Can. J. Math., Vol. XXXI, No. 1, 1979, pp. 130–138

EMBEDDING SMOOTH DENDROIDS IN HYPERSPACES

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1. Preliminaries. A *continuum* will be a connected, compact, metric space. By a *mapping* we mean a continuous function. By a *partially ordered* space X we mean a continuum X together with a partial order which is closed when regarded as a subset of $X \times X$. We let 2^X (resp. $C(X)$) denote the hyperspace of closed subsets (resp. subcontinua) of X with the Vietoris topology which coincides with the topology induced by the Hausdorff metric. The hyperspaces 2^X and $C(X)$ are arcwise connected metric continua (see [**3**, Theorem 2.7]). If $A \subset X$ we let $C(A)$ denote the subspace of subcontinua of X which lie in A.

If X is a partially ordered space we define two functions $L, M : X \to 2^X$ by setting for each $x \in X$

$$L(x) = \{y \in X | y \leqq x\} \quad \text{and} \quad M(x) = \{y \in X | x \leqq y\}.$$

Then L and M are upper semi-continuous. If $A \subset X$ we let $L(A) = \bigcup \{L(x) | x \in A\}$ and $M(A) = \bigcup \{M(x) | x \in A\}$. We also let Max $(X) = \{x \in X | M(x) = \{x\}\}$ and Min $(X) = \{x \in X | L(x) = \{x\}\}$.

A *chain* is a totally ordered set and an *order arc* is a compact and connected chain. If for each $x \in X$, $L(x)$ is an order arc and Min (X) is a closed set, then L is continuous (see, [**6**, Proposition 3.2]).

Let X be a uniquely arcwise connected continuum and let $p \in X$. We define a partial order $\leqq_p$ on X by $x \leqq_p y$ if x lies on the irreducible arc from p to y. If $\leqq_p$ is a closed partial order we say that X is *smooth at* p. For $x, y \in X$, we denote by $[x, y]$ the unique arc from x to y.

A metric ρ for the partially ordered space X is said to be *radially convex* if $x \leqq y \leqq z$ implies that $\rho(x, z) = \rho(x, y) + \rho(y, z)$.

THEOREM 1.1. (Carruth, [**1**]). *Every partially ordered space admits a radially convex metric.*

We denote by Cl (A) (resp. Bd (A)) the closure (resp. the boundary) of a subset A of X.

A continuum X is said to be *unicoherent* if whenever X is written as the union of two subcontinua P and Q, then $P \cap Q$ is connected. It is said to be *hereditarily unicoherent* if each subcontinuum is unicoherent. A *dendroid* is an arcwise connected, hereditarily unicoherent continuum. Clearly, a dendroid is uniquely arcwise connected. A dendroid is said to be *smooth* if it is smooth at some point. A continuum is said to be *indecomposable* if it cannot be written

Received September 1, 1977. This research was supported by a University of Saskatchewan Postdoctoral Fellowship and in part by NRC Grant A5616.

as the union of two proper subcontinua. A continuum is said to be *hereditarily indecomposable* if each of its subcontinua is indecomposable.

Nadler has proved in [**5**, (1.70)] that a dendroid which is embeddable in $C(X)$, where X is a hereditarily indecomposable continuum, is smooth. He asked in [**5**, (1.74)] whether every smooth dendroid can be embedded in $C(X)$ for every hereditarily indecomposable continuum X. Nadler has answered his question in the affirmative for the case of smooth fans, that is, smooth dendroids with only one ramification point (see [**5**, (1.73)]), as well as for the case of dendrites (see [**5**, (1.74.1)]). Our purpose in this paper is to give an affirmative answer to this question for the general case.

Finally, we wish to give our sincere thanks to Professor S. B. Nadler, Jr. for introducing us to this problem and for many very interesting discussions.

2. Smooth dendroids. For any subset of A of X and $\epsilon > 0$ we denote by $S(A, \epsilon)$ the open ϵ-ball about A.

LEMMA 2.1. *Let D be a dendroid smooth at p. Let ρ be a radially convex metric for X. If $r, \epsilon > 0$, then each component of* $\mathrm{Cl}\,(S(p, r + \epsilon)\backslash S(p, r))$ *has diameter less than or equal to 2ϵ and meets the boundary of $S(p, r)$ in exactly one point.*

Proof. Let K be a component of $\mathrm{Cl}\,(S(p, r + \epsilon)\backslash S(p, r))$. Let $a \in K$ and let $c \in \mathrm{Bd}\,(S(p, r)) \cap L(a))$. Since K is arcwise connected and D is uniquely arcwise connected, it follows that $c \in L(b)$ for each $b \in K$. Since ρ is radially convex, c is the unique point in $K \cap \mathrm{Bd}\,(S(p, r))$. If $b \in K$, then $r + \epsilon \geqq \rho(p, b) = \rho(p, c) + \rho(c, b)$. Since $\rho(p, c) = r$, we have $\rho(c, b) \leqq \epsilon$. The Lemma now follows from the triangle inequality.

The following lemma improves somewhat a result of J. B. Fugate [**2**, Theorem 1]. This lemma may be used to obtain a simple proof of Fugate's result [**2**, Theorem 2] that smooth dendroids admit small retractions onto finite trees.

LEMMA 2.2. *Let D be a dendroid smooth at p, and let ρ be a radially convex metric for D such that $\rho(x, p) \leqq 1$ for each $x \in D$. Then there exists a sequence of finite closed converings $\mathscr{U}_1, \mathscr{U}_2, \ldots$ of D such that the following conditions are satisfied:*

(i) $\mathscr{U}_1 > \mathscr{U}_2 > \ldots,$
(ii) $\mathscr{U}_i = \mathscr{U}_{i,1} \cup \ldots \cup \mathscr{U}_{i,2^i},$
(iii) $\mathscr{U}_{i,1} = \{\mathrm{Cl}\,(S(p, 1/2^i))\},$
(iv) $\mathscr{U}_{i,j} = \{U_{i,j,1}, \ldots, U_{i,j,n_{i,j}}\},$
(v) $U_{i,j,k} \cap U_{i,j,k'} = \phi$ for $k \neq k'$,
(vi) $U_{i,j,1} \cup \ldots \cup U_{i,j,n_{i,j}} = \mathrm{Cl}\,(S(p, j/2^i)\backslash S(p, (j-1)/2^i)),$
(vii) *diameter of* $U_{i,j,k} < 3/2^i$,
(viii) *if* $x, y \in U_{i,j,k}$ *and* $x \in L(y)$, *then* $L(y) \cap M(x) \subset U_{i,j,k}$,
(ix) *if* $L(U_{i,j+1,k}) \cap U_{i,j,k}. 00\, \phi$, *then* $U_{i,j+1,k} \subset M(U_{i,j,k}.)$.

https://doi.org/10.4153/CJM-1979-014-4 Published online by Cambridge University Press

Proof. Suppose that for some $i \geqq 1$ and for some j such that $1 \leqq j < 2^i$, $\mathscr{U}_1, \ldots, \mathscr{U}_{i-1}$ and $\mathscr{U}_{i,1}, \ldots, \mathscr{U}_{i,j}$ have been defined to satisfy the conditions (i) – (ix).

By Lemma 2.1, each of the sets

$$U_{i-1,(j+2)/2,r} \cap M(U_{i,j,k}) \cap \mathrm{Cl}\,(S(p, (j+1)/2^i)\backslash S(p, j/2^i))$$

may be decomposed into a collection $\mathscr{V}_{k,r}$ of finitely many disjoint closed sets each of diameter $<3/2^i$. Let $\mathscr{U}_{i,j+1} = \bigcup \{\mathscr{V}_{k,r} | k \in \{1, \ldots, n_{i,j}\}$ and

$$r \in \{1, \ldots, n_{i-1,(j+2)/2}\}\}.$$

To show that the members of $\mathscr{U}_{i,j+1}$ satisfy (viii) notice that if K is a component of $\mathrm{Cl}\,(S(p, (j+1)/2^i))\backslash S(p, j/2^i))$ and $x, y \in K$ with $x \in L(y)$, then $L(y) \cap M(x) \subset K$. The rest of the properties (i) – (ix) are clearly satisfied. By induction the proof of Lemma 2.2 is complete.

THEOREM 2.3. *If D is a dendroid smooth at p, then D can be embedded in a dendroid Y which is smooth at q,* Max (Y) *is closed and Y admits a radially convex metric ρ such that $\rho(q, x) = 1$ for each $x \in$* Max (Y).

Proof. Let $f : C \to D$ be a mapping of the Cantor set C onto D. Let d be a radially convex metric for D such that $d(p, x) \leqq 1/2$ for each $x \in D$. Define an equivalence relation $\sim$ on $C \times [0, 1]$ by setting $(c, x) \sim (e, y)$ in $C \times [0, 1]$ if and only if $x = y$ and $c = e$ or there exists $z \in L(f(c)) \cap L(f(e))$ such that $d(p, z) = x = y$. Suppose that $(c_i, x_i))_{i \in \omega}$ and $(e_i, y_i))_{i \in \omega}$ are sequences in $C \times [0, 1]$ which converge to (c, x) and (e, y), respectively, and such that $(c_i, x_i) \backsim (e_i, y_i)$ for each i. For each i let $z_i \in L(f(c_i)) \cap L(f(e_i))$ such that

$$\lim_{i \to \infty} d(p, z_i) = \lim_{i \to \infty} x_i = x = \lim_{i \to \infty} y_i = y.$$

Let $z = \lim_{i \to \infty} z_i$. Then $z \in L(f(c)) \cap L(f(e))$ and $d(z, p) = x$. Thus, $(c, x) \sim (e, y)$, and hence, $\sim$ is upper semi-continuous. Let $\pi : C \times [0, 1] \to (C \times [0, 1])/\sim\, = X$ be the natural projection of $C \times [0, 1]$ onto X. Define $g : D \to X$ by letting $g(x) = \pi((c, d(p, x)))$ for $c \in f^{-1}(x)$. Then it is easy to check that g is a well-defined mapping which carries D homeomorphically onto $g(X) \subset X$, and that X is arcwise connected. The set $X\backslash g(X)$ is homeomorphic to an open subset of $C \times [0, 1]$ which meets $\{c\} \times [0, 1]$ in a connected set for each $c \in C$. To see that X is hereditarily unicoherent notice that if K is a subcontinuum of X, then $K \cap g(D)$ is connected and also that if $K \cap g(D) \neq \phi$ and $\pi((c, a)) \in K\backslash g(D)$, then $\pi(\{c\} \times [d(p, f(c)), a]) \subset K$. It follows, now, that since $g(D)$ is hereditarily unicoherent, X is also. Thus, X is a dendroid. It is easy to check that X is smooth at the point $q = \pi(C \times \{0\})$, and that

$$\mathrm{Max}\,(X) = \pi(C \times \{1\}).$$

Since Max $(X) \cap g(D) = \phi$, it is easy to see that Carruth's proof of Theorem

1.1 (see [**1**]) may be modified to give a radially convex matric ρ on X such that $\rho(q, x) > 1$ for each $x \in \text{Max}(X)$ and $\rho(q, x) < 1/2$ for each $x \in g(D)$. Let $Y = \{x \in X | \rho(q, x) \leqq 1\}$. Then Y is easily seen to be the required smooth dendroid.

3. Some facts on hereditarily indecomposable continua. If X is a continuum, then a *Whitney map* for X is a mapping $\mu : C(X) \to [0, 1]$ such that $\mu(\{X\}) = 1$, $\mu(\{x\}) = 0$ for each $x \in X$, and if $A \subset B$, $A \neq B$ then $\mu(A) < \mu(B)$. It is known that Whitney maps exist for each continuum X (see [**8**]). They are monotone (see [**5**]). In the sequel we shall use the mapping $\nu : C(X) \to [0, 1]$ defined by $\nu(A) = 1 - \mu(A)$ for each $A \in C(X)$.

The following simple results seem to be known but not all of them, as far as the authors are aware, appear in the literature.

If X is a hereditarily indecomposable continuum, then $C(X)$ is uniquely arcwise connected (see [**3**, Theorem 8.4]) and a partial order is defined on $C(X)$ by reverse inclusion. Then, $\text{Min}(C(X)) = \{X\}$, $\text{Max}(C(X)) = \{\{x\} | x \in X\}$ and for each $A \in C(X) \backslash \{X\}$ $L(A)$ is an order arc. This partial order is closed, and hence, $C(X)$ is an order-isomorphism on each maximal order arc.

PROPOSITION 3.1. *If X is a hereditarily indecomposable continuum, then the function $M : C(X) \to C(C(X))$ defined by $M(A) = C(A)$ for each $A \in C(X)$ is continuous.*

Proof. We already know that M is upper semi-continuous. It suffices to prove, therefore, that if $A_1, A_2, \ldots$ is a sequence of subcontinua of X converging to A_0, then every subcontinuum B_0 of A_0 is the limit of a sequence $B_1, B_2, \ldots$ where, for each n, B_n is a subcontinuum of A_n. Let $a_n \in A_n$ for $n = 0, 1, 2, \ldots$ be such that $\lim_{n\to\infty} a_n = a_0$. Consider the unique arcs $[A_1, a_1], [A_2, a_2], \ldots$ in $C(X)$. Then $\text{Lim}_{n\to\infty} [A_n, a_n] = [A_0, a_0]$, since $C(X)$ is smooth at $\{X\}$. Therefore, if $B_0 \in [A_0, a_0]$ there exists a sequence $B_1, B_2, \ldots$ such that $B_n \in [A_0, a_n]$ for each $n = 1, 2, \ldots$ and $\lim_{n\to\infty} B_n = B_0$.

Remark. Nadler [**5**] calls the continua for which the function M of Proposition 3.1 is continuous *C^*-smooth.*

PROPOSITION 3.2. *Let X be a hereditarily indecomposable continuum and let $A \in C(X)$. If U is an open neighbourhood of A in $C(X)$, then $L(U) = \{B \in C(X) | B \supset C$ for some $C \in U\}$ is a neighbourhood of $[X, A]$.*

Proof. It suffices to prove that if $Q \in L(U)$ and $Q_n)_{n\in\omega}$ is a sequence of points of $C(X)$ such that $\lim_{n\to\infty} Q_n = Q$, then $Q_n)_{n\in\omega}$ is eventually in $L(U)$. For this consider the sequence $M(Q_n))_{n\in\omega}$. Since M is continuous, we have

$$\lim_{n\to\infty} M(Q_n) = M(Q).$$

But $Q \in L(U)$ implies that $M(Q) \cap U \neq \phi$, and hence, there exists some

https://doi.org/10.4153/CJM-1979-014-4 Published online by Cambridge University Press

integer n_0 such that $M(Q_n) \cap U \neq \phi$ for each $n \geqq n_0$. Let $A_n \in M(Q_n) \cap U$ for each $n \geqq n_0$. Then $Q_n \in L(A_n)$ for each $n \geqq n_0$. Thus, the sequence $Q_n)_{n\in\omega}$ is eventually in $L(U)$.

LEMMA 3.3. *Let X be a hereditarily indecomposable continuum, $R, Q \in C(X)$ such that $Q \in M(R)\backslash\{R\}$, and $Q_1 \ldots, Q_n \in M(Q) \cap \nu^{-1}\nu(Q_1)$ where $\nu : C(X) \to [0, 1]$ is the function defined above. Let $U_1, \ldots, U_n$ be disjoint neighbourhoods of $Q_1, \ldots, Q_n$, respectively, and let V be a neighbourhood of Q in $C(X)$. If m is a positive integer, then there exist distinct points $P_1, \ldots, P_m \in \mathscr{V} \cap \nu^{-1}\nu(Q)$ such that*

$$M(P_i) \cap U_j \neq \phi, \text{ for each } i \in \{1, \ldots, m\}; j \in \{1, \ldots, n\}, \quad \text{and}$$
$$L(P_i) \cap L(P_j) = L(R), \text{ for each } i \neq j; i, j \in \{1, \ldots, m\}.$$

Proof. Let S denote the composant of Q in R. Then by using the proof of Proposition 3.1 we have that $\mathrm{Cl}\,[C(R\backslash S)] = C(R)$ and there is a sequence $P_1, P_2, \ldots$ of subcontinua of $R\backslash S$, which lie in distinct composants of R and such that $\nu(P_i) = \nu(Q)$ for each $i \in \{1, 2, \ldots\}$, and $\lim_{t\to\infty} P_t = Q$. By Proposition 3.1, M is a continuous function, and hence, $\lim_{t\to\infty} M(P_t) = M(Q)$. By Proposition 3.2,

$$T = \bigcap_{j=1}^{n} L(U_j)$$

is a neighbourhood of $L(Q)$. Then, the sequence $P_t)_{i\in\omega}$ is eventually in $\mathscr{V} \cap T$, and hence, we can pick up distinct elements $P_{i_1}, \ldots, P_{i_m}$ in $\mathscr{V} \cap T$. Clearly, $M(P_{i_k}) \cap U_j \neq \phi$ for each $k \in \{1, \ldots, m\}$; $j \in \{1, \ldots, n\}$, and by the choice of the sequence $P_1, P_2, \ldots$ we have

$$L(P_{i_k}) \cap L(P_{i_s}) = L(R) \quad \text{for each } k \neq s;\ k, s \in \{1, \ldots, m\}.$$

4. Embedding smooth dendroids in hyperspaces. We are now ready to prove our main result.

THEOREM 4.1. *Let X be a hereditarily indecomposable continuum, and let D be a smooth dendroid. Then D is embeddable in $C(X)$.*

Proof. Let $p \in D$ such that D is smooth at p and let ρ be a radially convex metric for D (see [**1**]). Let $\nu : C(X) \to [0, 1]$ be as before. By Theorem 2.3, we may assume that Max (D) is a closed subset of D and that $\rho(p, x) = 1$ for each $x \in$ Max (D).

Let $\mathscr{U}_1, \mathscr{U}_2, \ldots$ be a sequence of finite closed covers of D which satisfies the hypotheses of Lemma 2.2.

Let $q_{1,1,1} \in \nu^{-1}(1/2)$ and let $Q_{1,1} = \{q_{1,1,1}\}$. Let

$$q_{1,2,1}, \ldots q_{1,2,n_{1,2}} \in \nu^{-1}(1) \cap M(q_{1,1,1})$$

be distinct points such that

$$L(q_{1,2,i}) \cap L(q_{1,2,j}) = L(q_{1,1,1}) \quad \text{for each } i \neq j$$

(see Lemma 3.3). For each $i = 1, \ldots, n_{1,2}$ let $V_{1,i}$ be a neighbourhood of $q_{1,2,i}$ such that

$$L(\mathrm{Cl}\,(V_{1,i})) \cap L(\mathrm{Cl}\,(V_{1,j})) \subset S(L(q_{1,1,1}), \tfrac{1}{4}), \quad \text{and}$$
$$\text{diameter } L(V_{i,j}) \cap \nu^{-1}(\epsilon) < \tfrac{1}{4}$$

for each $j = 1, \ldots, n_{1,2}$ and for each $0 < \epsilon \leqq 1$. Let $Q_1 = Q_{1,1} \cup Q_{1,2}$, where

$$Q_{1,2} = \{q_{1,2,1}, \ldots, q_{1,2,n_{1,2}}\}.$$

Let $F_1 = L(Q_{1,2})$ and let $f_1 : D \to F_1$ be the mapping such that $\nu(f_1(x)) = \rho(p, x)$ for each $x \in D$ and such that $f_1(U_{1,j,k}) \subset L(q_{1,j,k})$ for each $j \in \{1, 2\}$, and each $k \in \{1, \ldots, n_{1,j}\}$.

Suppose that for each $i \in \{1, \ldots, m\}$, $Q_i = Q_{i,1} \cup \ldots \cup Q_{i,2^i}$, $F_i = L(Q_{i,2^i})$, $V_{i,1}, \ldots, V_{i,n_{i,2^i}}$ and $f_i : D \to F_i$ have been defined so that the following conditions are satisfied:

1) $Q_{i,1} = \{q_{i,1,1}\} \subset \nu^{-1}(1/2^i) \cap [X, q_{1,1,1}]$, $Q_{i,j} = \{q_{i,j,1}, \ldots, q_{i,j,n_{i,j}}\} \subset \nu^{-1}(j/2^i)$,

2) if $U_{i,j,k} \subset L(U_{i-1,2^{i-1},s}) \cap M(U_{i,j-1,r})$, then $q_{i,j,k} \in L(V_{i-1,2^{i-1},s}) \cap M(q_{i,j-1,r})$ for each $j = 1, \ldots, 2^i$ and each $k \in \{1, \ldots, n_{i,j}\}$, $s \in \{1, \ldots, n_{i-1,2^{i-1}}\}$, $r \in \{1, \ldots, n_{i,j-1}\}$.

3) $L(q_{i,j,k}) \cap L(q_{i,j,t}) = L(q_{i,h,r})$ if $U_{i,j,k} \cup U_{i,j,t} \subset M(U_{i,h,r})$ but $U_{i,j,k} \cup U_{i,j,t} \not\subset M(U_{i,h+1,s})$ for $j \in \{1, \ldots, 2^i\}$, $k, t \in \{1, \ldots, n_{i,j}\}$ with $k \neq t$, $h \in \{1, \ldots, j-1\}$, $r \in \{1, \ldots, n_{i,h}\}$ and any $s \in \{1, \ldots, n_{i,h+1}\}$.

4) $q_{i,2^i,j} \in V_{i,j}$, $V_{i,j}$ is open in $C(X)$, diameter of $L(V_{i,j}) \cap \nu^{-1}(\epsilon) < 1/2^{i+1}$ for each $0 < \epsilon \leqq 1$, and such that if $U_{i,2^i,j} \cup U_{i,2^i,k} \subset M(U_{i,r,s})$ and $U_{i,2^i,j} \cup U_{i,2^i,k} \not\subset M(U_{i,r+1,t})$ for each $j, k \in \{1, \ldots, n_{i,2^i}\}$, $r \in \{1, \ldots, 2^i - 1\}$ with $j \neq k$, $s \in \{1, \ldots, n_{i,r}\}$, $t \in \{1, \ldots, n_{i,r+1}\}$, then $L(\mathrm{Cl}\,(V_{i,j})) \cap L(\mathrm{Cl}\,(V_{i,k})) \subset S(L(q_{i,r,s}), 1/2^{i+1})$,

5) $\bigcup_{j=1}^{n_{i,2^i}} L(\mathrm{Cl}\,(V_{i,j})) \subset \bigcup_{j=1}^{n_{i-1,2^{i-1}}} L(V_{i-1,j})$, and

6) $f_i : D \to F_i$ is the mapping such that $\nu(f_i(x)) = \rho(p, x)$ for each $x \in D$ and such that $f_i(U_{i,j,k}) \subset L(q_{i,j,k})$ for each $j \in \{1, \ldots, 2^i\}$ and each $k \in \{1, \ldots, n_{i,j}\}$.

Let $Q_{m+1,1} = \{q_{m+1,1,1}\} \subset \nu^{-1}(1/2^{m+1}) \cap L(q_{1,1,1})$. Suppose that $Q_{m+1,j}$ has been defined for some $1 \leqq j < 2^{m+1}$. Let

$$Q_{m+1,j+1} = \{q_{m+1,j+1,1}, \ldots, q_{m+1,j+1,n_{m+1,j+1}}\} \subset \nu^{-1}\left(\frac{j+1}{2^{m+1}}\right)$$

such that if $U_{m+1,j+1,k} \subset L(U_{m,2^m,r}) \cap M(U_{m+1,j,s})$, then

$$q_{m+1,j+1,k} \in L(V_{m,2^m,r}) \cap M(q_{m+1,j,s}),$$

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and such that

$$L(q_{m+1,j+1,k}) \cap L(q_{m+1,j+1,t}) = L(q_{m+1,h,r})$$

if

$$U_{m+1,j+1,k} \cup U_{m+1,j+1,t} \subset M(U_{m+1,h,r})$$

but

$$U_{m+1,j+1,k} \cup U_{m+1,j+1,t} \not\subset M(U_{m+1,h+1,s})$$

for $k, t \in \{1, \ldots, n_{m+1,j+1}\}$ with $k \neq t$, $h \in \{1, \ldots, j\}$, $r \in \{1, \ldots, n_{m+1,h}\}$ and any $s \in \{1, \ldots, n_{m+1,h+1}\}$.
This is possible by Lemma 3.3. Hence, we may assume that $Q_{m+1,i}$ is defined for each $i \in \{1, \ldots, 2^{m+1}\}$.

Let $F_{m+1} = L(Q_{m+1,2^{m+1}})$ and define $f_{m+1} : D \rightarrow F_{m+1}$ to be the mapping such that $\nu(f_{m+1}(x)) = \rho(p, x)$ for each $x \in D$ and such that $f_{m+1}(U_{m+1,j,k}) \subset L(q_{m+1,j,k})$ for each $j \in \{1, \ldots, 2^{m+1}\}$ and each $k \in \{1, \ldots, n_{m+1,j}\}$. Let

$$V_{m+1,1}, \ldots, V_{m+1,n_{m+1,2^{m+1}}}$$

be neighbourhoods of

$$q_{m+1,2^{m+1},1}, \ldots, q_{m+1,2^{m+1},n_{m+1,2^{m+1}}},$$

respectively, such that diameter $L(\mathscr{V}_{m+1,i}) \cap \nu^{-1}(\epsilon) < 1/2^{m+2}$ for each $i \in \{1, \ldots, n_{m+1,2m+1}\}$ and each $0 < \epsilon \leqq 1$, if

$$U_{m+1,2^{m+1},j} \cup U_{m+1,2^{m+1},k} \subset M(U_{m+1,r,s}) \quad \text{and}$$
$$U_{m+1,2^{m+1},j} \cup U_{m+1,2^{m+1},k} \not\subset M(U_{m+1,r+1,t})$$

for $j, k \in \{1, \ldots, n_{m+1,2^{m+1}}\}$ with $j \neq k$, $r \in \{1, \ldots, 2^{m+1} - 1\}$, $s \in \{1, \ldots, n_{m+1,r}\}$, $t \in \{1, \ldots, n_{m+1,r+1}\}$, then

$$L(\text{Cl}\ (V_{m+1,j})) \cap L(\text{Cl}\ (V_{m+1,k})) \subset S(L(q_{m+1,r,s}), 1/2^{m+2}),$$

and if

$$U_{m+1,2^{m+1},j} \subset U_{m,2^m,k},$$

then

$$L(\text{Cl}\ (V_{m+1,j})) \subset L(V_{m,k}).$$

By induction, the mappings $f_i : D \rightarrow C(X)$ of D into $C(X)$ are defined for each positive integer i. The mappings $f_1, f_2, \ldots$ form a Cauchy sequence, since for each i and each $x \in D, j > i$ implies that $f_j(x) \in \nu^{-1}(\rho(p, x)) \cap L(V_{i,k})$ for some $k \in \{1, \ldots, n_{i,2^i}\}$ and the set on the right has diameter $<1/2^{i+1}$. Hence, $f = \lim_{i\rightarrow\infty} f_i$ exists and is a mapping.

If $z \in f(D)$, then $f^{-1}(z) \subset \{x \in D | \rho(p, x) = \nu(z)\}$. Let $x, y \in D$ such that $\rho(p, x) = \rho(p, y)$. Let $3 < i$ be a positive integer so that $1/2^{i-3} < \rho(x, y)$. If j, k are integers such that $x \in L(U_{i,2^i,j})$ and $y \in L(U_{i,2^i,k})$ and r is the

https://doi.org/10.4153/CJM-1979-014-4 Published online by Cambridge University Press

smallest integer such that $x \in U_{i,r,s}$ for some s, then

$$L(\mathrm{Cl}\,(V_{i,j})) \cap L(\mathrm{Cl}\,(V_{i,k})) \cap \nu^{-1}(\nu(f(x))) = \phi,$$

since

$$L(q_{i,2^i,j}) \cap L(q_{i,2^i,k}) \subset L(q_{i,t,s})$$

for some $t \leqq r - 2$. We have

$$L(\mathrm{Cl}\,(V_{i,j})) \cap L(\mathrm{Cl}\,(V_{i,k})) \subset S(L(q_{i,t,s}), 1/2^{i+1}) \subset S(\{X\}, (t+1)/2^i), \quad \text{and}$$

$$(t+1)/2^i \leqq (r-1)/2^i < \nu(f(x)).$$

Hence, $f(x) \in L(\mathrm{Cl}\,(V_{i,j})) \cap \nu^{-1}\nu(f(x)))$ and $f(y) \in L(\mathrm{Cl}\,(V_{i,k})) \cap \nu^{-1}(\nu(f(x)))$ imply that $f(x) \neq f(y)$. This shows that f is one-to-one. Thus, f is an embedding of D into $C(X)$.

COROLLARY 4.2. *Let D be an arcwise connected one-dimensional continuum. Then the following are equivalent:*

(i) *D is smooth dendroid,*

(ii) *D is embeddable in $C(X)$ for some hereditarily indecomposable continuum X,*

(iii) *D is embeddable in $C(X)$ for every hereditarily indecomposable continuum X.*

Proof. J. Krasinkiewicz proved in [**4**, Corollary 4.2] that an arcwise connected one-dimensional continuum which is embeddable in $C(X)$ for some hereditarily indecomposable continuum X is a contractible dendroid. Nadler proved in [**5**, (1.70)] that it is smooth. The result, now, follows from Theorem 4.1.

Remarks. 1. Corollary 4.2 generalizes Theorem 1.73 in [**5**], where it is proved that a fan is smooth if and only if it is embeddable in $C(X)$ for any hereditarily indecomposable continuum X.

2. Theorem 4.1 together with the result that the hyperspaces of one-dimensional hereditarily indecomposable plane continua (e.g. pseudo-arc) are embeddable in E^3 in such a way that Whitney levels embed into horizontal planes (see [**7**]) gives a proof that every smooth dendroid embeds into E^3 so that all order arcs project into the z-axis in a one-to-one fashion.

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