefixing rule, $\vdash (\top \to \forall^0 x Bx) \to (\top \to Bx)$. By UnivGen $\vdash \forall^1 x[(\top \to \forall^0 x Bx) \to (\top \to Bx)]$, and by (1$\star$), $\vdash (\top \to \forall^0 x Bx) \to \forall^0 x(\top \to Bx)$.

The proofs of (8b) and (8c) are analogous, using the K-rules for \$ and $\$^\omega$ instead of the prefixing rule.

Rule versions of converses of these (that is, rule-analogs of the Barcan formula) are trivial.

One-way interchange laws: $\vdash \forall^1 x \forall^0 y Bxy \to \forall^0 x \forall^1 y Bxy$, and analogously with $\forall^\omega$ for $\forall^1$ in both occurrences.

Proof. Immediate from (8b) and (8c), taking Bx to be $\forall^0 y Bxy$. □

Further analogs of converse Barcan:

(9) $\vdash \$\forall^1 x Bx \to \forall^1 x \$Bx$
(10) $\vdash \$\forall^\omega x Bx \to \forall^\omega x \$Bx$.

Proof of (9): use \$-Introduction and K-law for \$ on (8b).

Proof of (10): $\$\forall^\omega x Bx$ is $\$(\forall^0 n \in N)[True(\langle \$^n \forall^0 x Bx \rangle)]$, which is equivalent to $\$(\forall^0 n \in N)[True(\langle \$^{n+1} \forall^0 x Bx \rangle)]$, i.e., $\$(\forall^0 n \in N)[True(\langle \$^n \$\forall^0 x Bx \rangle)]$. But by (8b) that entails $\$(\forall^0 n \in N)[True(\langle \$^n \forall^0 x \$Bx \rangle)]$, which is $\forall^\omega x \$Bx$.

Importation: $\vdash \exists x(C \to Bx) \to (C \to \exists x Bx)$ when x isn't free in C.

Proof. $\vdash Bx \to \exists x Bx$ by Q1, so $(C \to Bx) \to (C \to \exists x Bx)$ by Prefixing rule; then by UnivGen and Q2, $\exists x(C \to Bx) \to (C \to \exists x Bx)$. □

(#): $\$^\omega(\neg A \vee B) \to \$^\omega(A \to B)$.

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Proof. S8 gives $\vdash (\top \rightarrow \neg A \vee B) \rightarrow (A \rightarrow B)$. By K-rule for $\$^\omega$, $\vdash \$^\omega(\top \rightarrow \neg A \vee B) \rightarrow \$^\omega(A \rightarrow B)$, so by the "crucial feature", we get the result. □

(RQ): $\forall^\omega x(\neg Ax \vee Bx) \rightarrow \forall^\omega x(Ax \rightarrow Bx)$.

Proof. S8 gives $\vdash (\top \rightarrow \neg Ax \vee Bx) \rightarrow (Ax \rightarrow Bx)$. By UnivGen, $\vdash \forall^1 x[(\top \rightarrow \neg Ax \vee Bx) \rightarrow (Ax \rightarrow Bx)]$, and so by (1), $\vdash \forall^0 x(\top \rightarrow \neg Ax \vee Bx) \rightarrow \forall^0 x(Ax \rightarrow Bx)$. By the K-rule for $\$^\omega$ and the "crucial feature", we get the result. □

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