Proceedings of the Edinburgh Mathematical Society (1996) 39, 271–284 ©

THE SOLUTION VIA MONOTONICITY METHODS OF SOME NONSCALAR REACTION–DIFFUSION PROBLEMS

by MANUEL DELGADO

(Received 24th May 1994)

We present some applications of monotonicity methods to the solution of certain nonscalar reaction–diffusion problems. In particular we prove existence under appropriate conditions and we introduce a convergent algorithm.

1991 *Mathematics subject classification*: 35J55, 47H07, 47N60, 92D25.

1. Introduction

During the past few years nonlinear systems of reaction–diffusion equations have been intensively studied, motivated by recent developments in ecology, biology, biochemistry, etc. In many cases, these systems are of the form

$$\begin{cases} U_t - d_1 \Delta U = a_1(x,t)U - b_1(x,t)U^2 - c_1(x,t)UV \\ V_t - d_2 \Delta V = a_2(x,t)V - b_2(x,t)V^2 - c_2(x,t)UV \end{cases}$$

Here, the unknowns U and V are functions of x and t. These equations are supposed to be satisfied in a cylinder $x \in \bar{\Omega}$, $0 < t < +\infty$, where $\Omega \subset \mathbb{R}^N$ is an open bounded and regular domain. Of course, the equations are supplemented by appropriate boundary conditions. The functions a_i, b_i and c_i $(i=1,2)$ are smooth and nonnegative and the diffusion constants d_i $(i=1,2)$ are positive.

The previous system is rather representative, not only because it can be used to model various situations (as will be shown at once), but also because it deals with quadratic approximations of the second members of more general systems.

One of the main questions that can be considered in connection with these systems concerns the existence and multiplicity of stationary solutions (this is meaningful only if the previous coefficients do not depend on t, in particular if they are constant). In most cases, the interest is centred on positive solutions, the only physically meaningful solutions: densities of populations, concentrations, ...

For instance, systems of the kind

$$\begin{cases} -d_1\Delta U = a_1 U - b_1 U^2 - c_1 UV \\ -d_2\Delta V = a_2 V - b_2 V^2 - c_2 UV \end{cases} \tag{1.1}$$

model the situation in which two competing species, whose population densitities are U and V, coexist in Ω. The diffusion coefficients d_1 and d_2, are related to the diffusion of the species in Ω (here, diffusion means motion from high to low population density regions). a_1 and a_2 are the corresponding growth rates and b_1 and b_2 account for the self-regulation of each species (they cannot grow up indefinitely). Finally, the signs of the coefficients of UV in the equations determine the kind of interaction between the species. The negative sign in both equations indicate that these are competing species. This model is referred to in the literature as the Volterra–Lotka model with diffusion [**3**, **6**].

Other types of interaction between the species are very similar. Again, the differences are indicated by the sign of the last terms in the equations. For example, the sign will be negative in the first equation and positive in the second one in the case of a prey–predator model.

Other related (but different) models can also be mentioned. Thus, [**4**] (following the methods of [**3**]) deals with

$$\begin{cases} -d_1\Delta U = a_1 U - b_1 U^2 - H_1(U,V) \\ -d_2\Delta V = a_2 V - b_2 V^2 - H_2(U,V) \end{cases}$$

where $H(U,V)$ is the Holling–Tanner term, given by

$$H_i(U,V) = \frac{c_i UV}{1+mU} \quad i=1,2.$$

Once again, this problem has to be completed with suitable boundary conditions. This also models prey–predator competition with diffusion. The Holling–Tanner interaction term is introduced because with the "usual" term $-c_1UV$, one has $\lim_{U\to\infty} c_1UV = +\infty$ for any fixed $V>0$, i.e. predators are able to consume preys at an infinitely large rate. With the use of H

$$\lim_{U\to\infty} \frac{c_1UV}{1+mU} = \frac{c_1V}{m}$$

and this difficulty disappears.

In this paper, we will consider Dirichlet problems for reaction–diffusion systems of the kind (1.1) with constant coefficients a_i, b_i and c_i. Most papers dealing with these problems are devoted to the existence and multiplicity of positive solutions in terms of the various coefficients which appear in the model. As usual, these will be called "coexistence states". It can be proved (see [**10**]) that nonnegative nontrivial solutions are in fact strictly positive.

Notice that, in (1.1), the coefficients a_i, b_i and c_i can be reduced to four parameters. Indeed, the change of variables

$$U=\alpha u, \quad V=\beta v,$$

leads to the new system

$$\begin{cases} -d_1\Delta u=a_1u-b_1\alpha u^2-c_1\beta uv \\ -d_2\Delta v=a_2v-b_2\beta v^2-c_2\alpha uv \end{cases} \tag{1.2}$$

Thus, if we choose

$$\alpha=\frac{d_2}{c_2}, \quad \text{and} \quad \beta=\frac{d_1}{c_1},$$

we arrive at

$$\begin{cases} -\Delta u=au-bu^2-uv \\ -\Delta v=cv-dv^2-uv \end{cases} \tag{1.3}$$

Contrarily, by choosing

$$\alpha=\frac{d_1}{b_1}, \quad \text{and} \quad \beta=\frac{d_2}{b_2},$$

we obtain

$$\begin{cases} -\Delta u=au-u^2-buv \\ -\Delta v=cv-v^2-duv \end{cases} \tag{1.4}$$

with a suitable change of notation for the coefficients. We find these simplifications in some papers (for instance, see [5]; see also [6]).

Also, notice that a particular (important) family of solutions to (1.3) can be obtained by solving the single equation

$$-\Delta u=au-bu^2$$

in Ω and setting $v\equiv 0$. These are the semitrivial solutions. A similar family of semitrivial solutions arises assuming that $u\equiv 0$. These solutions are known as "extinction states".

Among the techniques that can be used to analyze the existence of coexistence states, let us mention the following:

(a) **Comparison techniques.** Sufficient conditions for the existence of co-existence states which involve sub-supersolutions are derived, for example, in [6]. Also, when $a=c$ in (1.3) necessary and sufficient conditions are given.

(b) **Decoupling techniques.** In [3, 9] the system is reduced to a single equation; essentially this is made by fixing one of the functions, say u, solving the second equation (this gives $v(u)$) and, then, replacing v by $v(u)$ for v in the first equation. Then, global bifurcation results (mainly due to P. H. Rabinowitz) can be applied.

(c) **Topological techniques.** By this we mean index theory [7], topological degree theory [**10**], etc. . . .

From the viewpoint of numerical computations, the monotonicity methods associated with the sub-supersolutions techniques have the advantage that they lead to iterative algorithms. Consequently, they are appropriate for computing numerical approximations.

In this paper we first prove that, under certain assumptions (concerning only the coefficients and the size of Ω), the well known necessary conditions for the existence of coexistence states (see [**6**]) are also sufficient conditions. Second, we see that our arguments lead to an iterative algorithm which can be used for the computation of the coexistence states. Also monotonicity provides some error estimates.

Regarding one of the coefficients as a parameter, the bifurcation phenomenon arises in a natural way. This happens because there exist semitrivial solutions of the system once the parameter attains a first critical value and coexistence states at a second one. In a forthcoming paper we will study the finite-dimensional approximation of the branches of nonsingular solutions which emanate from bifurcation points.

2. The main result: the existence of coexistence states

We consider the general problem

$$\begin{cases} -\Delta u = au - bu^2 - uv \\ -\Delta v = cv - dv^2 - uv \end{cases} \text{in } \Omega, \tag{2.1}$$

$$u\rfloor_{\partial\Omega} = v\rfloor_{\partial\Omega} = 0.$$

We assume the following is satisfied:

$$a > \lambda_1 \quad \text{and} \quad c > \lambda_1$$

Here λ_1 is the first eigenvalue of $-\Delta$ in Ω (it is well known that these inequalities are necessary for existence; see [**6**]). Notice that these necessary conditions restrict the size of the domain, because if it is sufficiently small then it can be possible that $\lambda_1 < \min(a,c)$. We want to give sufficient conditions about the size of Ω to secure the existence of positive solutions which can be determined by means of a monotone algorithm. We could state them in terms of bounds for λ_1 but is more intuitive giving conditions on Ω.

For each fixed $v \in C^0(\bar{\Omega})$, consider the nonlinear scalar problem

$$\begin{cases} -\Delta u + bu^2 + v(x)u = au \quad \text{in } \Omega, \\ u\rfloor_{\partial\Omega} = 0. \end{cases} \tag{2.2}$$

It is well known (see [3]) that this possesses exactly one positive solution if $a>\lambda_1(v)$ and that only the trivial solution $u\equiv 0$ is a nonnegative solution if $a\leqq\lambda_1(v)$. Here, we denote by $\lambda_1(v)$ the first eigenvalue of the scalar Dirichlet problem

$$\begin{cases} -\Delta u+v(x)u=\lambda u & \text{in } \Omega, \\ u\rfloor_{\partial\Omega}=0 \end{cases}.$$

As in [9] let us introduce the mapping $B\colon C^0(\bar\Omega)\to C^0(\bar\Omega)$, by setting

$$B(v)=\begin{cases} 0 & \text{if} \quad a\leqq\lambda_1(v) \\ u(v) & \text{if} \quad a>\lambda_1(v) \end{cases} \quad \text{for each } v,$$

with $u(v)$ being the unique solution mentioned before. B is continuous and reverses the order, i.e.

$$v_1\leqq v_2 \Rightarrow B(v_1)\geqq B(v_2).$$

It is then clear that solving (2.1) is equivalent to solving the following non-linear problem:

$$\begin{cases} -\Delta v+B(v)v+dv^2=cv & \text{in } \Omega, \\ v\rfloor_{\partial\Omega}=0. \end{cases} \tag{2.3}$$

Now a unique positive solution exists if $c>\lambda_1(B(v))$. On the contrary, if $c\leqq\lambda_1(B(v))$, then the trivial solution $v\equiv 0$ is the unique nonnegative solution. Hence we introduce the operator $T\colon C^0(\bar\Omega)\to C^0(\bar\Omega)$, with

$$T(v)=\begin{cases} 0 \text{ if } c\leqq\lambda_1(B(v)), \\ \text{the positive solution of } \begin{cases} -\Delta w+B(v)w=cv-dv^2 \text{ in } w \\ w\rfloor_{\partial\Omega}=0. \end{cases} & \text{otherwise.} \end{cases}$$

From this definition and the fact that $a>\lambda_1$, it is readily seen that a fixed point of T is a positive function and provides a solution of (2.1). It follows from standard results (see e.g. [1]) that T is a compact continuous operator from $C^0(\bar\Omega)$ into itself.

The main result in this paper is the following:

Theorem 1. *Consider the reaction–diffusion system*

$$\begin{cases} -\Delta u=au-bu^2-uv \\ -\Delta v=cv-dv^2-uv \end{cases} \text{in } \Omega,$$
$$u\rfloor_{\partial\Omega}=v\rfloor_{\partial\Omega}=0.$$

Assume the following are satisfied:

(H1) $a > \lambda_1 + \frac{c}{2d}$, *with* λ_1 *being the first eigenvalue of* $-\Delta$ *in* Ω,

(H2) $a \leqq c\left(1 - \frac{1}{b+1}\right)$,

(H3) Ω *is in a slab of width* δ, *with*

$$c\delta^2 \leqq 16.$$

Then, there exists at least one coexistence state for this problem. Furthermore, the iterates

$$\begin{cases} v_0 = \frac{c}{2d}, \\ v_{k+1} = T(v_k) \quad \textit{for } k \geqq 1 \end{cases} \tag{2.4}$$

converge monotonically to a fixed point of T.

The proof of this result will be given in the next section. Notice that the sufficient (not necessary) conditions

$$a > \lambda_1 + \frac{c}{d} \quad c > \lambda_1 + \frac{a}{b}$$

for the existence of coexistence states are known (see [**6**]); (H1) is a weaker condition than the first one. (H1) and (H2), that together imply that both coefficients a and c are greater than λ_1, give the conditions about the coefficients and the size of the domain. Also, notice that, in (2.4), the task is reduced (at most) to the solution of two scalar problems similar to (2.2) and (2.3).

In order to prove Theorem 1, the following result, which is due to H. Amann (see [**2**, Corollary 6.2]), will be useful:

Theorem 2. *Let E be an ordered Banach space and let* $[\bar{y}, \hat{y}]$ *be an nonempty order interval in E. Suppose that* $f: [\bar{y}, \hat{y}] \to E$ *is an increasing compact mapping such that* $\bar{y} \leqq f(\bar{y})$ *and* $f(\hat{y}) \leqq \hat{y}$. *Then f has a minimal fixed point* $\bar{x}$ *and a maximal fixed point* $\hat{x}$. *Moreover,* $\bar{x} = \lim_{k\to\infty} f^k(\bar{y})$, $\hat{x} = \lim_{k\to\infty} f^k(\hat{y})$ *and* $\{f^k(\bar{y})\}$ *(resp.* $\{f^k(\hat{y})\}$*) is an increasing (resp. decreasing) sequence.*

We argue as follows:

(a) First, we prove there exists $v_* \in C^1(\bar{\Omega})$ such that $v_* \leqq T(v_*)$.

(b) Then, we will find a nonempty interval $[v_*, v^*]$ in the ordered Banach space $C^0(\bar{\Omega})$ where T is increasing.

(c) Finally, we will check that $T(v^*) \leqq v^*$.

3. The proof of the main result

We first search for a function v_* such that $T(v_*) \geqq v_*$. We will need the following lemma, whose proof is given in an Appendix:

Lemma 3.1. *The boundary value problem for the logistic equation with positive coefficients*

$$\begin{cases} -\Delta z = pz - qz^2 & \text{in } \Omega, \\ z \rfloor_{\partial\Omega} = k \geqq 0, \end{cases}$$

possesses at least one solution if the following conditions are satisfied:

(1) $p > \lambda_1$, *with* λ_1 *being the first eigenvalue of* $-\Delta$ *in* Ω,

(2) $p \geqq qk$.

Furthermore, there exists a solution θ_{pqk} *with*

$$k \leqq \theta_{pqk} \leqq \frac{p}{q}. \tag{3.1}$$

We look for a function v_* of the form $v_* = \alpha\theta_{pq0}$ for some positive α. According to the notation introduced in Lemma 3.1, θ_{pq0} is a solution of a boundary value problem for a logistic equation whose coefficients p and q are chosen appropriately. For $w_* = T(\alpha\theta_{pq0})$, one will have:

$$\begin{cases} -\Delta w_* + B(\alpha\theta_{pq0})w_* = c\alpha\theta_{pq0} - d\alpha^2\theta_{pq0}^2 & \text{in } \Omega, \\ w_* \rfloor_{\partial\Omega} = 0. \end{cases} \tag{3.1a}$$

This problem possesses a constant supersolution $\bar{w}$. On the other hand, $\alpha\theta_{pq0}$ itself is a subsolution provided the following holds:

$$-\alpha\Delta\theta_{pq0} + B(\alpha\theta_{pq0})\alpha\theta_{pq0} - c\alpha\theta_{pq0} + d\alpha^2\theta_{pq0}^2 \leqq 0$$

$$\Leftrightarrow \alpha\theta_{pq0}[p - q\theta_{pq0} + B(\alpha\theta_{pq0}) - c + d\alpha\theta_{pq0}] \leqq 0,$$

i.e. provided p, q and α are chosen such that

$$p - c + (d\alpha - q)\theta_{pq0} + B(\alpha\theta_{pq0}) \leqq 0 \quad \text{in } \Omega. \tag{3.2}$$

Hence, if (3.2) is satisfied, then $v_* = \alpha\theta_{pq0}$ has the desired property. Notice that, setting $u_* = B(\alpha\theta_{pq0})$, (3.2) reads

$$u_* \leqq c - p + (q - d\alpha)\theta_{pq0} \quad \text{in } \Omega. \tag{3.3}$$

By definition, $u_* = B(\alpha\theta_{pq0})$ is the maximal solution to the nonlinear problem

$$\begin{cases} -\Delta u_* + b(u_*)^2 + \alpha\theta_{pq0} u_* = a u_* & \text{in } \Omega, \\ u_* \rfloor_{\partial\Omega} = 0, \end{cases} \tag{3.4}$$

and is positive if $a \leqq \lambda_1(v)$.

Furthermore, if $M > 0$ and $k > 0$ are such that

$$M \geqq \frac{a - p + q\theta_{pqk} - \alpha\theta_{pq0}}{b\theta_{pqk}}, \tag{3.4a}$$

then the function $\bar{u} = M\theta_{pqk}$ is a supersolution of (3.4), i.e.

$$u_* \leqq \bar{u} \quad \text{in } \Omega. \tag{3.5}$$

As a consequence, to show that v_* is a subsolution of (3.1a), it suffices to find positive constants p, q, α, k and M satisfying (3.4a) and

$$M \leqq \frac{c - p + (q - d\alpha)\theta_{pq0}}{\theta_{pqk}}. \tag{3.4b}$$

In other words, all we have to do is to prove that there exist positive constants p, q, α and k such that θ_{pq0} and θ_{pqk} exist and

$$\sup_{x \in \bar{\Omega}} \frac{a - p + q\theta_{pqk} - \alpha\theta_{pq0}}{b\theta_{pqk}} \leqq \inf_{x \in \bar{\Omega}} \frac{c - p + (q - d\alpha)\theta_{pq0}}{\theta_{pqk}}. \tag{3.6}$$

To this end, we will first choose p such that

$$\lambda_1 < p < a, \quad \frac{p(a-p)}{bc - p(b+1)} < p. \tag{3.7}$$

This is possible in view of Hypotheses (H1) and (H2). It is true if

$$a < c\left(1 - \frac{1}{b+1}\right);$$

if

$$a = c\left(1 - \frac{1}{b+1}\right);$$

then

https://doi.org/10.1017/S0013091500023002 Published online by Cambridge University Press

$$\frac{p(a-p)}{bc-p(b+1)}=\frac{p}{(b+1)}$$

and (3.7) is also permitted.

Then, we choose $q>0$ and $k>0$ such that

$$\frac{p(a-p)}{bc-p(b+1)}<qk\leqq p. \tag{3.8}$$

From Lemma 3.1, it is clear that the functions θ_{pq0} and θ_{pqk} exist. Finally, we choose α such that

$$0<\alpha<\frac{q}{d}\min\left(\frac{c}{2p},1\right). \tag{3.9}$$

This leads to the desired inequalities

$$\sup_{x\in\bar{\Omega}}\frac{a-p+q\theta_{pqk}-\alpha\theta_{pq0}}{b\theta_{pqk}}\leqq\frac{a-p}{bk}+\frac{q}{b}$$

$$\leqq\frac{(c-p)q}{p}\leqq\inf_{x\in\bar{\Omega}}\frac{c-p+(q-d\alpha)\theta_{pq0}}{\theta_{pqk}}.$$

Remark 1. Interchanging the role of the equations, a similar analysis shows that (H1), (H2) can be replaced by

$$c>\lambda_1 \quad \text{and} \quad c\leqq\frac{ad}{d+1}=a\left(1-\frac{1}{d+1}\right).$$

Now we will try to find an interval $[v_*,v^*]$ such that $T([v_*,v^*])\subset[v_*,v^*]$ where T is increasing. Assume $v_1\leqq v_2$ and set $T(v_1)=w_1$ and $T(v_2)=w_2$. Then,

$$\begin{cases}-\Delta w_1+B(v_1)w_1=cv_1-dv_1^2\\ -\Delta w_2+B(v_2)w_2=cv_2-dv_2^2\end{cases}.$$

Subtracting, one easily obtains:

$$-\Delta(w_1-w_2)+B(v_1)(w_1-w_2)+[B(v_1)-B(v_2)]w_2$$
$$=c(v_1-v_2)-d(v_1^2-v_2^2).$$

But we know that $B(v_1)\geqq B(v_2)$; consequently,

https://doi.org/10.1017/S0013091500023002 Published online by Cambridge University Press

$$\begin{cases} -\Delta(w_1-w_2)+B(v_1)(w_1-w_2)\leqq(v_1-v_2)[c-d(v_1+v_2)] & \text{in } \Omega, \\ (w_1-w_2)\rfloor_{\partial\Omega}=0. \end{cases}$$

Since $B(v_1)\geqq 0$, the weak maximum principle, leads to the fact that

$$w_1\leqq w_2 \quad \text{in } \Omega,$$

provided

$$c-d(v_1+v_2)\geqq 0 \quad \text{in } \Omega. \tag{3.10}$$

Notice that (3.10) is satisfied if, for example,

$$v_k\leqq\frac{c}{2d} \quad \text{in } \Omega \quad \text{for} \quad k=1,2. \tag{3.11}$$

Accordingly, let us set $c^*\equiv\frac{c}{2d}$ and consider the corresponding interval

$$[v_*,v^*]=\left\{f\in C^0(\bar{\Omega});\quad \alpha\theta_{pq0}(x)\leqq f(x)\leqq\frac{c}{2d}\quad \forall x\in\bar{\Omega}\right\} \tag{3.12}$$

in the ordered Banach space $C^0(\bar{\Omega})$. Notice that, whenever α satisfies (3.9), the interval in (3.12) is nonempty. Indeed, from (3.9) we see that

$$v_*=\alpha\theta_{pq0}\leqq\alpha\frac{p}{q}<\frac{c}{2d}.$$

On the other hand, it is clear that T is increasing in $[v_*,v^*]$.

In order to end the proof of Theorem 1, let us check that under hypothesis (H1)–(H3), one has

$$T\left(\frac{c}{2d}\right)\leqq\frac{c}{2d}.$$

Let us set

$$w^*=T\left(\frac{c}{2d}\right);$$

we will find a regular function $\bar{w}$ such that

$$\begin{cases} -\Delta\bar{w}+B\left(\frac{c}{2d}\right)\bar{w}\geqq\frac{c^2}{4d} & \text{in } \Omega \\ \bar{w}\rfloor_{\partial\Omega}\geqq 0 \end{cases}$$

and $\bar{w} \leqq \frac{c}{2d}$. In that case, from the maximum principle we will have

$$w^* \leqq \bar{w},$$

whence the desired inequality

$$w^* \leqq v^*$$

will hold. Suppose that Ω lies in a slab of width δ, say $0 < x_1 < \delta$. Let us check that

$$\bar{w} = -\frac{c^2}{8d}\left(\frac{\delta}{2} - x_1\right)^2 + \frac{c^2\delta^2}{32d}$$

is a suitable choice; indeed,

$$-\Delta\bar{w} + B\left(\frac{c}{2d}\right)\bar{w} \geqq -\Delta\bar{w} = \frac{c^2}{4d} \quad \text{in } \Omega.$$

On the other hand,

$$\bar{w} \leqq \bar{w}\left(\frac{\delta}{2}\right) = \frac{c^2\delta^2}{32d} \leqq \frac{c}{2d}.$$

For the monotonicity of the eigenvalues, if $\bar{v}$ is the fixed point of T,

$$\lambda_1(\bar{v}) \leqq \lambda_1\left(\frac{c}{2d}\right) = \lambda_1 + \frac{c}{2d} < a$$

and so, $B(\bar{v})$ is positive. Accordingly, Theorem 1 is proven.

Notice that, for suitable domains Ω and suitable coefficients, (H1), (H2) and (H3) are simultaneously satisfied. Indeed, in a slab of width δ, the first eigenvalue λ_1 is given by

$$\lambda_1 = \frac{\pi^2}{\delta^2}.$$

If Ω is an open set in such a slab, then the corresponding first eigenvalue λ_1 satisfies

$$\lambda_1 \leqq \frac{\pi^2}{\delta^2} < \frac{16}{\delta^2}.$$

Consequently, for appropriate positive values of the coefficients a, b, c and d one has

$$\lambda_1 < a \leqq c\left(1 - \frac{1}{b+1}\right) \quad \text{and} \quad c^2\delta^2 < 16.$$

Remark 2. A sufficient condition for (H1) is

$$a > \frac{\pi^2}{\delta^2} + \frac{c}{2d}.$$

Appendix I: proof of Lemma 3.1

We use the sub-supersolution techniques, choosing as a subsolution the function $z_* = \varepsilon\phi_1$, with ϕ_1 being an eigenfunction associated to λ_1 and $\varepsilon > 0$ small enough. On the other hand, the role of a supersolution is played by the constant function $z^* \equiv \frac{p}{q}$.

Of course, $z_* \leqq k$ on $\partial\Omega$. Also,

$$-\Delta z_* - pz_* + qz_*^2 = \varepsilon\lambda_1\phi_1 - p\varepsilon\phi_1 + q\varepsilon^2\phi_1^2 = \varepsilon\phi_1[\lambda_1 - p + q\varepsilon\phi_1] \leqq 0 \quad \text{in } \Omega$$

if ε is sufficiently small (this happens because $p > \lambda_1$). For the function z^*, it is clear that

$$-\Delta z^* - pz^* + qz^{*2} = 0 \quad \text{in } \Omega$$

and $z^* \geqq k$ on $\partial\Omega$. Finally, observe that ε can be chosen in such a way that $z_* \leqq z^*$. Thus, we deduce that a solution θ_{pqk} exists and satisfies

$$0 < \theta_{pqk} \leqq \frac{p}{q} \quad \text{in } \Omega.$$

Finally, remark that $ps - qs^2 \geqq 0$ for $0 \leqq s \leqq p/q$; hence, $-\Delta\theta_{pqk} \geqq 0$ in Ω and, from the weak maximum principle, we also have

$$\theta_{pqk} \geqslant k \quad \text{in } \Omega.$$

Appendix II

In our problem, the coefficients are extremely regular. We can say, however, something about the regularity of the boundary of bounded domain $\Omega \subset \mathbb{R}^N$ that we consider.

It is classical the validity of the results stated above if $\partial\Omega$ is C^2 (see, for example [8]). But we can prove the existence and unicity of solutions of the problem

$$\begin{cases} -\Delta u = f \\ u\rfloor_{\partial\Omega} = 0 \end{cases}$$

and the compacity of the operator defined in (4.2) if $\partial\Omega$ is $C^{1,\gamma}$.

Indeed, following [13], we denote

$$\Omega_\delta = \{[x : x + y \in \Omega \quad \text{if} \quad |y| \leqq \delta\}$$

and for $b \in \mathbb{R}$, $k \in \mathbb{N}$, $0 < \alpha < 1$ such that $k + \alpha + b \geqq 0$ we define

$$C^{(b)}_{k+\alpha} = \{u : \Omega \to \mathbb{R}, \quad \forall \delta > 0, u \in C^{k,\alpha}(\Omega_\delta) \quad \text{and} \quad |u|^{(b)}_{k+\alpha} < +\infty\}$$

where

$$|u|^{(b)}_{k+\alpha} = \sup_{\delta > 0} \delta^{k+\alpha+b} |u|_{k,\alpha,\Omega\delta}.$$

Note that $C^{(-k-\alpha)}_{k+\alpha} = C^{k,\alpha}$.

The above definitions are applicable if Ω is a bounded open set in $\mathbb{R}^N$ with $\partial\Omega \in C^1$. One can prove (see [13, Lemma 2.1]) that, putting $a = k + \alpha$ and $a' = k' + \alpha'$,

Lemma. *If $0 \leqq a' \leqq a$, $a' + b \geqq 0$ and b is not an integer $\leqq 0$, then*

$$|u|^{(b)}_{a'} \leqq C|u|^{(b)}_a$$

for some constant C that may depend on Ω, a, $a'$and b.

This implicates the continuity of the imbedding

$$C^{(b)}_a \hookrightarrow C^{(b)}_{a'}.$$

In particular, we can choose $a = 2 + \alpha$ with $0 < \alpha < 1$, $a' = 1 + \gamma$ with $0 < \gamma < 1$ and $b = -a' = -1 - \gamma$. Results the continuity of the imbedding

$$C^{(-1-\gamma)}_{2+\alpha} \hookrightarrow C^{1,\gamma}.$$

On the other hand, in the same paper there is proven a result of existence and uniqueness of solutions for the problem

$$\begin{cases} -\Delta u = f \\ u\rfloor_{\partial\Omega} = 0 \end{cases}$$

when $\partial\Omega \in C^{1,\gamma}$. It is justified (see [**13**], Theorem 6.1).

Theorem. *Let Ω be an open bounded whose boundary is $C^{1,\gamma}$. The above problem of Dirichlet has a unique solution $u \in C^{(-b)}_{2+\alpha}$ for every $f \in C^{(2-b)}_{0,\alpha}$, being $0<\alpha<1$ and $0<b \leqq 1+\gamma$.*

Taking $b=1+\gamma$ results that $u \in C^{(-1-\gamma)}_{2+\alpha} \hookrightarrow C^{1,\gamma}$. And finally, the compact imbedding $C^{(1,\gamma)}(\bar{\Omega}) \hookrightarrow C^1(\bar{\Omega})$ holds for Ω bounded [**12**].

Acknowledgement. The author is grateful to Prof. E. Fernández-Cara for several fruitful comments.

REFERENCES

1. H. Amann, On the number of solutions of nonlinear equations in ordered Banach spaces, *J. Funct. Anal.* **11** (1972), 346–384.

2. H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Rev.* **18** (1976), 620–709.

3. J. Blat and K. J. Brown, Bifurcation of steady-state solutions in predator–prey and competition systems, *Proc. Roy. Soc. Edinburgh* **97A** (1981), 21–34.

4. J. Blat and K. J. Brown, Global bifurcation of positive solutions in some systems of elliptic equations, *SIAM J. Math. Anal.* **17** (1986), 1339–1353.

5. R. S. Cantrell and C. Cosner, On steady-state problem for the Volterra–Lotka competition model with diffusion, *Houston J. Math.* **13** (1987), 337–352.

6. C. Cosner and A. C. Lazer, Stable coexistence states in the Volterra–Lotka competition model with diffusion, *SIAM J. Appl. Math.* **44** (1984), 1112–1132.

7. E. N. Dancer, On positive solutions of some pairs of differential equations, *J. Differential Equations* **60** (1985), 236–258.

8. D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order (Springer-Verlag, New York 1977).

9. J. Hernandez, Maximum principles and decoupling for positive solutions of reaction–diffusion systems, in *Reaction–diffusion Equations* (K. J. Brown and A. A. Lacey eds., Clarendon Press, Oxford 1990).

10. L. Li, Coexistence theorems of steady states for predator–prey interacting systems, *Trans. Amer. Math. Soc.* **305** (1988), 143–166.

11. C. V. Pao, On nonlinear reaction–diffusion systems, *J. Math. Anal. Appl.* **87** (1982), 165–198.

12. R. Adams, *Sobolov spaces* (Academic Press, New York, 1975).

13. D. Gilbarg and L. Hörmander, Intermediate Schauder estimates, *Arch. Rational Mech. Anal.* **74** (1980), 297–318.

Universidad de Sevilla
C/Tarfia S/N
41012 Sevilla
Spain