

A NOTE ON RESOLVENT CONVERGENCE ON A THIN DOMAIN

RICARDO P. SILVA

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Abstract

In this paper we provide a new proof of strong convergence of resolvent operators associated with boundary value problems on thin domains.

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1. Introduction

Since the pioneer work by Hale and Raugel [15], the investigation of the asymptotic behaviour of the nonlinear dynamics of semilinear reaction–diffusion equations in thin domains has received the attention of several authors, mainly interested in the continuity of global attractors with respect to specific metrics; see [1, 7, 10–19] and their references.

Arrieta and Carvalho [3] and Carvalho and Piskarev [9] systematised a process to obtain the continuity of global attractors with respect to singular perturbations. Their ideas are based on a rigorous study of convergence properties of the linear part of the equation. Motivated by their ideas, in this paper we present a simple and direct proof of the convergence of resolvent operators of the singular problem considered in [17]. Although we are aware that the result itself has been proved by other authors, for example [1, 17], the point is that our combination of ideas presented in [1, 4] opens up the possibility of treating other classes of singular problems, such as those in [2, 4, 8].

The paper is organised as follows: in Section 2 we present the functional analytic setting of the problem and we state the main result, Theorem 2.1, which is proved in Section 3.

2. Preliminaries

Let Ω be a smooth and bounded domain in \mathbb{R}^{m+n} . A generic point of Ω will be denoted by $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$ and ϵ will represent a small positive parameter.

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Considering the squeezing operator $\Phi_\epsilon : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^{m+n}$ defined by $\Phi_\epsilon(x, y) = (x, \epsilon y)$, we set up the *thin domain* $\Omega_\epsilon := \Phi_\epsilon(\Omega)$, $\epsilon \in (0, 1)$, and we analyse convergence properties of the solutions of the family of linear equations

$$\begin{aligned} -\Delta u^\epsilon + u^\epsilon &= f^\epsilon \quad \text{in } \Omega_\epsilon, \\ \frac{\partial u^\epsilon}{\partial \eta_\epsilon} &= 0 \quad \text{on } \partial\Omega_\epsilon, \end{aligned}$$

where η_ϵ denotes the outward unitary normal vector field to $\partial\Omega_\epsilon$.

In order to capture the limiting behaviour of u^ϵ as $\epsilon \rightarrow 0$, we perform a dilatation of the domain Ω_ϵ by a factor ϵ in the y -direction. Thus we obtain in Ω the equivalent equations

$$\begin{aligned} -\Delta_x u - \frac{1}{\epsilon^2} \Delta_y u + u &= \hat{f}^\epsilon \quad \text{in } \Omega, \\ \nabla_x u \cdot \eta_x + \frac{1}{\epsilon^2} \nabla_y u \cdot \eta_y &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $\hat{f}^\epsilon(x, y) = f^\epsilon(x, \epsilon y)$, and $\eta = (\eta_x, \eta_y)$ is the outward unitary normal vector field to $\partial\Omega$.

The relation between the spaces of functions defined in Ω_ϵ and those defined in Ω is given by

$$\Phi_\epsilon^* : L^2(\Omega_\epsilon) \rightarrow L^2(\Omega), \quad u \mapsto u \circ \Phi_\epsilon,$$

an isomorphism which restricts an isomorphism from $H^1(\Omega_\epsilon)$ onto $H^1(\Omega)$.

Stressing the fact that the domains Ω_ϵ vary in accordance with the parameter ϵ , collapsing themselves to a lower dimensional subset as $\epsilon \rightarrow 0$, to preserve the relative capacity of a measurable subset of Ω_ϵ we perform a dilatation of the $(m + n)$ -dimensional Lebesgue measure by a factor $1/\epsilon$. With this measure, namely

$$\rho_\epsilon := \frac{1}{\epsilon} \times \text{Lebesgue measure},$$

we introduce the Lebesgue and Sobolev spaces $L^2(\Omega_\epsilon; \rho_\epsilon)$ and $H^1(\Omega_\epsilon; \rho_\epsilon)$, respectively. By considering in $H^1(\Omega)$ the equivalent norm

$$\|u\|_\epsilon := \left(\int_\Omega \left(|\nabla_x u|^2 + \frac{1}{\epsilon^2} |\nabla_y u|^2 + |u|^2 \right) dx dy \right)^{1/2},$$

it follows that the isomorphism

$$\Phi_\epsilon^* : H^1(\Omega_\epsilon; \rho_\epsilon) \rightarrow H^1_\epsilon(\Omega)$$

is indeed an isometry, where $H^1_\epsilon(\Omega) := (H^1(\Omega), \|\cdot\|_\epsilon)$. A similar observation can be made regarding $\Phi_\epsilon^* : L^2(\Omega_\epsilon; \rho_\epsilon) \rightarrow L^2(\Omega)$.

We notice that $H^1_\epsilon(\Omega)$ is a Hilbert space with respect to the inner product

$$\begin{aligned} a_\epsilon &: H^1_\epsilon(\Omega) \times H^1_\epsilon(\Omega) \rightarrow \mathbb{R}, \\ a_\epsilon(u, v) &:= \int_\Omega \left(\nabla_x u \cdot \nabla_x v + \frac{1}{\epsilon^2} \nabla_y u \cdot \nabla_y v + uv \right) dx dy. \end{aligned}$$

This inner product induces a self-adjoint unbounded linear operator with compact resolvent

$$A_\epsilon : D(A_\epsilon) \subset L^2(\Omega) \rightarrow L^2(\Omega),$$

defined by

$$a_\epsilon(u, v) = \langle A_\epsilon u, v \rangle, \quad \forall v \in H^1_\epsilon(\Omega), \forall u \in D(A_\epsilon),$$

where $\langle u, v \rangle = \int_\Omega uv \, dx \, dy$. It follows from the smoothness of $\partial\Omega$ that

$$D(A_\epsilon) = \left\{ u \in H^2(\Omega) : \nabla_x u \cdot \eta_x + \frac{1}{\epsilon^2} \nabla_y u \cdot \eta_y = 0 \text{ almost everywhere in } \partial\Omega \right\},$$

and

$$A_\epsilon u = -\Delta_x u - \frac{1}{\epsilon^2} \Delta_y u + u, \quad \forall u \in D(A_\epsilon).$$

Now, for $u \in H^1(\Omega)$, we notice that

$$a_\epsilon(u, u) \xrightarrow{\epsilon \rightarrow 0} \begin{cases} \int_\Omega (|\nabla_x u|^2 + |u|^2) \, dx \, dy & \text{if } \nabla_y u = 0, \\ \infty & \text{if } \nabla_y u \neq 0. \end{cases}$$

Therefore, in order to establish a formal limit for a_ϵ , it is necessary to work in a proper subspace of $H^1(\Omega) \times H^1(\Omega)$. In [17] the authors introduce the subspace $H^1_s(\Omega) := \{u \in H^1(\Omega) : \nabla_y u = 0 \text{ almost everywhere in } \Omega\}$ as well as the bilinear form $a_0 : H^1_s(\Omega) \times H^1_s(\Omega) \rightarrow \mathbb{R}$ given by

$$a_0(u, v) := \int_\Omega (\nabla_x u \cdot \nabla_x v + uv) \, dx \, dy, \quad \forall u, v \in H^1_s(\Omega).$$

The subspace $H^1_s(\Omega)$ is an infinite dimensional closed subspace of $H^1(\Omega)$.

Defining $L^2_s(\Omega) := \overline{H^1_s(\Omega)}^{L^2(\Omega)}$, it follows that a_0 defines a self-adjoint unbounded linear operator with compact resolvent

$$A_0 : D(A_0) \subset L^2_s(\Omega) \rightarrow L^2_s(\Omega)$$

defined by

$$a_0(u, v) = \langle A_0 u, v \rangle, \quad \forall v \in H^1_s(\Omega), \forall u \in D(A_0).$$

Analogously, it follows from the smoothness of $\partial\Omega$ that

$$D(A_0) = \left\{ u \in H^2(\Omega) \cap H^1_s(\Omega) : \frac{\partial u}{\partial \eta} = 0 \text{ almost everywhere in } \partial\Omega \right\},$$

and

$$A_0 u = -\Delta_x u + u, \quad \forall u \in D(A_0).$$

Now we have the elements to state the main result of this paper.

THEOREM 2.1. *Given a bounded family $f^\epsilon \in L^2(\Omega)$, there exists $f^0 \in L^2(\Omega)$ such that, up to subsequence,*

$$\lim_{\epsilon \rightarrow 0} \|A_\epsilon^{-1} f^\epsilon - A_0^{-1} f^0\|_\epsilon = 0.$$

REMARK 2.2. After these considerations, at the limit where $\epsilon = 0$, we must seek solutions $u \in H_s^1(\Omega)$ of the equations

$$\begin{aligned} -\Delta_x u + u &= f^0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \eta} &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

If each x -section $\Omega_x := \{y \in \mathbb{R}^n : (x, y) \in \Omega\}$ is a connected set, then such solutions are of the form $u(x, y) = v(x)$ almost everywhere in Ω , for a suitable function v defined in $P(\Omega) := \{x \in \mathbb{R}^m : \Omega_x \neq \emptyset\}$. This justifies the expression ‘*reduction of dimension*’ in such problems. Details can be found in [17].

3. Convergence results

In this section we are concerned with convergence properties of solutions of (2.1). This means that, given $f^\epsilon \in L^2(\Omega)$, we will deal with the family of linear problems

$$A_\epsilon u^\epsilon = f^\epsilon,$$

investigating convergence of u^ϵ as $\epsilon \rightarrow 0$.

At first glance, we are able to prove the following weak convergence result.

LEMMA 3.1. *Let $f^\epsilon \in L^2(\Omega)$ be such that $\|f^\epsilon\|_{L^2(\Omega)} \leq c$ for some constant c not dependent on $\epsilon \in (0, 1)$. There exists $f^0 \in L_s^2(\Omega)$ and $u^0 = A_0^{-1} f^0 \in H_s^1(\Omega)$ such that, up to subsequence,*

$$A_\epsilon^{-1} f^\epsilon \xrightarrow{\epsilon \rightarrow 0} u^0, \quad w - H^1(\Omega) \quad \text{and} \quad s - L^2(\Omega).$$

PROOF. Setting $u^\epsilon = A_\epsilon^{-1} f^\epsilon \in H_\epsilon^1(\Omega)$,

$$a_\epsilon(u^\epsilon, v) = \langle f^\epsilon, v \rangle, \quad \forall v \in H_\epsilon^1(\Omega).$$

In particular, taking $v = u^\epsilon$ as a test function, it follows from Hölder’s inequality that

$$\|u^\epsilon\|_{H^1(\Omega)}^2 \leq a_\epsilon(u^\epsilon, u^\epsilon) \leq \|f^\epsilon\|_{L^2(\Omega)} \|u^\epsilon\|_{H^1(\Omega)},$$

which means

$$\|u^\epsilon\|_{H^1(\Omega)} \leq c, \quad \forall \epsilon \in (0, 1).$$

Since $H^1(\Omega)$ is a reflexive space and $L^2(\Omega) \hookrightarrow H^1(\Omega)$ compactly, taking a subsequence if necessary, there exists $u^0 \in H^1(\Omega)$ such that $u^\epsilon \xrightarrow{\epsilon \rightarrow 0} u^0$, weakly

in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. Moreover, noticing that $a_\epsilon(u^\epsilon, u^\epsilon) \leq c^2$, one has that $\|\nabla_y u^\epsilon\|_{L^2(\Omega)^n} \leq c\epsilon$. Therefore, $\nabla_y u^\epsilon \xrightarrow{\epsilon \rightarrow 0} 0$ in $L^2(\Omega)^n$ which means that $\nabla_y u^0 = 0$, proving that actually $u^0 \in H^1_s(\Omega)$.

On the other hand, since $\|f^\epsilon\|_{L^2(\Omega)} \leq c$, there exists $f^0 \in L^2(\Omega)$ such that, up to a subsequence, $f^\epsilon \rightharpoonup f^0$ in $L^2(\Omega)$. Moreover, since

$$\int_{\Omega} (\nabla_x u^\epsilon \cdot \nabla_x \varphi + u^\epsilon \varphi) \, dx \, dy \xrightarrow{\epsilon \rightarrow 0} \int_{\Omega} (\nabla_x u^0 \cdot \nabla_x \varphi + u^0 \varphi) \, dx \, dy, \quad \forall \varphi \in H^1_s(\Omega),$$

and

$$\int_{\Omega} (\nabla_x u^\epsilon \cdot \nabla_x \varphi + u^\epsilon \varphi) \, dx \, dy = \int_{\Omega} f^\epsilon \varphi \, dx \, dy, \quad \forall \varphi \in H^1_s(\Omega),$$

we obtain

$$\int_{\Omega} (\nabla_x u^0 \cdot \nabla_x \varphi + u^0 \varphi) \, dx \, dy = \int_{\Omega} f^0 \varphi \, dx \, dy, \quad \forall \varphi \in H^1_s(\Omega),$$

that is, $u^0 = A_0^{-1} f^0$. □

In the following lemma, similar to [1, Lemma 3.1], we strengthen the previous convergence result.

LEMMA 3.2. *Let $f^\epsilon, u^\epsilon, f^0$ and u^0 be as in Lemma 3.1. Then*

$$\lim_{\epsilon \rightarrow 0} \|u^\epsilon - u^0\|_{H^1(\Omega)} = 0.$$

PROOF. It follows from the weak convergence, $u^\epsilon \rightharpoonup u^0$ in $H^1(\Omega)$ (obtained in Lemma 3.1), that

$$\begin{aligned} \|u^0\|_{H^1(\Omega)}^2 &= \int_{\Omega} (|\nabla_x u^0|^2 + |u^0|^2) \, dx \, dy \\ &\leq \liminf_{\epsilon \rightarrow 0} \int_{\Omega} (|\nabla_x u^\epsilon|^2 + |\nabla_y u^\epsilon|^2 + |u^\epsilon|^2) \, dx \, dy \\ &\leq \limsup_{\epsilon \rightarrow 0} \int_{\Omega} (|\nabla_x u^\epsilon|^2 + |\nabla_y u^\epsilon|^2 + |u^\epsilon|^2) \, dx \, dy \\ &\leq \lim_{\epsilon \rightarrow 0} \int_{\Omega} \left(|\nabla_x u^\epsilon|^2 + \frac{1}{\epsilon^2} |\nabla_y u^\epsilon|^2 + |u^\epsilon|^2 \right) \, dx \, dy \\ &= \lim_{\epsilon \rightarrow 0} \int_{\Omega} f^\epsilon u^\epsilon \, dx \, dy \\ &= \int_{\Omega} f^0 u^0 = \int_{\Omega} (|\nabla_x u^0|^2 + |u^0|^2) \, dx \, dy = \|u^0\|_{H^1(\Omega)}^2, \end{aligned}$$

which proves the statement. □

REMARK 3.3. Although we can obtain Theorem 2.1 immediately from Lemma 3.2, which is actually done in [1], we prefer to use the technique employed in [4–6] to obtain the expected convergence result. We think it is possible in this way to consider more classes of singular perturbations problems as reaction–diffusion equations in dumbbell type domains [4–6] and reaction–diffusion equations with large diffusion phenomena [8], besides estimating their rates of convergence [2].

LEMMA 3.4. *Let f^ϵ , u^ϵ , f^0 and u^0 be as in Lemma 3.1. If*

$$\lambda^\epsilon = \frac{1}{2} \|u^\epsilon\|_\epsilon^2 - \int_\Omega f^\epsilon u^\epsilon \, dx \, dy = \min_{\varphi \in H^1(\Omega)} \left\{ \frac{1}{2} \|\varphi\|_\epsilon^2 - \int_\Omega f^\epsilon \varphi \, dx \, dy \right\},$$

$$\lambda^0 = \frac{1}{2} \|u^0\|_{H^1(\Omega)}^2 - \int_\Omega f^0 u^0 \, dx \, dy = \min_{\varphi \in H^1(\Omega)} \left\{ \frac{1}{2} \|\varphi\|_{H^1(\Omega)}^2 - \int_\Omega f^0 \varphi \, dx \, dy \right\},$$

then $\lim_{\epsilon \rightarrow 0} \lambda_\epsilon = \lambda_0$.

PROOF. Taking u^0 as a test function for λ^ϵ and recalling that $\nabla_y u^0 = 0$,

$$\begin{aligned} \lambda^\epsilon &\leq \frac{1}{2} \|u^0\|_{H^1(\Omega)}^2 - \int_\Omega f^\epsilon u^0 \, dx \, dy \\ &= \frac{1}{2} \|u^0\|_{H^1(\Omega)}^2 - \int_\Omega f^0 u^0 \, dx \, dy - \int_\Omega (f^\epsilon - f^0) u^0 \, dx \, dy \\ &= \lambda^0 - \int_\Omega (f^\epsilon - f^0) u^0 \, dx \, dy, \end{aligned}$$

which leads to

$$\limsup_{\epsilon \rightarrow 0} \lambda_\epsilon \leq \lambda_0.$$

On the other hand,

$$\begin{aligned} \lambda_0 &= \frac{1}{2} \int_\Omega (|\nabla_x u^0|^2 + |u^0|^2) \, dx \, dy - \int_\Omega f^0 u^0 \, dx \, dy \\ &= \frac{1}{2} \int_\Omega (|\nabla_x u^0 - \nabla_x u^\epsilon + \nabla_x u^\epsilon|^2 + |u^0 - u^\epsilon + u^\epsilon|^2) \, dx \, dy - \int_\Omega f^0 (u^0 - u^\epsilon) \, dx \, dy \\ &\quad - \int_\Omega (f^0 - f^\epsilon) u^\epsilon \, dx \, dy - \int_\Omega f^\epsilon u^\epsilon \, dx \, dy \\ &= \frac{1}{2} \int_\Omega (|\nabla_x u^0 - \nabla_x u^\epsilon|^2 + |u^0 - u^\epsilon|^2) \, dx \, dy + \int_\Omega (\nabla_x u^0 - \nabla_x u^\epsilon) \cdot \nabla_x u^\epsilon \, dx \, dy \\ &\quad + \frac{1}{2} \int_\Omega |\nabla_x u^\epsilon|^2 \, dx \, dy + \int_\Omega (u^0 - u^\epsilon) u^\epsilon \, dx \, dy + \frac{1}{2} \int_\Omega |u^\epsilon|^2 \, dx \, dy \\ &\quad - \int_\Omega f^0 (u^0 - u^\epsilon) \, dx \, dy - \int_\Omega (f^0 - f^\epsilon) u^\epsilon \, dx \, dy - \int_\Omega f^\epsilon u^\epsilon \, dx \, dy \\ &\leq \frac{1}{2} \|u^0 - u^\epsilon\|_{H^1(\Omega)}^2 + \lambda_\epsilon + \int_\Omega (\nabla_x u^0 - \nabla_x u^\epsilon) \cdot \nabla_x u^\epsilon \, dx \, dy + \int_\Omega (u^0 - u^\epsilon) u^\epsilon \, dx \, dy \\ &\quad - \int_\Omega f^0 (u^0 - u^\epsilon) \, dx \, dy - \int_\Omega (f^0 - f^\epsilon) u^\epsilon \, dx \, dy \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{2} \|u^0 - u^\epsilon\|_{H^1(\Omega)}^2 + \lambda^\epsilon + \|\nabla_x u^0 - \nabla_x u^\epsilon\|_{L^2(\Omega)} \|\nabla_x u^\epsilon\|_{L^2(\Omega)} + \|u^0 - u^\epsilon\|_{L^2(\Omega)} \|u^\epsilon\|_{L^2(\Omega)} \\ &\quad + \|f^0\|_{L^2(\Omega)} \|u^0 - u^\epsilon\|_{L^2(\Omega)} - \int_{\Omega} (f^\epsilon - f^0) u^\epsilon \, dx \, dy, \end{aligned}$$

which leads to $\lambda^0 \leq \liminf_{\epsilon \rightarrow 0} \lambda^\epsilon$, proving the lemma. \square

COROLLARY 3.5 (Proof of Theorem 2.1). *If f^ϵ , u^ϵ , f^0 and u^0 are as in Lemma 3.1 then*

$$\lim_{\epsilon \rightarrow 0} \|u^\epsilon - u^0\|_\epsilon = 0.$$

PROOF. Inspired by the proof of Lemma 3.4 and recalling that $\nabla_y u^0 = 0$,

$$\begin{aligned} \lambda^\epsilon &= \frac{1}{2} \|u^\epsilon\|_\epsilon^2 - \int_{\Omega} f^\epsilon u^\epsilon \, dx \, dy \\ &= \frac{1}{2} \|u^\epsilon - u^0 + u^0\|_\epsilon^2 - \int_{\Omega} f^\epsilon (u^\epsilon - u^0) \, dx \, dy - \int_{\Omega} (f^\epsilon - f^0) u^0 \, dx \, dy - \int_{\Omega} f^0 u^0 \, dx \, dy \\ &= \frac{1}{2} \|u^\epsilon - u^0\|_\epsilon^2 + \frac{1}{2} \|u^0\|_{H^1(\Omega)}^2 + \int_{\Omega} (\nabla_x u^\epsilon - \nabla_x u^0) \cdot \nabla_x u^0 \, dx \, dy + \int_{\Omega} (u^\epsilon - u^0) u^0 \, dx \, dy \\ &\quad - \int_{\Omega} f^0 u^0 \, dx \, dy - \int_{\Omega} f^\epsilon (u^\epsilon - u^0) \, dx \, dy - \int_{\Omega} (f^\epsilon - f^0) u^0 \, dx \, dy \\ &= \frac{1}{2} \|u^0 - u^\epsilon\|_\epsilon^2 + \lambda^0 + \int_{\Omega} (\nabla_x u^0 - \nabla_x u^\epsilon) \cdot \nabla_x u^\epsilon \, dx \, dy + \int_{\Omega} (u^0 - u^\epsilon) u^\epsilon \, dx \, dy \\ &\quad - \int_{\Omega} f^\epsilon (u^\epsilon - u^0) \, dx \, dy - \int_{\Omega} (f^\epsilon - f^0) u^0 \, dx \, dy. \end{aligned}$$

The result follows by arguing as in Lemma 3.4, recalling that $\lambda^\epsilon \xrightarrow{\epsilon \rightarrow 0} \lambda^0$. \square

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RICARDO P. SILVA, Instituto de Geociências e Ciências Exatas,
UNESP - Univ Estadual Paulista, Departamento de Matemática,
13506-900, Rio Claro SP, Brazil
e-mail: rpsilva@rc.unesp.br