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MINIMAL $\mathbf{F}_2$-FLOWS

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Abstract

Let σ be an ergodic endomorphism of the r-dimensional torus and Π a semigroup generated by two affine transformations lying above σ. We show that the flow defined by Π admits minimal sets of positive Hausdorff dimension and we give necessary and sufficient conditions for this flow to be minimal.

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1. Introduction

Semigroups of endomorphisms of the circle group $\mathbf{T}$ correspond to multiplicative semigroups of integers. Let Σ be such a semigroup. If Σ is generated by a single integer $a \geqslant 2$ the the flow $(\mathbf{T}, \Sigma)$ is essentially the one-sided shift on a-ary sequences. In this case $(\mathbf{T}, \Sigma)$ has uncountably many minimal subflows. For larger semigroups the situation is different. Calling Σ *lacunary* if all its positive elements are powers of a single integer and *non-lacunary* otherwise, Furstenberg [3, Lemma IV.2, Theorem IV.1] proved the following

THEOREM. *If Σ is non-lacunary then the only infinite closed Σ-invariant subset of* $\mathbf{T}$ *is* $\mathbf{T}$ *itself.*

In [1] this result was generalized for commutative semigroups of endomorphisms of $\mathbf{T}^r$. Namely, those semigroups Σ, possessing the property that the only infinite closed Σ-invariant subset of $\mathbf{T}^r$ is $\mathbf{T}^r$ itself, were fully characterized. It was furthermore shown in [2] that if Π is any semigroup of affine transformations,

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lying above such a semigroup Σ, then Π also has this property. It can now be asked which other semigroups of affine transformations, lying above commutative semigroups of endomorphisms, satisfy this property. In this paper the question is settled for semigroups generated by a pair of affine transformations lying above a single ergodic endomorphism. The theorem stating the conditions under which such a flow possesses the property in question is presented in Section 2 (see Theorem 2.1). The conditions are fairly mild and imply that the flow is even minimal. It also turns out that flows of this type admit in any case minimal sets of positive Hausdorff dimension, and usually also of positive topological dimension. Thus these actions of the free group of two generators $\mathbf{F}_2$ (or free semigroup of two generators) display in this respect a type of behavior different from that of $\mathbf{Z}$- and $\mathbf{R}$-expansive flows (see [4], [5]). The proof of the theorem is carried out in Section 3.

2. The main theorem

The r-dimension torus will be denoted by $\mathbf{T}^r$ and considered as an additive group: $\mathbf{T}^r = \mathbf{R}^4/\mathbf{Z}^r$. Its points are given by column r-vectors. Continuous endomorphisms of $\mathbf{T}^r$ can be lifted to points and to linear transformations of $\mathbf{R}^r$, respectively.

An *affine transformation* of $\mathbf{T}^r$ is a transformation π given by $\pi(x) = \sigma(x) + \alpha$, $x \in \mathbf{T}^r$, for some endomorphism σ of $\mathbf{T}^r$ and $\alpha \in \mathbf{T}^r$. We denote π by $\sigma + \alpha$; π is said to be an affine transformation *lying above* σ, and σ is the endomorphism *lying below* π.

Given a semigroup Π of affine transformations of $\mathbf{T}^r$ we put $\Pi A = \bigcup_{\pi \in \Pi} \pi(A)$ for $A \subseteq \mathbf{T}^r$. A set E is Π-*invariant* if $\Pi E \subseteq E$. A closed Π-invariant set M is *minimal* if it contains no closed Π-invariant proper subset. The flow $(\mathbf{T}^r, \Pi)$ is *minimal*, or more briefly Π is *minimal*, if $\mathbf{T}^r$ is Π-minimal.

The set of eigenvalues of an endomorphism σ of $\mathbf{T}^r$ is denoted by $\operatorname{spec}(\sigma)$. Put

$$V_\lambda(\sigma) = \{v \in \mathbf{C}^r : (\sigma - \lambda I)^r v = 0\}, \qquad \lambda \in \operatorname{spec}(\sigma),$$

where I is the identity endomorphism, and $V_{<1}(\sigma) = \bigoplus_{|\lambda|<1} V_\lambda(\sigma)$. The set $V_{<1}(\sigma) \cap \mathbf{R}^r$ and its projection in $\mathbf{T}^r$ will also be denoted by $V_{<1}(\sigma)$. The union of all closed σ-invariant proper subgroups of $\mathbf{T}^r$ will be denoted by $UH(\sigma)$. With this we can formulate our principal result.

THEOREM 2.1. *Let* σ *be an ergodic endomorphism of* $\mathbf{T}^r$, $\alpha_1, \alpha_2 \in \mathbf{T}^r$ *and* Π *the semigroup of affine transformations generated by* $\sigma + \alpha_1$ *and* $\sigma + \alpha_2$. *Then*

(1) Π *admits minimal sets of positive Hausdorff dimension.*

(2) Π *is minimal if and only if* $\alpha_2 - \alpha_1 \notin V_{<1}(\sigma) + UH(\sigma)$.

THEOREM 2.2. *Let* σ *be an ergodic automorphism of* $\mathbf{T}^r$, $\alpha_1, \alpha_2 \in \mathbf{T}^r$, *and* Π *the semigroup generated by* $\sigma + \alpha_1$, $\sigma + \alpha_2$ *and* $(\sigma + \alpha_1)^{-1}$. *Then* Π *is minimal if and only if* $\alpha_2 - \alpha_1 \notin UH(\sigma)$.

The proof of the last theorem is rather easy. The next section is devoted to that of the first one.

3. The proof of Theorem 2.1.

The following lemma allows a slight reduction in the proof of Theorem 2.1.

LEMMA 3.1. *If Theorem* 2.1 *holds in the case* $\alpha_1 = 0$ *then it holds in the general case as well.*

PROOF. Suppose, in general, that $E \subseteq \mathbf{T}^r$ is a π-invariant set, where $\pi = \tau + \beta$ is an affine transformation. It is then easy to check that for any $\gamma \in \mathbf{T}^r$ the set $E + \gamma$ is π'-invariant, where $\pi' = \tau + \beta + \gamma - \tau(\gamma)$. Thus in our case instead of Π we may consider the semigroup generated by the transformations $\sigma + \alpha_1 + \gamma - \sigma(\gamma)$ and $\sigma + \alpha_2 + \gamma - \sigma(\gamma)$, γ being an arbitrary element of $\mathbf{T}^r$. Since σ is ergodic $I - \sigma$ is surjective, whence an appropriate choice of γ yields the transformations σ and $\sigma + \alpha_2 - \alpha_1$. This proves the lemma.

Consequently, from now on Π will denote the semigroup generated by σ and $\sigma + \alpha$.

PROPOSITION 3.1. *If* $\alpha \notin V_{<1}(\sigma) + UH(\sigma)$ *then* Π *is minimal.*

PROOF. Put $\pi_0 = \sigma$, $\pi_1 = \sigma + \alpha$ and

$$\Pi^{(n)} = \left\{ \pi_{k_1}\pi_{k_2} \cdots \pi_{k_n} : k_i \in \{0,1\}, 1 \leqslant i \leqslant n \right\}, \qquad n = 0, 1, 2, \ldots .$$

Obviously

$$\Pi^{(n)}x = \sigma^n(x) + \Pi^{(n)}0, \qquad x \in \mathbf{T}^r,$$

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so that it suffices to show that $\Pi^{(n)}0 \to \mathbf{T}^r$, as $n \to \infty$, in the Hausdorff metric. Now it is easy to verify that

$$\Pi^{(n)}0 = \sum_{k=0}^{n-1} \{0, \sigma^k(\alpha)\}, \qquad n = 0, 1, 2, \ldots.$$

Define on $\mathbf{T}^r$ a sequence $(\nu_n)_{n=0}^{\infty}$ of measures by

$$\nu_n = \mathop{*}_{k=0}^{n-1} \left(\tfrac{1}{2}\delta_0 + \tfrac{1}{2}\delta_{\sigma^k(\alpha)}\right), \qquad n = 0, 1, 2, \ldots,$$

where δ_t denotes the Dirac measure at t. We have $\operatorname{supp}\nu_n = \Pi^{(n)}0$, and it is therefore sufficient to prove that, as $n \to \infty$,

(3.1) $$\nu_n \to \mu$$

weakly, where μ is the Haar measure on $\mathbf{T}^r$. Suppose, to the contrary, that (3.1) is not satisfied. Using Fourier transforms we find that there exists a non-zero character $\mathbf{l} \in \mathbf{Z}^r$ such that $\hat{\nu}_n(\mathbf{l}) \nrightarrow 0$ as $n \to \infty$. Thus, as $n \to \infty$,

$$\prod_{k=0}^{n-1} \left(\tfrac{1}{2} + \tfrac{1}{2}e^{-2\pi i(\mathbf{l}, \sigma^k(\alpha))}\right) \nrightarrow 0$$

where (u, v) denotes the inner product of the vectors $u, v \in \mathbf{C}^r$. This implies that, as $n \to \infty$,

(3.2) $$(\mathbf{l}, \sigma^k(\alpha)) \to 0 \pmod 1$$

It is easy to construct a basis of $\mathbf{C}^r$ having the following properties:

(a) σ is in Jordan form with respect to B;

(b) if $v \in B \cap V_\lambda(\sigma)$ for a real $\lambda \in \operatorname{spec}(\sigma)$ then $v \in \mathbf{R}^r$;

(c) if $v \in B$ then $\bar{v} \in B$, where the bar denotes complex-conjugation.

Evidently, a vector $u \in \mathbf{C}^r$ actually lies in $\mathbf{R}^r$ if and only if in its representation $u = \sum_{v \in B} t_v v$ with respect to the basis B we have $t_{\bar{v}} = \bar{t}_v$ for every $v \in B$. Given any $\lambda \in \operatorname{spec}(\sigma)$ put

$$h(\lambda) = \max\left\{ j\text{: there exists } v \in V_\lambda(\sigma) \text{ such that } \left(\mathbf{l}, (\sigma - \lambda I)^j v\right) \neq 0\right\}.$$

It is clear that if λ and λ' are conjugate over $\mathbf{Q}$ then $h(\lambda) = h(\lambda')$. Fix an eigenvalue λ of σ. Suppose $h(\lambda) = m$. Take a vector $v^{(0)} \in B \cap V_\lambda(\sigma)$ with $(\mathbf{l}, (\sigma - \lambda I)^m v^{(0)}) \neq 0$. Let $v^{(j)} = (\sigma - \lambda I)^j v^{(0)}$ for $j = 0, 1, \ldots, m$. Then

$$\sigma^k(v^{(j)}) = \sum_{i=0}^{k} \binom{k}{i} \lambda^{k-i}(\sigma - \lambda I)^i v^{(j)} = \sum_{i=0}^{m-j} \binom{k}{i} \lambda^{k-i} v^{(j+i)} + u^{k,j},$$
$$k \geq 0, \quad 0 \leq j \leq m,$$

where $(\mathbf{l}, u^{k,j}) = 0$. Employing only linear combinations of the vectors $v^{(j)}$, $0 \leq j \leq m$, we observe that for any polynomial $P \in \mathbf{C}[x]$ of degree not exceeding

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m there exists a vector $v \in V_\lambda(\sigma)$ such that

$$(3.3) \qquad (\mathbf{l}, \sigma^k(v)) = P(k)\bar{\lambda}^k, \qquad k = 0, 1, 2, \ldots.$$

On the other hand, for any $v \in V_\lambda(\sigma)$, (3.3) is satisfied for some $P \in \mathbf{C}[x]$ with $\deg P \leqslant m$.

Returning to our sequence $(\mathbf{l}, \sigma^k(\alpha))$ we get

$$(\mathbf{l}, \sigma^k(\alpha)) = \sum_{\lambda \in \operatorname{spec}(\sigma)} P_\lambda(k)\lambda^k, \qquad k = 0, 1, 2, \ldots,$$

for some polynomials P_λ, $\lambda \in \operatorname{spec}(\sigma)$, with $\deg P_\lambda \leqslant h(\lambda)$ and $P_{\bar{\lambda}} = \bar{P}_\lambda$ for each $\lambda \in \operatorname{spec}(\sigma)$. By (3.2) we have, as $k \to \infty$, that

$$\sum_{\lambda \in \operatorname{spec}(\sigma)} P_\lambda(k)\lambda^k \to 0 \pmod 1.$$

In view of some results of Pisot [6] this implies that, denoting $K = \mathbf{Q}(\operatorname{spec}(\sigma))$, we have

(a) if $|\lambda| \geqslant 1$ then $P_\lambda \in \mathbf{Q}(\lambda)[x]$,

(b) if $|\lambda_1|, |\lambda_2| \geqslant 1$ and $\psi(\lambda_1) = \lambda_2$ for some $\psi \in \operatorname{Gal}(K/\mathbf{Q})$ then $\psi(P_{\lambda_1}) = P_{\lambda_2}$.

From our earlier observations concerning the general form of the sequences $((\mathbf{l}, \sigma^k v))_{k=0}^\infty$, $v \in V_\lambda(\sigma)$, it now follows that a vector $\beta \in V_{<1}(\sigma)$ can be found such that

$$(\mathbf{l}, \sigma^k(\alpha - \beta)) = \sum_{\lambda \in \operatorname{spec}(\sigma)} P'_\lambda(k)\lambda^k, \qquad k = 0, 1, 2, \ldots,$$

where $P'_\lambda = P_\lambda$ if $|\lambda| \geqslant 1$ and the two properties (a) and (b) are valid for all the polynomials P'_λ, $\lambda \in \operatorname{spec}(\sigma)$. Putting $\gamma = \alpha - \beta$ we see therefore that $(\mathbf{l}, \sigma^k(\gamma))$ is rational for all k. Since the sequence $(\mathbf{l}, \sigma^k(\gamma))$ satisfies the recurrence derived from the characteristic polynomial of σ, it follows that there exists a positive integer d such that $(d\mathbf{l}, \sigma^k(\gamma)) \in \mathbf{Z}$ for all k. Let Γ denote the subgroup of $\mathbf{Z}^r$ generated by $d\mathbf{l}$, and H its annihilator in $\mathbf{T}^r$. Set $H_\sigma = \bigcap_{k=0}^\infty \sigma^{-k}(H)$. H_σ is a closed σ-invariant proper subgroup of $\mathbf{T}^r$ containing γ. Hence

$$\alpha = \beta + \gamma \in V_{<1}(\sigma) + UH(\sigma)$$

leading to a contradiction. This completes the proof.

We denote the Hausdorff dimension of a set $A \subseteq \mathbf{T}^r$ by $\dim A$.

Proposition 3.2. *If $\alpha \in V_{<1}(\sigma) + UH(\sigma)$ then there exists a Π-minimal subset M of $\mathbf{T}^r$ with* $\dim M > 0$.

Proof. Express α in the form

$$(3.4) \qquad \alpha = \beta + \gamma, \qquad \beta \in V_{<1}(\sigma), \qquad \gamma \in H,$$

where H is a closed σ-invariant proper subgroup of $\mathbf{T}^r$.

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We first construct a closed subset of $\mathbf{T}^r$, invariant under the transformations σ and $\sigma + \beta$, as follows. Denote

$$\rho = \max\{|\lambda| : \lambda \in \operatorname{spec}(\sigma),\ |\lambda| < 1\}.$$

Choose $\varepsilon > 0$ so that $\rho + \varepsilon < 1$. Let $\|\cdot\|$ be some norm on $\mathbf{R}^r$. Bringing σ into Jordan form we easily find that

$$\|\sigma^n(\beta)\| \leqslant C(\rho + \varepsilon)^n, \qquad n = 0, 1, 2, \ldots,$$

for an appropriately chosen constant C. Putting $\Delta = \{0, 1\}^{\mathbf{N}}$ we see that function $F: \Delta \to \mathbf{R}^r$ given by

$$F(\eta) = \sum_{i=1}^{\infty} \eta_i \sigma^{i-1}(\beta), \qquad \eta = (\eta_1, \eta_2, \ldots) \in \Delta,$$

is well-defined and continuous. Denote by P the natural projection of $\mathbf{R}^r$ on $\mathbf{T}^r$. Let $S = P(F(\Delta))$. Then S is clearly invariant under both σ and $\sigma + \beta$. Set

$$\Delta_m = \{\eta \in \Delta : \eta_1 = \eta_2 = \cdots = \eta_m = 0\}, \quad m \in \mathbf{N},$$
$$S_m = P(F(\Delta_m)), \quad m \in \mathbf{N}.$$

Then S_m is closed and σ-invariant, and its diameter can be made arbitrarily small by selecting m sufficiently large. We observe that $S = \bigcup_{i=1}^{2^m} (S_m + x_i)$ for appropriately chosen $x_i \in S$, $1 \leqslant i \leqslant 2^m$.

Suppose first that H in (3.4) is finite. Let M be any σ-minimal set with $\dim M > 0$. The set $E = M + S + H$ is a closed Π-invariant subset of $\mathbf{T}^r$. For sufficiently large m the set $M + S_m$ is a proper subset of $\mathbf{T}^r$, whence it is of measure 0. It follows that E is also a proper subset of $\mathbf{T}^r$. It now suffices to show that if $M_1 \subseteq E$ is any Π-minimal set then $\dim M_1 > 0$. Select a point $x \in M_1$. Replacing x by any limit point of the sequence $(\sigma^n(x))_{n=0}^{\infty}$ we may assume that $x \in M + H$. The σ-minimality of M now implies that $(y + H) \cap M_1 \neq \varnothing$ for every $y \in M$. Thus $M \subseteq M_1 + H$, which gives $\dim M_1 \geqslant \dim M > 0$.

Now suppose that H is infinite. Write

$$\alpha = \beta + \gamma_C + \gamma_F, \qquad \beta \in V_{<1}(\sigma), \qquad \gamma_C \in C, \qquad \gamma_F \in F, \tag{3.5}$$

where C is an infinite closed connected σ-invariant proper subgroup of $\mathbf{T}^r$ and F a finite σ-invariant subgroup of $\mathbf{T}^r$. We may assume that no decomposition of this form of α is possible with C replaced by another subgroup of $\mathbf{T}^r$ of a smaller dimension. Consider the semigroup Π_C of affine transformations of C generated by the restrictions of σ and of $\sigma + \gamma_C$ to C. since the restriction of σ to C is easily seen to be ergodic it follows from Proposition 3.1 that Π_C is minimal. In fact, otherwise we obtain a decomposition of γ_C analogous to the decomposition (3.5) of α. Such a decomposition of γ_C leads to a decomposition of α with the γ_C term replaced by an element lying in a σ-invariant proper subgroup of C, thus contradicting our choice of C.

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The set $E = C + S + F$ is obviously a closed Π-invariant subset of $\mathbf{T}^r$, and as in the former case is easily shown to be properly contained in $\mathbf{T}^r$. Let $M \subseteq E$ be Π-minimal. The minimality of Π_C implies that for every $c \in C$ there exist $s \in S$ and $f \in F$ such that $c + s + f \in M$. Select now any $c \in C$. The restriction of σ to C being onto, we can find a sequence $(c_n)_{n=1}^{\infty}$ in C with $\sigma^n(c_n) = c$. Choose sequences $(s_n)_{n=1}^{\infty}$ and $(f_n)_{n=1}^{\infty}$ in S and in F, respectively, such that $c_n + s_n + f_n \in M$. Taking any limit point of the sequence $(\sigma^n(c_n + s_n + f_n))$ we arrive at a point of $c + F$ lying in M. Consequently $M + F \supseteq C$, which gives $\dim M \geqslant \dim C > 0$. This completes the proof.

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