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Abstract

In this paper, we investigate Frobenius eigenvalues of the compactly supported rigid cohomology of a variety defined over a finite field with q elements, using Dwork's method. Our study yields several arithmetic consequences. First, we establish that the zeta functions of a set of related affine varieties can reveal all Frobenius eigenvalues of the rigid cohomology of the variety up to a Tate twist. This result does not seem to be known for the ℓ-adic cohomology. As a second application, we provide several q-divisibility lower bounds for the Frobenius eigenvalues of the rigid cohomology of the variety, in terms of the dimension and multi-degrees of the defining equations. These divisibility bounds for rigid cohomology are generally better than what is suggested from the best known divisibility bounds in ℓ-adic cohomology, both before and after the middle cohomological degree.

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1. Introduction

Dwork [Dwo62] engineered a cohomology theory in order to study the zeta function of a projective hypersurface over a finite field $\mathbf{F}_q$ of q elements with characteristic p. In this paper, we revisit his construction, and study some problems on Frobenius eigenvalues of affine varieties in rigid cohomology.

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1.1 Visibility of Frobenius eigenvalues

Our first theorem is about the *visibility* of Frobenius eigenvalues in the zeta function of an affine variety.

Let $\mathbf{F}_q$ be a finite field with q elements and characteristic p. Given an algebraic variety Z over $\mathbf{F}_q$, its *zeta function* is defined as

$$\zeta_Z(t) = \exp\left\{\sum_{m=1}^{\infty} \frac{|Z(\mathbf{F}_{q^m})|}{m} t^m\right\}.$$

Weil conjectured, and Dwork subsequently proved [Dwo60], that $\zeta_Z(t)$ is a rational function.

By trace formulae in rigid cohomology (due to Étesse and Le Stum [ELS93]) and ℓ-adic cohomology (see, e.g., [Del77, Rapport, §§ 4–6]), the zeta function is an alternating product

$$\zeta_Z(t) = \prod_{i=0}^{2\dim Z} \det(1 - t\cdot \mathrm{Frob}_q | \mathrm{H}^i_c(Z))^{(-1)^{i+1}}. \tag{1.1}$$

Here, $\mathrm{H}^i_c(Z)$ could mean either Berthelot's compactly supported rigid cohomology $\mathrm{H}^i_{\mathrm{rig},c}(Z)$, or compactly supported ℓ-adic cohomology $\mathrm{H}^i_c(Z_{\overline{\mathbf{F}}_q}, \mathbf{Q}_\ell)$ ($\overline{\mathbf{F}}_q$ is a fixed algebraic closure of $\mathbf{F}_q$, $Z_{\overline{\mathbf{F}}_q} = Z \otimes_{\mathbf{F}_q} \overline{\mathbf{F}}_q$, and $\ell \neq p$ is a prime number). Note that the finite dimensionality of $\mathrm{H}^i_c(Z)$ also yields a cohomological proof for the rationality of the zeta function.

By the trace formulae, reciprocal roots and poles of $\zeta_Z(t)$ constitute a subset of the Frobenius eigenvalues of $\mathrm{H}^*_c(Z)$. When Z is *smooth* and *proper* over $\mathbf{F}_q$, the converse is also true, as a result of Deligne's resolution of Weil's conjecture [Del74, Théorème 1.6] (for rigid cohomology, see Katz–Messing [KM74]). In such cases, the Frobenius eigenvalues of $\mathrm{H}^i_c(Z)$ are algebraic integers having archimedean absolute value $q^{i/2}$ (with respect to any abstract embedding $W(\mathbf{F}_q)[1/p] \hookrightarrow \mathbf{C}$ or $\mathbf{Q}_\ell \hookrightarrow \mathbf{C}$). Therefore, the denominator and numerator of the right-hand side of (1.1) do not have common factors; the zeta function alone can recover the Frobenius eigenvalues.

Without the smooth proper condition, the linear factors of the determinants in (1.1) could cancel out. If a cancellation happens, $\zeta_Z(t)$ may not be capable of witnessing all the Frobenius eigenvalues, not even up to Tate twist. Here is a simple example. Let X be a general nonsingular cubic curve in $\mathbf{A}^2_{\mathbf{F}_q}$, and let $Y = \mathbf{A}^2_{\mathbf{F}_q} - X$ be its complement. Then the zeta function of the affine variety $Z = X \sqcup Y$ equals that of $\mathbf{A}^2_{\mathbf{F}_q}$, namely, $(1-q^2t)^{-1}$. On the other hand, there exist Frobenius eigenvalues of $\mathrm{H}^1_c(Z) = \mathrm{H}^1_c(X) \oplus \mathrm{H}^1_c(Y)$ of absolute value $q^{1/2}$.

Our first theorem asserts that, if we are willing to take the *defining equations* of an affine variety Z into the consideration, then we can *recover all the Frobenius eigenvalues of Z up to Tate twist* from zeta functions of finitely many varieties related to Z. In order to give the precise statement, let us introduce some terminologies.

Definition 1.1. Let $\Gamma = \{\Gamma_a(t), \Gamma_b(t), \ldots\} \subset 1 + t\mathbf{C}_p[\![t]\!]$ be a collection of p-adic meromorphic function on $\mathbf{C}_p$ (i.e., fractions of p-adic entire functions). We say that a p-adic number $\gamma \in \mathbf{C}_p - \{0\}$ is *visible* in Γ, if $\Gamma_a(\gamma^{-1}) = 0$ or ∞ for some $\Gamma_a \in \Gamma$. We say that γ is *weakly visible* in Γ, if there exists $m \in \mathbf{Z}$ such that $q^m\gamma$ is visible in Γ.

Now let $f_1, \ldots, f_r \in \mathbf{F}_q[x_1, \ldots, x_n]$ be a collection of polynomials. For every subset $I \subset \{1, 2, \ldots, r\}$, set $Z_I = \operatorname{Spec} \mathbf{F}_q[x_1, \ldots, x_n]/(f_i : i \in I) \subset \mathbf{A}^n_{\mathbf{F}_q}$ and $Z_I^* = Z_I \cap \mathbf{G}^n_{\mathrm{m}}$. Write $Z = Z_{\{1,2,\ldots,r\}}$.

Theorem 1.2. *Let Z be the vanishing locus of $f_1, \ldots, f_r \in \mathbf{F}_q[x_1, \ldots, x_n]$ in $\mathbf{A}^n$. Then any Frobenius eigenvalue of $\mathrm{H}^\bullet_{\mathrm{rig},c}(Z)$ is weakly visible in the finite set $\{\zeta_{Z_I^*}(t) : I \subset \{1, 2, \ldots, r\}\}$.*

The theorem is already interesting when $Z \subset \mathbf{A}^n_{\mathbf{F}_q}$ is an affine hypersurface. In this special situation, it asserts that if a Frobenius eigenvalue λ is canceled out in the zeta function, then either it equals q^m for some m, or there must exist a reciprocal root or reciprocal pole of $\zeta_{Z\cap\mathbf{G}^n_{\mathrm{m}}}(t)$ that equals a Tate twist of λ. Thus, the zeta function $\zeta_{Z\cap\mathbf{G}^n_{\mathrm{m}}}$ alone can recover all the Frobenius eigenvalues of $\mathrm{H}^\bullet_{\mathrm{rig},c}(Z)$ up to Tate twist.

In view of the 'motivic' philosophy, the same result should also hold for ℓ-adic cohomology. But our method, which is p-adic in nature, depends upon an explicit chain model of rigid cohomology and does not work in the ℓ-adic context.

1.2 Divisibility of Frobenius eigenvalues

The second result of this paper concerns the notion of *q-divisibility* as algebraic integers of Frobenius eigenvalues of affine and projective varieties.

Let $f_1, \ldots, f_r \in \mathbf{F}_q[x_1, \ldots, x_n]$ be a collection of polynomials. Write $d_j = \deg f_j$. Without loss of generality, we will assume that all the degrees d_j are positive. By rearranging, we can and will assume that $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_r$. Let

$$Z = \operatorname{Spec} \mathbf{F}_q[x_1, \ldots, x_n]/(f_1, \ldots, f_r),$$

be the vanishing scheme of these polynomials in $\mathbf{A}^n_{\mathbf{F}_q}$. For any integer $j \geqslant 0$, we define a non-negative integer

$$\mu_j(n; d_1, \cdots, d_r) = j + \max\left\{0, \left\lceil \frac{n - j - \sum_{i=1}^r d_i}{d_1} \right\rceil\right\}.$$

Recall that the classical Ax–Katz theorem [Ax64, Kat71] states that all the reciprocal roots and poles of the zeta function of Z are divisible by $q^{\mu_0(n;d_1,\ldots,d_r)}$ as algebraic integers. This divisibility was later upgraded to a divisibility on Frobenius eigenvalues on ℓ-adic cohomology. Esnault and Katz [EK05] showed that the Frobenius eigenvalues of $\mathrm{H}^\bullet_c(Z_{\overline{\mathbf{F}}_q}, \mathbf{Q}_\ell)$ are divisible by $q^{\mu_0(n;d_1,\ldots,d_r)}$; furthermore, for $j \geqslant 0$, the Frobenius eigenvalues of $\mathrm{H}^{n-1+j}_c(Z_{\overline{\mathbf{F}}_q}, \mathbf{Q}_\ell)$ are divisible by $q^{\mu_j(n;d_1,\ldots,d_r)}$.

The theorem of Esnault and Katz does not give the most optimal bound when $r > 1$. Recently, Esnault and the first author [EW22] revisited this theme. Based on their study for projective varieties, they suggested a divisibility bound beyond the middle cohomological degree better than the Esnault–Katz bound.

Question A. Is it true that any Frobenius eigenvalue of $\mathrm{H}^{\dim Z+j}_c(Z_{\overline{\mathbf{F}}_q}, \mathbf{Q}_\ell)$ is divisible by $q^{\mu_j(n;d_1,\ldots,d_r)}$, in the ring of algebraic integers, for all integers j satisfying $0 \leqslant j \leqslant \dim Z$?

This question, if answered affirmatively, would simultaneously improve the results of [EK05] and Deligne's integrality theorem [DK73, Exposé XXI, § 5] beyond the middle cohomological degree.

Since Frobenius eigenvalues are supposed to be 'motivic', one is led to ask the same question for the Frobenius eigenvalues of rigid cohomology.

Question B. Is it true that any Frobenius eigenvalue of $\mathrm{H}^{\dim Z+j}_{\mathrm{rig},c}(Z)$ is divisible by $q^{\mu_j(n;d_1,\ldots,d_r)}$, in the ring of algebraic integers, for all integers j satisfying $0 \leqslant j \leqslant \dim Z$?

In this article, we show the following theorem.

Theorem. *Question B has an affirmative answer.*

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Unlike in the ℓ-adic situation, where theorems are usually proved via dévissage, we pursue the divisibility for rigid cohomology using a different method, via Dwork's p-adic theory, refining and upgrading the chain-level approach in [Wan00] to Dwork cohomology, and then by comparison with rigid cohomology.

Somewhat surprisingly, the bounds we obtain through the p-adic methods are sharper than anticipated by Question B. Also, our approach gives divisibility bounds before middle cohomological degree, improving the Ax–Katz type bounds of [EK05].

To state our bounds beyond middle cohomological degree, we define

$$d_i^* = \begin{cases} d_i & \text{if } 1 \leqslant i \leqslant n - \dim Z, \\ 1 & \text{if } i > n - \dim Z \text{ and } d_i = d_1, \\ 0 & \text{if } i > n - \dim Z \text{ and } d_i < d_1. \end{cases}$$

For integer $j \geqslant 0$, define another non-negative integer

$$\nu_j(n; d_1, \ldots, d_r) = j + \max\left\{0, \left\lceil \frac{n - j - \sum_{i=1}^r d_i^*}{d_1} \right\rceil \right\}.$$

Note that $d_i \geqslant d_i^*$, and thus $\nu_j(n; d_1, \ldots, d_r) \geqslant \mu_j(n; d_1, \ldots, d_r)$; also the numbers $\nu_j(n; d_1, \ldots, d_r)$ form an increasing sequence in j. These numbers depend on the degrees of the defining equations of Z and also on the dimension of Z.

If Z is a complete intersection by $f_1, \ldots, f_r$, namely, if $n - \dim Z = r$, one checks that $d_i^* = d_i$ and $\nu_j(n; d_1, \ldots, d_r) = \mu_j(n; d_1, \ldots, d_r)$.

Theorem 1.3 (Divisibility beyond middle cohomological degree). *Let the notation be as above. For every $0 \leqslant j \leqslant \dim Z$:*

- *the Frobenius eigenvalues of $\mathrm{H}^{\dim Z + j}_{\mathrm{rig},c}(Z)$ are divisible by $q^{\nu_j(n; d_1, \ldots, d_r)}$ in the ring of algebraic integers; and*
- *the Frobenius eigenvalues of $\mathrm{H}^{\dim Z + 1 + j}_{\mathrm{rig},c}(\mathbf{A}^n_{\mathbf{F}_q} - Z)$ are divisible by $q^{\nu_j(n; d_1, \ldots, d_r)}$ in the ring of algebraic integers.*

Remark 1.4.

(a) The second item is the consequence of the first, thanks to the long exact sequence for compactly supported cohomology.

(b) For any separated variety Z over $\mathbf{F}_q$, the Frobenius eigenvalues of $\mathrm{H}^*_{\mathrm{rig},c}(Z)$ are always algebraic integers. When Z is smooth proper, we use the Weil conjecture and the integrality of the zeta function (see [KM74, Theorem 1]). If Z is proper but possibly singular, we can produce a proper hypercovering using smooth proper varieties by alteration [dJ96], and then apply cohomological descent [Tsu03]. If Z is not proper, we can embed Z into a proper variety $\overline{Z}$, and conclude by using the assertion for proper varieties and the long exact sequence

$$\cdots \to \mathrm{H}^i_{\mathrm{rig},c}(Z) \to \mathrm{H}^i_{\mathrm{rig}}(\overline{Z}) \to \mathrm{H}^i_{\mathrm{rig}}(\overline{Z} - Z) \to \cdots.$$

Our method is capable of seeing this too (see p. 1246).

Since $\nu_j(n; d_1, \ldots, d_r) \geqslant \mu_j(n; d_1, \ldots, d_r)$, Theorem 1.3 establishes an enhanced positive answer to Question B. In the complete intersection case, the two bounds are identical. If Z is not a complete intersection, then the divisibility bound in Theorem 1.3 can be strictly better.

What about before the middle cohomological degree? In this range, the only known divisibility for ℓ-adic cohomology is the theorem of Esnault–Katz which says that the divisibility is by $q^{\mu_0(n; d_1, \ldots, d_r)}$. We have an improved p-adic companion in this range as well.

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Since Z is cut out by r equations, $\mathrm{H}^i_{\mathrm{rig},c}(Z)$ and $\mathrm{H}^i_c(Z_{\overline{\mathbf{F}}_q}, \mathbf{Q}_\ell)$ all vanish if $i < n-r$ (see Lemma 6.2). So, we will assume that $i \geqslant n-r$. If $n-r = \dim Z$, i.e., Z is a complete intersection, then Theorem 1.3 already covers all the possible cohomological degrees i such that $\mathrm{H}^i_{\mathrm{rig},c}(Z)$ is nontrivial. However, if Z is not a complete intersection, then $\mathrm{H}^i_{\mathrm{rig},c}(Z)$ and $\mathrm{H}^i_c(Z_{\overline{\mathbf{F}}_q}, \mathbf{Q}_\ell)$ could be nonzero for $n-r \leqslant i < \dim Z$. A novelty of our approach is that we can provide improved divisibility information of Frobenius eigenvalues in these degrees as well; of course, only for rigid cohomology.

THEOREM 1.5 (Divisibility before middle cohomological degree). *Let the notation be as above. For every $0 \leqslant m \leqslant \dim Z - (n-r)$, the Frobenius eigenvalues of $\mathrm{H}^{n-r+m}_{\mathrm{rig},c}(Z)$ are divisible by $q^{\epsilon_m(n;d_1,\ldots,d_r)}$ in the ring of algebraic integers, where*

$$\epsilon_m(n; d_1, \ldots, d_r) = \max\left\{0, \left\lceil \frac{n - (d_1 + \cdots + d_{r-m} + d^*_{r-m+1} + \cdots + d^*_r)}{d_1} \right\rceil \right\}.$$

The Frobenius eigenvalues of $\mathrm{H}^{n-r+1+m}_{\mathrm{rig},c}(\mathbf{A}^n_{\mathbf{F}_q} - Z)$ are divisible by $q^{\epsilon_m(n;d_1,\ldots,d_r)}$ in the ring of algebraic integers as well.

The numbers $\epsilon_m(n; d_1, \ldots, d_r) (m = 0, 1, \ldots, \dim Z - (n-r))$ form an increasing sequence in the closed interval $[\mu_0(n; d_1, \ldots, d_r), \nu_0(n; d_1, \ldots, d_r)]$, the smallest one $\epsilon_0(n; d_1, \ldots, d_r) = \mu_0(n; d_1, \ldots, d_r)$ being responsible for the Ax–Katz theorem and the Esnault–Katz theorem; and we have

$$\epsilon_{\dim Z-(n-r)}(n; d_1, \ldots, d_r) = \nu_0(n; d_1, \ldots, d_r).$$

Theorems 1.3 and 1.5 have projective analogues. For a closed subvariety Z of $\mathbf{P}^n_{\mathbf{F}_q}$, set

$$\mathrm{H}^*_{\mathrm{rig}}(Z)_{\mathrm{prim}} = \mathrm{Coker}(\mathrm{H}^*_{\mathrm{rig}}(\mathbf{P}^n_{\mathbf{F}_q}) \to \mathrm{H}^*_{\mathrm{rig}}(Z)).$$

THEOREM 1.6. *Let $f_1, \ldots, f_r \in \mathbf{F}_q[x_0, \ldots, x_n]$ be homogeneous polynomials of positive degrees $d_1 \geqslant \cdots \geqslant d_r$. Let Z be the vanishing scheme of $f_1, \ldots, f_r$ in $\mathbf{P}^n_{\mathbf{F}_q}$. Then, for $0 \leqslant j \leqslant \dim Z$, as algebraic integers:*

- *the Frobenius eigenvalues of $\mathrm{H}^{\dim Z+j}_{\mathrm{rig}}(Z)_{\mathrm{prim}}$ are divisible by $q^{\nu_j(n+1;d_1,\ldots,d_r)}$; and*
- *the Frobenius eigenvalues of $\mathrm{H}^{\dim Z+1+j}_{\mathrm{rig},c}(\mathbf{P}^n_{\mathbf{F}_q} - Z)$ are divisible by $q^{\nu_j(n+1;d_1,\ldots,d_r)}$.*

For $0 \leqslant m \leqslant \dim Z - (n-r)$, as algebraic integers:

- *the Frobenius eigenvalues of $\mathrm{H}^{n-r+m}_{\mathrm{rig}}(Z)_{\mathrm{prim}}$ are divisible by $q^{\epsilon_m(n+1;d_1,\ldots,d_r)}$; and*
- *the Frobenius eigenvalues of $\mathrm{H}^{n-r+1+m}_{\mathrm{rig},c}(\mathbf{P}^n_{\mathbf{F}_q} - Z)$ are divisible by $q^{\epsilon_m(n+1;d_1,\ldots,d_r)}$.*

Theorem 1.6 is a formal consequence of Theorems 1.3 and 1.5. For the proof, see [WZ25, pp. 21–23].

Remark 1.7. These divisibility theorems in rigid cohomology now raise new questions for ℓ-adic cohomology and Hodge theory, through the 'motivic' philosophy.

- The Frobenius eigenvalues of ℓ-adic cohomology groups of affine or projective Z should also satisfy the divisibility stated in Theorems 1.3, 1.5 and 1.6.
- For affine or projective varieties defined over the field $\mathbf{C}$ of complex numbers, the numbers

  $$\nu_j(n; d_1, \ldots, d_r) \quad \text{and} \quad \epsilon_m(n; d_1, \ldots, d_r)$$

  should give lower bounds for Hodge levels of compactly supported singular cohomology of complex varieties cut out by a set of polynomial equations of degrees $d_1, \ldots, d_r$.

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Our method, purely analytic, is certainly not applicable for ℓ-adic cohomology. But the chain-level considerations may be useful for the problem on Hodge levels.

Note. After submitting this paper and during its refereeing process, we resolved both questions positively in [WZ25]. In particular, we established the validity of Question A for ℓ-adic cohomology. Independently, Rai and Shuddhodan [RS23] also provided a confirmation of Question A.

The paper is organized as follows. Section 2 introduces the overconvergent Dwork cohomology and provides the construction of a specific chain model for computing it. It also contains the statement of a theorem of Baldassarri and Berthelot [BB04] which allows us to relate the overconvergent Dwork cohomology with the compactly supported rigid cohomology of an affine variety. Section 3 proves Theorem 1.2. Section 4 explains how to transplant the theory of Adolphson and Sperber, which uses a more complicated model of Dwork cohomology, to the situation that concerns us. After proving two lemmas in § 5, we provide the proofs of Theorems 1.3 and 1.5 in § 6.

1.3 Notation and conventions

Throughout the main body of the paper, $\mathbf{N}=\{0,1,\ldots\}$ denotes the set of *non-negative* integers. We fix a prime number p, and we let q be a power of p. Let $\mathcal{O}_K=W(\mathbf{F}_q)[\zeta_p]$, where $\zeta_p\neq 1$ is a pth root of unity. Note that $\mathcal{O}_K$ contains an element π satisfying $\pi^{p-1}+p=0$. Let K be the field of fractions of $\mathcal{O}_K$. Let $|\cdot|$ be the ultrametric on K extending that of $\mathbf{Q}_p$. The p-power Frobenius map of $\mathbf{F}_q$ lifts to an automorphism $\tau\in\mathrm{Gal}(K/\mathbf{Q}_p(\zeta_p))$ such that $\tau(\zeta_p)=\zeta_p$ (thus $\tau(\pi)=\pi$), and $\tau^q=\mathrm{Id}$.

We consider exclusively rigid cohomology over the base field K. Thus, for an algebraic variety X over $\mathbf{F}_q$ and an overconvergent F-isocrystal $\mathcal{E}$ on X, the rigid cohomology groups $\mathrm{H}^i_{\mathrm{rig,c}}(X;\mathcal{E})$ and $\mathrm{H}^i_{\mathrm{rig}}(X;\mathcal{E})$ are all finite dimensional K-vector spaces (by Kedlaya [Ked06]). When $\mathcal{E}=\mathcal{O}_X$ is the constant isocrystal, we simply write $\mathrm{H}^i_{\mathrm{rig}}(X)$ or $\mathrm{H}^i_{\mathrm{rig,c}}(X)$ for its rigid cohomology.

2. Overconvergent Dwork cohomology

After establishing the rationality of the zeta function, Dwork proceeded to pioneer the study of p-adic absolute values of the reciprocal roots and zeros of zeta functions of an algebraic variety Z over $\mathbf{F}_q$. He proved the famous 'Newton above Hodge' theorem when Z is a nonsingular hypersurfaces in a projective space [Dwo62]. He accomplished this by devising some explicit chain complexes of p-adic Banach spaces, which are equipped with some chain-level representations of the Frobenius operation. A substantial portion of his work was centered around the chain-level investigations. He only delved into the cohomology spaces under special occasions where finite dimensionality can be shown. But even without knowing finiteness of the cohomology spaces, the chain-level approach can still extract numerous properties of the zeta function.

Actually, Dwork did not design just one chain complex, but rather infinitely many different chain complexes, each corresponding to a specific 'splitting function' in his terminology. The chain-level Frobenius operators associated with these chain models are all capable of calculating the zeta function of a variety, regardless of whether the variety is nonsingular or not. In some applications, such as the 'visibility theorem' 1.2, knowledge of some formal properties of these operators suffices. For more intricate analysis, such as many important results of Adolphson and Sperber [AS87a, AS87b, AS89], and the 'divisibility theorem' that relies on them, some specific chain model needed to be used.

Since our first goal is to prove the visibility theorem, in this section we focus exclusively on a most straightforward (overconvergent) Dwork complex. In effect, it is the overconvergent de Rham complex of an exponentially twisted integrable connection on a p-adic polydisk. We state a theorem of Baldassarri and Berthelot that relates the overconvergent Dwork cohomology of an affine variety Z with the rigid cohomology of Z with compact support.

The role of a subtler model, the *Artin–Hasse model*, is clarified in a subsequent section, before we prove the divisibility theorem.

Definition 2.1. On the structure sheaf of the rigid analytic affine line $\mathbf{A}_K^{1,\mathrm{an}}$ (with coordinate z), we define an integrable connection ∇_π by the formula

$$\nabla_\pi \xi = \mathrm{d}\xi + \pi\xi \mathrm{d}z.$$

The connection is overconvergent. It is equipped with a Frobenius structure

$$\varphi_\pi : \xi \mapsto \xi^\tau(z^p) \cdot \theta(z)^{-1},$$

where $\theta(z)$ is the Dwork exponential

$$\theta(z) = \exp(\pi z - \pi z^p).$$

It is well known that the radius of convergence of $\theta(z)$ is $|p|^{-((p-1)/p^2)}$, and thus overconvergent (cf. [Rob00, p. 396, Theorem (b)]). The pair $(\nabla_\pi, \varphi_\pi)$ defines an overconvergent F-isocrystal $\mathcal{L}_\pi$ on $\mathbf{A}_k^1$, called the *Dwork crystal*. See [LS07, § 4.2.1 and § 8.3] for more details. The dual isocrystal of $\mathcal{L}_\pi$ is $\mathcal{L}_{-\pi}$.

Situation 2.2. Let $f_1, \dots, f_r \in \mathbf{F}_q[x_1, \dots, x_n]$. Let $Z = \mathrm{Spec}(\mathbf{F}_q[x_1, \dots, x_n]/(f_1, \dots, f_r))$. Introducing new variables $x_{n+1}, \dots, x_{n+r}$, let $g = x_{n+1}f_1 + \cdots + x_{n+r}f_r$. We refer to the rigid cohomology with twisted coefficient $\mathrm{H}^\bullet_{\mathrm{rig}}(\mathbf{A}^{n+r}_{\mathbf{F}_q}; g^*\mathcal{L}_\pi)$, as the 'overconvergent Dwork cohomology' of Z.

As we have said, the benefit of being able to work with the overconvergent Dwork cohomology is that it has a rather explicit chain-level model. We now elaborate on the de Rham complex that is used to compute the overconvergent Dwork cohomology.

Construction 2.3 (Overconvergent Dwork complex). Consider the Monsky–Washnitzer algebra

$$\begin{aligned} B &= K\langle x_1, \dots, x_{n+r}\rangle^\dagger \\ &= \left\{ \sum_{u \in \mathbf{N}^{n+r}} c_u x^u \in K[\![x_1, \dots, x_{n+r}]\!] : \exists \rho > 1, |c_u|\rho^{|u|} \xrightarrow{|u|\to\infty} 0 \right\}, \end{aligned}$$

where $|u| = |u_1| + \cdots + |u_{n+r}|$, $x^u = \prod_{i=1}^{n+r} x_i^{u_i}$. For a subset $I = \{i_1, \dots, i_m\}$ of $\{1, \dots, n+r\}$, with $i_1 < \cdots < i_m$, we write $x^I = \prod_{i\in I} x_i$, and $dx^I = dx_{i_1} \wedge \cdots \wedge dx_{i_m}$. We use the notation Ω^m to denote the space of 'overconvergent m-forms', that is,

$$\Omega^m = \bigoplus_{\substack{I \subset \{1,\dots,n+r\} \\ |I| = m}} B \cdot dx^I = \bigoplus_{\substack{I \subset \{1,\dots,n+r\} \\ |I| = m}} B_I \frac{dx^I}{x^I}, \tag{2.1}$$

where $B_I = x^I B$.

Write $g = \sum_{u \in \mathbf{N}^{n+r}} a_u x^u$, with $a_u \in \mathbf{F}_q$. Let A_u be the Teichmüller lift of a_u in $W(\mathbf{F}_q) \subset \mathcal{O}_K$, and let $G = \sum_{u\in\mathbf{N}^{n+r}} A_u x^u$. Then $G \equiv g \mod \pi$, and the overconvergent Dwork cohomology is computed by the exponentially twisted de Rham complex

$$\mathcal{D}^\bullet : \quad \Omega^0 \xrightarrow{d+\pi dG} \Omega^1 \xrightarrow{d+\pi dG} \cdots \xrightarrow{d+\pi dG} \Omega^{n+r}. \tag{2.2}$$

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That is,

$$\mathrm{H}^*_{\mathrm{rig}}(\mathbf{A}^{n+r}_{\mathbf{F}_q}, g^*\mathcal{L}_\pi) \simeq \mathrm{H}^*(\mathcal{D}^\bullet).$$

Remark 2.4. The complex $\mathcal{D}^\bullet$ is said to be exponentially twisted because, symbolically,

$$d + \pi dG = \exp(-\pi G) \circ d \circ \exp(\pi G).$$

Remark 2.5. Dwork cohomology originated in Dwork's study of zeta functions of projective hypersurfaces [Dwo62, Dwo64]. The overconvergent Dwork cohomology $\mathrm{H}^*_{\mathrm{rig}}(\mathbf{A}^{n+r}; g^*\mathcal{L}_\pi)$ is an overconvergent variant of Dwork's construction. Dwork used Banach spaces rather than weakly completed algebras in the sense of Monsky–Washnitzer. His construction was systematically generalized in the context of toric exponential sums by Adolphson and Sperber [AS89] (still using Banach spaces instead of weakly completed versions). Apparently, the nice properties of overconvergent Dwork cohomology were first studied by Monsky [Mon71].

Construction 2.6 (Frobenius action on the overconvergent Dwork complex). The inverse image isocrystal $g^*\mathcal{L}_\pi$ has a Frobenius structure, which can also be explained using the exponential twist. We denote by σ the endomorphism of B defined by

$$\sigma\colon \sum_{u\in\mathbf{N}^{n+r}} a_u x^u \mapsto \sum_{u\in\mathbf{N}^{n+r}} a_u^\tau x^{pu}.$$

Then the Frobenius structure on $g^*\mathcal{L}_\pi$ with respect to σ can be symbolically determined via exponential twist (Remark 2.5) using

$$\begin{aligned}\varphi(\xi) &= \prod_u \exp(-\pi A_u x^u)\cdot \sigma\left(\prod_u \exp(\pi A_u x^u)\cdot \xi\right)\\ &= \prod_u \exp(\pi A_u^\tau x^{pu} - \pi A_u x^u)\cdot \xi^\sigma\\ &= \prod_u \theta(A_u x^u)^{-1}\cdot \xi^\sigma,\end{aligned}$$

where u ranges in $\mathbf{N}^{n+r}$ and θ is the Dwork exponential defined above. Since θ is overconvergent, it follows that φ indeed takes B into B. Recall $\mathbf{F}_q = \mathbf{F}_{p^a}$. Let

$$F_1(x) = \prod_u \theta(A_u x^u) \quad \text{and} \quad F_a(x) = \prod_{i=0}^{a-1} F_1^{\tau^i}(x^{p^i}). \tag{2.3}$$

Then both $F_1(x)$ and $F_a(x)$ are overconvergent analytic functions in $(x_1,\dots,x_{n+r})$. Hence, $\varphi = F_1^{-1}\circ\sigma$.

The Frobenius φ induces an operation on the spaces Ω^m of differential forms.

$$\varphi^{(m)}\left(\sum_{\substack{I\subset\{1,\dots,n+r\}\\ |I|=m}} \xi_I(x)\frac{dx_I}{x_I}\right) = p^m\varphi(\xi_I(x))\frac{dx_I}{x_I}, \quad \xi_I\in B_I.$$

One can check that the above definition turns $(\varphi^{(m)})_{m=0}^{n+r}$ into a chain map $\varphi^{(\bullet)}\colon \mathcal{D}^\bullet\to\mathcal{D}^\bullet$.

$$\begin{array}{ccccccc}\Omega^0 & \xrightarrow{d+\pi dG} & \Omega^1 & \xrightarrow{d+\pi dG} & \cdots & \xrightarrow{d+\pi dG} & \Omega^{n+r}\\ \big\downarrow{\scriptstyle\varphi^{(0)}} & & \big\downarrow{\scriptstyle\varphi^{(1)}} & & & & \big\downarrow{\scriptstyle\varphi^{(n+r)}}\\ \Omega^0 & \xrightarrow[d+\pi dG]{} & \Omega^1 & \xrightarrow[d+\pi dG]{} & \cdots & \xrightarrow[d+\pi dG]{} & \Omega^{n+r}\end{array}$$

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It induces the semilinear Frobenius map on rigid cohomology

$$\varphi\colon \mathrm{H}^*_{\mathrm{rig}}(\mathbf{A}^{n+r}; g^*\mathcal{L}_\pi) \to \mathrm{H}^*_{\mathrm{rig}}(\mathbf{A}^{n+r}; g^*\mathcal{L}_\pi).$$

Remark 2.7. Although the Frobenius operator φ behaves well after taking cohomology, it is ill suited for chain-level manipulations due to being an 'expanding map', which is not completely continuous in the sense of Serre [Ser62]: that is, φ is not a uniform limit of finite rank linear maps.

This analytic issue is common to F-isocrystals in general, and is not limited to the Dwork isocrystal. Consider the simplest example of the standard Frobenius acting on the trivial isocrystal on $\mathbf{A}^1$, when $q = p$. To take advantage of overconvergence, one is led to examine the Frobenius operation on p-adic disks slightly larger than the unit disk. However, the p-power Frobenius map $a \mapsto a^p$ takes the one-dimensional closed ball of radius $|\pi|^{-1/N}$ to the *larger* closed ball of radius $|\pi|^{-p/N}$. The Frobenius pullback induces a map on functions:

$$\xi(x) \mapsto \xi(x^p)\colon \left\{\sum a_n x^n : a_n \pi^{-n/N} \to 0\right\} \to \left\{\sum b_n x^n : b_n \pi^{-pn/N} \to 0\right\}.$$

After extending the field K to a larger scalar field by adding Nth roots of π, an orthonormal basis of the Banach space $\left\{\sum a_n x^n : a_n \pi^{-n/N} \to 0\right\}$ is given by $1, \pi^{1/N}x, (\pi^{1/N}x)^2, \ldots$, and an orthonormal basis of $\left\{\sum a_n x^n : a_n \pi^{-pn/N} \to 0\right\}$ is given by $1, \pi^{p/N}x, (\pi^{p/N}x)^2, \ldots$. The columns of the matrix representing the linear mapping $\xi(x) \mapsto \xi(x^p)$ with respect to these bases have larger and larger absolute values. On the other hand, the norm of the column vectors of a completely continuous operator should converge to 0.

CONSTRUCTION 2.8 (Dwork operators). To fix this, following Dwork, we consider the following operator ψ on power series, given by

$$\psi\Bigg(\sum_{u \in \mathbf{N}^{n+r}} a_u x^u\Bigg) = \sum_{u \in \mathbf{N}^{n+r}} a_{pu} x^u. \tag{2.4}$$

Recall that $q = p^a$ and $\tau^a = \mathrm{Id}$. Then

$$\begin{aligned}(\tau^{-1} \circ \psi)^a \left(\sum a_u x^u\right) &= \sum a_{qu}^{\tau^{-a}} x^u = \sum a_{qu} x^u \\ &= \psi^a \left(\sum a_u x^u\right),\end{aligned} \tag{2.5}$$

as well as

$$\eta \cdot (\tau^{-1} \circ \psi)(\xi) = (\tau^{-1} \circ \psi)(\eta^\tau(x^p) \cdot \xi(x)).$$

Form the composition

$$\alpha_1 = \tau^{-1} \circ \psi \circ F_1 \quad \text{as well as} \quad \alpha_a = \alpha_1^a.$$

Then by (2.4) and (2.5),

$$\begin{aligned}\alpha_a = \alpha_1^a &= \underbrace{(\tau^{-1} \circ \psi \circ F_1) \circ \cdots \circ (\tau^{-1} \circ \psi \circ F_1)}_{a \text{ times}}, \\ &= (\tau^{-1} \circ \psi)^a \circ \left(\prod_{i=0}^{a-1} F_1^{\tau^i}(x^{p^i})\right) \\ &= \psi^a \circ F_a.\end{aligned}$$

It is clear that α_1 preserves B_I for any $I \subset \{1, 2, \ldots, n+r\}$ and is a left inverse to φ: $\alpha_1 \circ \varphi = \mathrm{Id}$. Similarly, α_a also preserves B_I and is a left inverse to the ath iteration of φ. Note that α_1

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is a τ^{-1}-semilinear map on the infinite-dimensional K-vector space B_I, and α_a is a K-linear operator on B_I. In the terminology of Monsky [Mon71, Definition 2.1], α_1 and α_a are 'Dwork operators' on the spaces B_I (the power series $F_1(x)$ and $F_a(x)$ are integrally defined, i.e., they are elements of $\mathcal{O}_K[\![x_1,\dots,x_{n+r}]\!]\cap B_I$). Moreover, both α_1 and α_a are 'nuclear operators' on the space B in the sense of Monsky [Mon71, Theorem 2.1]. Therefore, their 'characteristic power series' [Mon71, Theorem 1.2] are well defined.

The operators α_1 and α_a extend to endomorphsms of the Dwork complex. We define

$$\alpha_1^{(m)}\colon \Omega^m \to \Omega^m,$$

by the formula

$$\alpha_1^{(m)}\left(\sum_{\substack{I\subset\{1,\dots,n+r\}\\|I|=m}} \xi_I(x)\frac{dx_I}{x_I}\right) = \alpha_1(\xi_I(x))\frac{dx_I}{x_I}, \quad \xi_I\in B_I.$$

Then $\alpha_1^{(m)}\circ\varphi^{(m)} = p^m\mathrm{Id}$, and the following diagram is commutative.

$$\begin{array}{ccccccc}
\Omega^0 & \xrightarrow{d+\pi dG} & \Omega^1 & \xrightarrow{d+\pi dG} & \cdots & \xrightarrow{d+\pi dG} & \Omega^{n+r}, \\
\uparrow{\scriptstyle \alpha_1^{(0)}} & & \uparrow{\scriptstyle p^{-1}\alpha_1^{(1)}} & & & & \uparrow{\scriptstyle p^{-n-r}\alpha_1^{(n+r)}} \\
\Omega^0 & \xrightarrow[d+\pi dG]{} & \Omega^1 & \xrightarrow[d+\pi dG]{} & \cdots & \xrightarrow[d+\pi dG]{} & \Omega^{n+r}
\end{array}$$

Similarly, we may define $\alpha_a^{(m)}\colon\Omega^m\to\Omega^m$, satisfying $\alpha_a^{(m)}\circ(\varphi^{(m)})^a = q^m\mathrm{Id}$, and the operators $q^{-m}\alpha_a^{(m)}$ induce a chain map of $\mathcal{D}^\bullet$. Thus, the operators $(p^{-m}\alpha_1^{(m)})_{m=0}^{n+r}$ and $(q^{-m}\alpha_a^{(m)})_{m=0}^{n+r}$ induce maps on overconvergent Dwork cohomology,

$$\alpha_1,\alpha_a\colon \mathrm{H}^*_{\mathrm{rig}}(\mathbf{A}^{n+r};g^*\mathcal{L}_\pi)\to\mathrm{H}^*_{\mathrm{rig}}(\mathbf{A}^{n+r};g^*\mathcal{L}_\pi).$$

Since $\alpha_a^{(m)}$ (respectively, $\alpha_1^{(m)}$) is a left inverse to $\varphi^{(m)}$ (respectively, $(\varphi^{(m)})^a$) on the chain level, and since the overconvergent Dwork cohomology is finite dimensional, we find on the cohomology level that $q^{-m}\alpha_a^{(m)}$ is *equal* to the inverse to $(\varphi^{(m)})^a$. In particular,

$$\begin{aligned}&\det(1-t\cdot q^{-m}\alpha_a|\mathrm{H}^m_{\mathrm{rig}}(\mathbf{A}^{n+r};g^*\mathcal{L}_\pi))\\ &= \det(1-t\cdot(\varphi^{(m)})^{-a}|\mathrm{H}^m_{\mathrm{rig}}(\mathbf{A}^{n+r};g^*\mathcal{L}_\pi)).\end{aligned} \tag{2.6}$$

Remark 2.9 (Fredholm determinant). For every subset I of $\{1,2,\dots,n+r\}$, the collection

$$\{x^u : u\in\mathbf{N}^{n+r},\, u_i\geqslant 1 \text{ if } i\in I\} \tag{2.7}$$

is a 'basis' of B_I in the sense that every $\xi\in B_I$ can be written uniquely as an infinite linear combination $\xi=\sum a_u x^u$ for some $a_u\in K$. Furthermore, any continuous linear operator on B_I can be represented by a unique infinite matrix.

According to Monsky's theory [Mon71, Theorem 1.6], the matrix associated to the Dwork operator α_a has a well-defined Fredholm determinant, which equals the *characteristic power series* he defined. This is because B_I is a union of certain α_a-invariant Banach subspaces $B_I(b)$: $B_I=\bigcup_{b>0}B_I(b)$. Here, $B_I(b)$ is the space of rigid analytic functions converging on the closed polydisk of radius $|p|^{-b}$. The operators $\alpha_a|_{B_I(b)}$ are all completely continuous in the sense of Serre [Ser62]. An appropriately scaled version of (2.7) is an orthonormal basis of the Banach space $B_I(b)$. For this reason, in the remainder of this paper, we simply refer to the characteristic power series of $\alpha_a|_{B_I}$ as the *Fredholm determinant* of α_a, denoted as $\det(1-t\alpha_a|B_I)$.

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On occasion, we will also encounter other overconvergent spaces, such as the spaces C and C/B (which will show up in the proof of Theorem 3.7) among others. In these situations, when we refer to the term 'basis', it should be interpreted in a manner analogous to that explained in the preceding paragraph. Consequently, the characteristic power series of Dwork operators on these spaces can all be computed as the Fredholm determinants of the infinite matrices associated to the specified 'bases'. Therefore, we do not distinguish between the terms 'characteristic power series' and 'Fredholm determinant'.

Now that we have completed the groundwork concerning overconvergent Dwork cohomology, we are ready to present a theorem of Baldassarri and Berthelot that establishes a connection between overconvergent Dwork cohomology and the rigid cohomology of Z. This theorem serves as the cornerstone for our chain-level arguments.

THEOREM 2.10 [BB04, Theorem 3.1]. *Using notation as in Situation 2.2, we have a natural isomorphism*

$$\mathrm{H}^*_{\mathrm{rig}}\big(\mathbf{A}^{n+r}_{\mathbf{F}_q}; g^*\mathcal{L}_\pi\big) \simeq \mathrm{H}^*_{\mathrm{rig},Z}\big(\mathbf{A}^n_{\mathbf{F}_q}\big),$$

which is compatible with Frobenius actions.

In fact, the following local version of the theorem is true.

THEOREM 2.11 [BB04, Theorem 2.14]. *Let the notation be as in Situation 2.2. Let $\varpi\colon \mathbf{A}^{n+r} \to \mathbf{A}^n$ be the projection to the first n coordinates. Let $\mathcal{L}$ denote the arithmetic $\mathscr{D}$-module associated to the Dwork crystal $g^*\mathcal{L}_\pi$ on $\mathbf{A}^{n+r}$. Then $\varpi_+\mathcal{L}$ is isomorphic to the local cohomology complex $\mathbf{R}\underline{\Gamma}^\dagger_Z(\mathcal{O}_{\widehat{\mathbf{P}}^n,\mathbf{Q}}({}^\dagger H))[r]$.*

Here, $\mathcal{O}_{\widehat{\mathbf{P}}^n,\mathbf{Q}}({}^\dagger H)$ is the sheaf of functions on the formal projective space $\widehat{\mathbf{P}}^n$ with overconvergent singularities along the infinity hyperplane $H \subset \mathbf{P}^n_{\mathbf{F}_q}$. This is what was denoted by $\mathcal{O}_{\widehat{\mathbf{P}}^n,\mathbf{Q}}(\infty)$ in [BB04]. See Example 5.3 for more details.

Remark 2.12. Theorems 2.10 and 2.11 were originally proved by Baldassarri and Berthelot for the case where Z is a complete intersection. They noted [BB04, p. 208, Remark] that this hypothesis was only necessary to verify an equality (2.14.4) regarding the compatibility between the local cohomology functor and the extraordinary inverse image functor. This compatibility was later established by Caro unconditionally [Car04, (2.2.18.1)].

However, be aware that Caro's construction of the local cohomology complex [Car04, § 2] differs from Berthelot's approach when Z is not a divisor. Nevertheless, as noted in [Car04, Remarque 2.2.7], the two constructions coincide when the local cohomology is taken with respect to modules of the form $\mathcal{O}_{\mathcal{P},\mathbf{Q}}({}^\dagger H)$. Also, the proof of [BB04, Theorem 2.14] remains valid using Caro's version of the local cohomology functor.

We were informed by Steve Sperber that Nobuo Tsuzuki also prepared a proof of Theorem 2.10 in 2011 (unpublished). We appreciate Sperber for sharing Tsuzuki's manuscript with us.

Remark 2.13. If in the definition of Dwork cohomology one uses finite-type rings instead of using weakly completed algebra or Banach algebras, one gets the so-called 'algebraic Dwork cohomology'. The algebraic analogue of Theorem 2.10 is well known: it was proved by N. Katz [Kat68] when Z is a hypersurface, and by Adolphson–Sperber [AS00] when Z is a *smooth* complete intersection in a smooth affine variety. The algebraic analogue of Theorem 2.11 was shown by Dimca et al. [DMSS00] and Baldassarri–D'Agnolo [BD04].

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In addition to the work of Baldassarri and Berthelot, there are many previous works aiming to provide a comparison between rigid cohomology spaces and the cohomology spaces (and their variants) constructed by Dwork and by Adolphson and Sperber, such as Berthelot [Ber84] (for exponential sums over a one-dimensional torus), P. Bourgeois [Bou99] (for Newton nondegenerate toric exponentials sums) and Peigen Li [Li22] (for toric exponential sums).

Theorem 2.10 implies a result on cohomology with compact supports by taking Poincaré duality. Before stating it, we recall some standard notation about Tate twists.

For an integer w, let $K(w)$ be the one-dimensional vector space over K equipped with a τ-semilinear map $x \mapsto p^{-w}\tau(x)$. For a finite-dimensional K-vector space M equipped with a bijective τ-semilinear map φ, we use $M(w)$ to denote the tensor product $M \otimes_K K(w)$, whose τ-semilinear map is twisted by p^{-w}. For (M, φ) as above, we endow its dual space M^* the transpose inverse of φ. With this convention, we have $(M(w))^* \simeq M^*(-w)$ as spaces equipped with τ-semilinear maps.

Corollary 2.14. *Let $f_1, \dots, f_r \in \mathbf{F}_q[x_1, \dots, x_n]$ be polynomials in n variables. Let Z be the closed subscheme of $\mathbf{A}^n$ cut out by $f_1, \dots, f_r$. Let $g = \sum_{i=1}^r x_{n+i} f_i \in \mathbf{F}_q[x_1, \dots, x_{n+r}]$. Then we have an isomorphism of rigid cohomology spaces compatible with Frobenius actions:*

$$\mathrm{H}^{n-r+j}_{\mathrm{rig},c}(Z)(n) \xrightarrow{\sim} [\mathrm{H}^{n+r-j}_{\mathrm{rig}}(\mathbf{A}^{n+r}; g^*\mathcal{L}_\pi)]^*.$$

In other words, under the isomorphism, the Frobenius operator $q^{-n}\mathrm{Frob}_q|_{\mathrm{H}^{n-r+j}_{\mathrm{rig},c}(Z)}$ corresponds to the transpose of the inverse of the Frobenius operator $(\varphi^{(n+r-j)})^a$ on the overconvergent Dwork cohomology. Moreover,

$$\det\bigl(1 - t\mathrm{Frob}_q \big| \mathrm{H}^{n-r+j}_{\mathrm{rig},c}(Z)\bigr) = \det\bigl(1 - q^{j-r} t \alpha_a^{(n+r-j)} \big| \mathrm{H}^{n+r-j}_{\mathrm{rig}}(\mathbf{A}^{n+r}; g^*\mathcal{L}_\pi)\bigr).$$

Proof. Recall the statement of Poincaré duality for rigid cohomology (see [LS07, Corollary 8.3.14] and [Ked06, Theorem 1.2.3]; note that the latter article ignores the Tate twist). For a nonsingular, geometrically connected variety X of dimension N, and any closed subvariety Y of X, we have a perfect pairing

$$\mathrm{H}^i_{\mathrm{rig},Y}(X) \otimes_K \mathrm{H}^{2N-i}_{\mathrm{rig},c}(Y) \to K(-N).$$

Applying Poincaré duality and Theorem 2.10 to $X = \mathbf{A}^n$ and $Y = Z$, we find that the dual space of $\mathrm{H}^{n-r+j}_{\mathrm{rig},c}(Z)(n)$ is isomorphic to $\mathrm{H}^{n+r-j}_{\mathrm{rig},Z}(\mathbf{A}^n) \simeq \mathrm{H}^{n+r-j}_{\mathrm{rig}}(\mathbf{A}^{n+r}; g^*\mathcal{L}_\pi)$, in a way that commutes with semilinear Frobenius actions. This implies that the ath iterations of these semilinear maps, which are linear, also match.

The ath iteration of the semilinear Frobenius operator on $\mathrm{H}^{n-r+j}_{\mathrm{rig},c}(Z)(n)$ is the linear Frobenius operator twisted by q^{-n}, i.e., the map $q^{-n}\mathrm{Frob}_q \colon \mathrm{H}^{n-r+j}_{\mathrm{rig},c}(Z) \to \mathrm{H}^{n-r+j}_{\mathrm{rig},c}(Z)$. By the above discussion, it corresponds to the transpose of the inverse of the ath iteration of the semilinear Frobenius operator $\varphi^{(n+r-j)}$ on the overconvergent Dwork cohomology $\mathrm{H}^{n+r-j}_{\mathrm{rig}}(\mathbf{A}^{n+r}; g^*\mathcal{L}_\pi)$. Since the characteristic polynomial of the transpose of a linear operator is identical to that of the operator itself, we conclude that

$$\begin{aligned}\det(1 - t\mathrm{Frob}_q | \mathrm{H}^{n-r+j}_{\mathrm{rig},c}(Z)) &= \det(1 - tq^n (\varphi^{(n+r-j)})^{-a} | \mathrm{H}^{n+r-j}_{\mathrm{rig}}(\mathbf{A}^{n+r}; g^*\mathcal{L}_\pi)) \\ [\text{By } (2.6)] &= \det(1 - tq^{j-r}\alpha_a | \mathrm{H}^{n+r-j}_{\mathrm{rig}}(\mathbf{A}^{n+r}; g^*\mathcal{L}_\pi)).\end{aligned}$$

This completes the proof. ∎

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3. Visibility of Frobenius eigenvalues

Let $f_1, \ldots, f_r \in \mathbf{F}_q[x_1, \ldots, x_n]$ be a collection of polynomials. For every subset $I \subset \{1, 2, \ldots, r\}$, set

$$Z_I = \operatorname{Spec} \mathbf{F}_q[x_1, \ldots, x_n]/(f_i : i \in I),$$

and $Z_I^* = Z_I \cap \mathbf{G}_{\mathrm{m}}^n$. Write $Z = Z_{\{1,2,\ldots,r\}}$.

The purpose of this section is to prove Theorem 1.2; that is, we want to show that the Frobenius eigenvalues of Z are weakly visible in the zeta functions $\zeta_{Z_I^*}(t)$.

The idea of the proof of Theorem 1.2 can be explained as follows.

(i) By Corollary 2.14, the overconvergent Dwork cohomology associated to Z computes the rigid cohomology of Z. Hence, the Frobenius eigenvalues of Z can be manifestly computed using the operator α_a. This is our starting point of the proof of Theorem 1.2.

(ii) The chain-level operators $\alpha_a|_{B_I}$ are 'nuclear operators'. Since cohomology spaces are subquotients of $\bigoplus B_I$, Monsky's spectral theory (see Lemmas 3.1 and 3.2) implies that the cohomological eigenvalues are also 'eigenvalues' of $\bigoplus \alpha_a|_{B_I}$. In the latter context, 'eigenvalue' should be interpreted as the reciprocal roots of the Fredholm determinant of α_a. Thus, Frobenius eigenvalues are visible in the Fredholm determinants $\det(1 - t\alpha_a|B_I)$.

(iii) The spaces B_I are all α_a-invariant subspaces of B. Thus, $\det(1 - t\alpha_a|B)$ witnesses all the 'chain-level eigenvalues' of $\alpha_a|_{B_I}$. The Dwork trace formula, applying to $\alpha_a|_B$, equates an alternating product of $\det(1 - t \cdot q^m \alpha_a|B)$ with an alternating product of zeta functions (see Theorem 3.7 and Lemma 3.8). A Möbius inversion (see the formulae in Definition 3.4) then allows us to represent $\det(1 - t\alpha_a|B)$ as an infinite product of zeta functions. The Frobenius eigenvalues, which are visible in the Fredholm determinant, are thereby weakly visible in the zeta functions.

We carry out the details. To show that the Fredholm determinants $\det(1 - t\alpha_a|B_I)$ contain all the information of Frobenius eigenvalues, we need the following lemma, extracted from [Mon71, Theorem 1.4].

LEMMA 3.1. *Let X be a variable. Let $M' \xrightarrow{f} M \xrightarrow{g} M''$ be a complex of nuclear $K[X]$-modules in the sense of* [Mon71, Definition 1.4]. *Then $H = \operatorname{Ker} g / \operatorname{Im} f$ is also nuclear and $\det(1 - t \cdot X|H)$ is a factor of $\det(1 - t \cdot X|M)$.*

Proof. We follow Monsky's notation in [Mon71, § 1]. By [Mon71, Theorem 1.4], H is a nuclear $K[X]$-module. It follows that, for any bounded subset (cf. [Mon71, Definition 1.3]) $S \subset K[X] - XK[X]$, we have a decomposition $H = N(S, H) \oplus F(S, H)$ [Mon71, Theorem 1.1]. By [Mon71, Proof of Theorem 1.4, eighth line], $N(S, H)$ is a subquotient of $N(S, M)$. As both $N(S, H)$ and $N(S, M)$ are finite dimensional, $\det(1 - tX|N(S, H))$ divides $\det(1 - tX|N(S, M))$. Taking the limit by letting S run through all bounded subsets, we conclude that $\det(1 - tX|H)$ is a factor of $\det(1 - tX|M)$. □

LEMMA 3.2. *For each $I \subset \{1, 2, \ldots, n + r\}$ and each $i \in \mathbf{Z}$, $\det(1 - t \cdot q^{-i}\alpha_a|B_I)$ is a p-adic entire function. Moreover, the polynomial $\det(1 - t\mathrm{Frob}_q|\mathrm{H}^{n-r+j}_{\mathrm{rig,c}}(Z))$ is a factor of*

$$\prod_{|I|=n+r-j} \det(1 - t \cdot q^{j-r}\alpha_a|B_I).$$

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Proof. All summands B_I of (2.1) are nuclear $K[X]$-modules, where the action of X comes from that of α_a. Since $\alpha_a|_{B_I}$ is nuclear, the series $\det(1-t\cdot q^{-i}\alpha_a|B_I)$ is a p-adic entire function by Monsky's theory. Since $\Omega^{n+r-j}=\bigoplus_{|I|=n+r-j}B_I\frac{dx^I}{x^I}$, $\alpha_a^{(n+r-j)}=\bigoplus\alpha_a|_{B_I}$, and since the overconvergent Dwork cohomology $\mathrm{H}^{n+r-j}(\mathbf{A}^{n+r};g^*\mathcal{L}_\pi)$ is computed as the cohomology of $\Omega^{n+r-j-1}\to\Omega^{n+r-j}\to\Omega^{n+r-j+1}$, Corollary 2.14 and Lemma 3.1 imply immediately that $\det(1-t\mathrm{Frob}_q|\mathrm{H}^{n-r+j}_{\mathrm{rig},c}(Z))$ is a factor of $\prod_{|I|=n+r-j}\det(1-t\cdot q^{j-r}\alpha_a|B_I)$. □

Corollary 3.3. *Any Frobenius eigenvalue of* $\mathrm{H}^*_{\mathrm{rig},c}(Z)$ *is weakly visible in* $\det(1-t\alpha_a|B)$.

Proof. By Lemma 3.2, Frobenius eigenvalues of $\mathrm{H}^{n-r+j}_{\mathrm{rig},c}(Z)$ are weakly visible in the product

$$\prod_{|I|=j}\det(1-t\alpha_a|B_I).$$

The corollary follows since B_I is a nuclear submodule of B. □

Next, we explain how to read off the Fredholm determinants from the zeta functions.

Definition 3.4. Following Dwork, we introduce an operation δ on the set $1+t\mathbf{C}_p[\![t]\!]$ of formal power series with constant term one given by

$$\delta(\Gamma(t))\stackrel{\mathrm{def}}{=\!=}\frac{\Gamma(t)}{\Gamma(qt)}.$$

The endomorphism δ is invertible, and its inverse reads

$$\delta^{-1}(\Gamma(t))=\prod_{i=0}^{\infty}\Gamma(q^it).$$

Lemma 3.5. *If* $\Gamma(t)\in 1+t\mathbf{C}_p[\![t]\!]$ *is an entire function, then* $\delta^{-1}\Gamma(t)$ *is also an entire function.*

Proof. Let $\Gamma(t)=\prod_{j=1}^{\infty}(1-\gamma_it)$ be its infinite product expansion, where the reciprocal zero γ_j approaches zero as j goes to infinity. Then

$$\delta^{-1}(\Gamma(t))=\prod_{i=0}^{\infty}\prod_{j=1}^{\infty}(1-q^i\gamma_jt),$$

is also such an infinity product whose reciprocal zero approaches zero. □

Lemma 3.6. *Assume that* $\Gamma\in 1+t\mathbf{C}_p[\![t]\!]$ *is a* p*-adic meromorphic function on* $\mathbf{C}_p$ *and that* $\lambda\in\mathbf{C}_p$ *is weakly visible in* Γ. *Then* λ *is weakly visible in* $\delta(\Gamma)$.

Proof. Let $Z(t)=\delta(\Gamma)$. Then $\Gamma(t)=\prod_{i=0}^{\infty}Z(q^it)$. Write $Z(t)=u(t)/v(t)$, where u,v are entire functions, without common zeros. Then

$$\Gamma(t)=\prod_{i=0}^{\infty}\frac{u(q^it)}{v(q^it)}=\frac{\delta^{-1}(u(t))}{\delta^{-1}(v(t))}.$$

By Lemma 3.5, $\delta^{-1}u(t)$ and $\delta^{-1}v(t)$ are entire. If $q^m\lambda$ is a reciprocal zero of $\Gamma(t)$, then $1-q^m\lambda t$ must be a factor of the infinite product $\delta^{-1}(u(t))=\prod_i u(q^it)$. Hence, $q^m\lambda$ is a reciprocal zero of $u(q^it)$ for some i, i.e., $q^{m-i}\lambda$ is a reciprocal zero of $Z(t)$. Thus, λ is weakly visible in $Z(t)$. The polar case is similar. □

At this point, we recall the Dwork trace formula [Dwo60, p. 637, Lemma 2]. Remember that, for each choice of π, the value of the overconvergent function $\theta(t)=\exp(\pi t-\pi t^p)$ at $t=1$ is a

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primitive pth root of unity, and the function

$$\Psi(t) = \theta(1)^{\mathrm{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(t)} \tag{3.1}$$

is a nontrivial additive character on $\mathbf{F}_q$.

THEOREM 3.7 (Dwork trace formula). *For each positive integer m, define the mth toric exponential sum*

$$S_m^*(g) = \sum_{x\in\mathbf{G}_{\mathrm{m}}^{n+r}(\mathbf{F}_{q^m})} (\Psi\circ \mathrm{Tr}_{\mathbf{F}_{q^m}/\mathbf{F}_q})(g(x)).$$

Then

$$L^*(t) \overset{\mathrm{def}}{=\!=} \exp\left\{\sum_{m=1}^{\infty} S_m^*(g)\frac{t^m}{m}\right\} = \{\delta^{n+r}(\det(1-t\alpha_a|B))\}^{(-1)^{n+r-1}}.$$

Proof. Let $C = K\langle x_1^{\pm 1},\dots,x_{n+r}^{\pm 1}\rangle^\dagger$ be the weak completion of $\mathcal{O}_K[x_1^{\pm 1},\dots,x_{n+r}^{\pm 1}]$ with π inverted. Then the rigid cohomology $\mathrm{H}^*(\mathbf{G}_{\mathrm{m}}^{n+r}; g^*\mathcal{L}_\pi)$ is computed by the exponentially twisted de Rham complex

$$C \to \bigoplus_{|I|=1} C\frac{dx^I}{x^I} \to \cdots \to \bigoplus_{|I|=m} C\frac{dx^I}{x^I} \to \cdots,$$

whose differentials are given by $d+\pi dG$ as in (2.2), and it is equipped with Dwork operators $q^{-m}\alpha_a^{(m)}$ as in Construction 2.8. By [ELS93, Théorème 6.3II] (noticing that θ^i there is our $q^{-i}\alpha_a^{(i)}$), the L-function for the isocrystal $g^*\mathcal{L}_\pi$, which is $L^*(t)$ in our case, equals

$$\prod_{i=0}^{n+r} \det(1-tq^{n+r-i}\alpha_a^{(i)}|\mathrm{H}^i_{\mathrm{rig}}(\mathbf{G}_{\mathrm{m}}^{n+r}; g^*\mathcal{L}_\pi))^{(-1)^{i+1}},$$

(note that $\mathrm{H}^i_{\mathrm{rig}}(\mathbf{G}_{\mathrm{m}}^{n+r}, g^*\mathcal{L}_\pi) = 0$ for $i > n+r$). Since $\alpha_a^{(m)}$ are nuclear operators, the above quantity can also be written as

$$\prod_{i=0}^{n+r} \det(1-tq^{n+r-i}\alpha_a|C^{\binom{n+r}{i}})^{(-1)^{i+1}} = \{\delta^{n+r}(\det(1-t\alpha_a|C))\}^{(-1)^{n+r-1}}. \tag{3.2}$$

If we can substitute C with B in (3.2), we can readily derive the desired result. It suffices to show that $\det(1-t\alpha_a|C/B) = 1$. Recall the formula $\det(1-t\gamma) = \exp(-\sum_{m=1}^{\infty}\frac{\mathrm{Tr}(\gamma^m)m}{t}^m)$. We only have to show that $\mathrm{Tr}(\alpha_a^m|C/B) = 0$ for all positive integers m. A 'basis' for C/B comprises the images of x^u in C/B, where $u \in \mathbf{Z}^{n+r} - \mathbf{N}^{n+r}$. In particular, these u are never zero. Recall that $\alpha_a = \psi^a \circ F_a$ (see (2.3), Construction 2.8). Because $F_a \in B$, the monomials in its power-series expansion are x^v with $v \in \mathbf{N}^{n+r}$. Thus,

$$\alpha_a^m(x^u) = \sum_v b_v x^{(u+v)/q^m},$$

in which $b_v = 0$ if $q^m \nmid u+v$ or if $v \notin \mathbf{N}^{n+r}$. To compute the trace of $\alpha_a^m|_{C/B}$, we need to examine the coefficient of x^u in $\alpha_a^m(x^u)$ for $u \notin \mathbf{N}^{n+r}$. If $u = (u+v)/q^m$, then we have $v = (q^m-1)u$ in $\mathbf{Z}^{n+r}$. Since $u \notin \mathbf{N}^{n+r}$, we have $v \notin \mathbf{N}^{n+r}$. Therefore, if $u = (u+v)/q^m$, we must have $b_v = 0$. From this, we conclude that the diagonal entries of the matrix representation of the operator $\alpha_a^m|_{C/B}$ on the quotient with respect to the 'basis'

$$\{\text{The image of } x^u \text{ in } C/B : u \in \mathbf{Z}^{n+r} - \mathbf{N}^{n+r}\}$$

are all zero. The theorem follows. □

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Combining Corollary 3.3, Lemma 3.6 and Theorem 3.7, we infer that any Frobenius eigenvalue of $\mathrm{H}^{\bullet}_{\mathrm{rig},c}(Z)$ is weakly visible in $L^*(t)$.

Our next objective is to relate the function $L^*(t)$ to the zeta functions of Z_I.

Lemma 3.8. *We have*

$$L^*(t) = \prod_{J\subset\{1,2,\dots,r\}} \zeta_{Z_J^*}(q^{|J|}t)^{(-1)^{r-|J|}}.$$

Proof. For any subset J of $\{1,2,\dots,r\}$, let $\mathbf{A}^J$ denote the affine space with coordinates $(y_j)_{j\in J}$, and let g_J denote the regular function $\sum_{j\in J} y_j f_j(x)$. The orthogonality of characters implies that

$$S_J \stackrel{\text{def}}{=\!=} \sum_{x\in\mathbf{G}_{\mathrm{m}}^n(\mathbf{F}_q)} \sum_{y\in\mathbf{A}^J(\mathbf{F}_q)} \Psi(g_J(x,y)) = q^{|J|}|Z_J^*(\mathbf{F}_q)|.$$

Since $\mathbf{A}^n = \mathbf{G}_{\mathrm{m}}^n \sqcup D$, where D is the union of all coordinate hyperplanes, and since D admits the standard semisimplicial resolution

$$\cdots \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \bigsqcup_{\substack{J\subset\{1,2,\dots,r\}\\ |J|=r-2}} \mathbf{A}^J \;\substack{\longrightarrow\\ \longrightarrow}\; \bigsqcup_{\substack{J\subset\{1,2,\dots,r\}\\ |J|=r-1}} \mathbf{A}^J \longrightarrow D,$$

we deduce from the inclusion-exclusion that

$$\begin{aligned}\sum_{x\in\mathbf{G}_{\mathrm{m}}^{n+r}(\mathbf{F}_q)} \Psi(g(x)) &= \sum_{J\subset\{1,2,\dots,r\}} (-1)^{r-|J|} S_J \\ &= \sum_{J\subset\{1,2,\dots,r\}} (-1)^{r-|J|} q^{|J|} |Z_J^*(\mathbf{F}_q)|.\end{aligned}$$

Replacing $\mathbf{F}_q$ by $\mathbf{F}_{q^m}$ and Ψ by $\Psi\circ\mathrm{Tr}_{\mathbf{F}_{q^m}/\mathbf{F}_q}$ in the above calculation, and exponentiating, we get the desired result. □

Proof of Theorem 1.2. By Corollary 3.3, the Frobenius eigenvalues of $\mathrm{H}^*_{\mathrm{rig},c}(Z)$ are weakly visible in $\det(1-t\alpha_a|B)$. By Lemma 3.6 and Theorem 3.7, the Frobenius eigenvalues are weakly visible in $L^*(t)$. By Lemma 3.8, they are weakly visible in $\{\zeta_{Z_J^*}(t) : J\subset\{1,2,\dots,r\}\}$. □

Remark 3.9. The same proof also works for subvarieties in $\mathbf{G}_{\mathrm{m}}^n$. That is, if $\{f_1,\dots,f_r\}$ is a set of Laurent polynomials and Z_I is the vanishing scheme of $(f_i)_{i\in I}$, then the Frobenius eigenvalues of $Z_{\{1,2,\dots,r\}}$ are weakly visible in the set $\{\zeta_{Z_I}(t) : I\subset\{1,2,\dots,r\}\}$.

4. Reciprocal roots of Fredholm determinants

To establish Theorem 1.3 and Theorem 1.5, we provide a more robust result at the chain level. In this section, we apply Adolphson–Sperber's theory to the specific context of our interest and deduce a lower bound of the q-orders of 'eigenvalues' of the operator $\alpha_a : B_I \to B_I$ introduced in Construction 2.8. Given that rigid cohomologies are subquotients of the entries in the overconvergent Dwork complex (2.2), divisibility at the chain level will imply divisibility at the cohomological level, as per Lemma 3.2.

Notation 4.1. Assume that we are provided with a sequence of polynomials $f_1,\dots,f_r \in \mathbf{F}_q[x_1,\dots,x_n]$, where the degree of f_i is denoted by d_i. These polynomials are ordered such that $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_r$. Let Z be the vanishing scheme of these polynomials $f_1,\dots,f_r$ in the affine space $\mathbf{A}^n_{\mathbf{F}_q}$.

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For a given subset I of $\{1,2,\ldots,n+r\}$, we define I' as $I\cap\{1,2,\ldots,n\}$ and I'' as $I\cap\{n+1,\ldots,n+r\}$.

Recall that B_I represents the subspace $x^I B$ within the Monsky–Washnitzer algebra B (refer to Construction 2.3), and that α_1 and α_a are the Dwork operators acting on B_I (Construction 2.8).

The main result of this section is the following lemma, which is a straightforward consequence of Adolphson–Sperber's result (Theorem 4.10) once we have explained the relations between two different representations of Frobenius structures.

LEMMA 4.2. *Let the notation be as in Construction 2.8. Let λ be a reciprocal root of the p-adic entire function* $\det(1-t\alpha_a|B_I)$. *Then*

$$\operatorname{ord}_q\lambda \geqslant \frac{1}{d_1}\left(|I'|+\sum_{i\in I''}(d_1-d_i)\right)=\frac{1}{d_1}\left(|I|+\sum_{i\in I''}(d_1-d_i-1)\right),$$

where ord_q *is the p-adic valuation normalized so that* $\operatorname{ord}_q(q)=1$.

Remark 4.3. The combination of Lemma 4.2 and Lemma 3.2 already provides interesting bounds on the q-orders of the Frobenius eigenvalues of Z in all cohomological degrees. However, upon closer examination of these bounds, one finds that they are not as strong as asserted by our main theorems. To achieve stronger results, we need to further cut down some excess contributions, with the assistance of an algebraic lemma, Lemma 5.2. In addition, we require some arguments to elevate the q-order estimates to q-divisibility bounds in the ring of algebraic integers. These steps are addressed in the later sections.

Remark 4.4. Lemma 4.2 relies on the work of Adolphson and Sperber [AS87b, Proposition 4.2], which uses a more complicated representation of the Dwork operator. In this remark, we provide an informal explanation of why such a more intricate choice is necessary. For simplicity, we focus on the case when $q=p$, making $\alpha_a=\alpha_1$.

Our objective is to provide sharp lower bounds for the q-orders, or, equivalently, upper bounds for the p-adic absolute values of the reciprocal roots of the Fredholm determinant $\det(1-t\alpha_1|B_I)=\sum_m t^m\cdot\operatorname{Tr}(\wedge^m\alpha_1)$. By the theory of Newton polygons, we need to give upper bounds for $|\operatorname{Tr}(\wedge^m\alpha_1)|$.

We can view α_1 as an infinite matrix by fixing the standard 'basis' of the infinite-dimensional K-vector space B_I that comprises the monomials

$$\{x_1^{u_1}\cdots x_{n+r}^{u_{n+r}}:u_j\geqslant 0 \quad\text{and}\quad u_i\geqslant 1 \text{ if } i\in I\}. \tag{4.1}$$

A straightforward computation shows that the infinite matrix representing α_1 is $[F_{pu-v}]$ (u_j, $v_j\geqslant 0, u_i, v_i\geqslant 1$ if $i\in I$), where $F_1(x)=\sum_{u\in\mathbf{N}^{n+r}}F_u x^u$ is the power-series expansion of the function F_1 (2.3) that defines the Frobenius structure. The number $\operatorname{Tr}(\wedge^m\alpha_1)$ is just the sum of $m\times m$ principal minors of the infinite matrix $[F_{pu-v}]$. Therefore, ultimately, we need a good upper bound for $|F_u|$.

Since the function F_1 is derived from the Dwork exponential $\theta(t)=\exp(\pi t-\pi t^p)$, to get an upper bound for $|F_u|$, we need an upper bound for the coefficient a_j of t^j in the power-series expansion of θ. The most optimal bound is

$$|a_j|\leqslant |p|^{\frac{j(p-1)}{p^2}} \tag{4.2}$$

(cf. [Dwo62, Equation (4.7) and p. 57, line 1]). But this estimate falls short of meeting our need (see (4.4) below).

Therefore, the matrix of the Dwork operator with respect to the naive 'basis' (4.1) does not yield an ideal estimate. From an analytic perspective, the key to improving the estimate lies in finding a suitable 'basis' for B, such that the matrix of the Dwork operator under the new 'basis' has smaller entries. Equivalently, we would like to construct an isomorphism $\rho\colon B\to B$, inducing the following commutative diagram.

$$\begin{array}{ccc} B & \xrightarrow{\rho} & B \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \beta_1} \\ B & \xrightarrow{\rho} & B \end{array}$$

We hope that the matrix of β_1 under the 'basis' (4.1) should have smaller entries than those of α_1. In addition, we expect that β_1 should map B_I to B_I.

For every choice of $s=1,2,\ldots,\infty$, Dwork was able to construct a 'splitting function' of level s [Dwo62, Equation (4.1)]. These splitting functions then give rise to Dwork operators $\alpha_{1,s}$ on B, B_I, and hence Ω^m. All of them can play the role of β_1 above. For $s=1$, we are reduced to the exponentially twisted α_1 that we have been using so far. It is the choice $s=\infty$, which corresponds to what we refer to as the *Artin–Hasse representation* of the Dwork crystal, that has the most optimal convergence property (cf. (4.3) below) among all splitting functions, and this was extensively employed in the work of Adolphson and Sperber. Below, we introduce Dwork's construction for $s=\infty$.

Construction 4.5. Let $E(t)=\exp\{\sum_{j=0}^{\infty}t^{p^j}/p^j\}$ be the Artin–Hasse exponential. It is well known that $E(t)\in\mathbf{Z}_{(p)}[\![t]\!]$. By the theory of Newton polygons, the series $\sum_{j=0}^{\infty}t^{p^j}/p^j$ has a root γ satisfying $|\gamma|=|\pi|=|p|^{1/p-1}$. We define $\theta_\infty(t)=E(\gamma\cdot t)$. Write

$$\theta_\infty(t)=\sum_{j=0}^{\infty}c_jt^j.$$

Then (cf. (4.2))

$$|c_j|\leqslant|\pi|^j=|p|^{\frac{j}{p-1}}. \tag{4.3}$$

The twisting function $\exp(\pi t)$ appearing in the overconvergent Dwork complex may be explained as the infinite product

$$\widehat{\theta}(t)=\exp(\pi t)=\prod_{j=0}^{\infty}\theta(t^{p^j}).$$

Dwork introduced the twisting factor associated with θ_∞ as

$$\widehat{\theta}_\infty(t)=\prod_{j=0}^{\infty}\theta_\infty(t^{p^j})=\exp\left\{\sum_{j=0}^{\infty}\gamma_jt^{p^j}\right\},\quad \text{so we have } \theta_\infty(t)=\widehat{\theta}_\infty(t)/\widehat{\theta}_\infty(t^p).$$

Here, $\gamma_j=\sum_{i=0}^{j}\gamma^{p^i}/p^i$. It is easy to show that $\widehat{\theta}_\infty(t)$ is a rigid analytic function on the open unit disk bounded by 1 (see [Dwo62, Equation (4.13)]). For a conceptual explanation of this unusual exponential function in connection with Frobenii liftings on the formal multiplicative group, we recommend reading Pulita's article [Pul07].

Construction 4.6 (Dwork operators β_1 and β_a associated to the Artin–Hasse exponential). Recall the meaning of $G(x)=\sum_{u\in\mathbf{N}^{n+r}}A_ux^u$, B, B_I given in Construction 2.3 and the definitions of the operators ψ, α_1 and α_a given in Construction 2.8. Also recall that α_1 was defined as an

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exponential twist, i.e., the following diagram is commutative.

$$\begin{array}{ccc} B_I & \xrightarrow{\quad\alpha_1\quad} & B_I \\ {\scriptstyle \cdot\widehat{\theta}}\big\downarrow & & \big\downarrow{\scriptstyle \cdot\widehat{\theta}} \\ K[\![x_1,\dots,x_{n+r}]\!] & \xrightarrow{\tau^{-1}\circ\psi} & K[\![x_1,\dots,x_{n+r}]\!] \end{array}$$

The vertical arrows send ξ to $\xi\cdot\prod_u \widehat{\theta}(A_u x^u)$.

Define the Dwork operator associated to the 'Artin–Hasse representation' as a different twist using the commutativity of the following diagram.

$$\begin{array}{ccc} B_I & \xrightarrow{\quad\beta_1\quad} & B_I \\ {\scriptstyle \cdot\widehat{\theta}_\infty}\big\downarrow & & \big\downarrow{\scriptstyle \cdot\widehat{\theta}_\infty} \\ K[\![x_1,\dots,x_{n+r}]\!] & \xrightarrow{\tau^{-1}\circ\psi} & K[\![x_1,\dots,x_{n+r}]\!] \end{array}$$

The vertical arrows are now given by $\xi\mapsto\xi\cdot\prod_u \widehat{\theta}_\infty(A_u x^u)$. Using the relation between θ_∞ and $\widehat{\theta}_\infty$,

$$\beta_1(\xi)=(\psi(\xi\cdot\Phi))^{\tau^{-1}},$$

where $\Phi(x)=\prod_u \theta_\infty(A_u x^u)$. Since B_I are ideals of B and Φ is overconvergent, by definition, it is clear that $\beta_1\colon B_I\to B_I$ is well defined for any subset I of $\{1,2,\dots,n+r\}$. The operator β_1 is a Dwork operator in the sense of Monsky [Mon71] since $\Phi\in\mathcal{O}_K[\![x_1,\dots,x_{n+r}]\!]\cap B$. We define the operator β_a as the ath iteration of β_1: $\beta_a=\beta_1^a$. Then $\beta_1\colon B_I\to B_I$ is τ^{-1}-semilinear and β_a is K-linear.

The following lemma tells us that the we can use the operator β_a to study the Frobenius eigenvalues of the rigid cohomology.

Lemma 4.7. *Using notation as in Construction 4.6, we have* $\det(1-t\alpha_a|B_I)=\det(1-t\beta_a|B_I)$.

Proof. Since α_1 and β_1 are both defined as twists of $\tau^{-1}\circ\psi$, we have the following commutative diagram.

$$\begin{array}{ccccccc} B_I & \hookrightarrow & K[\![x_1,\dots,x_{n+r}]\!] & \xrightarrow{\rho} & K[\![x_1,\dots,x_{n+r}]\!] & \hookleftarrow & B_I \\ \big\downarrow{\scriptstyle\alpha_1} & & \big\downarrow{\scriptstyle\widetilde{\alpha}_1} & & \big\downarrow{\scriptstyle\widetilde{\beta}_1} & & \big\downarrow{\scriptstyle\beta_1} \\ B_I & \hookrightarrow & K[\![x_1,\dots,x_{n+r}]\!] & \xrightarrow{\rho} & K[\![x_1,\dots,x_{n+r}]\!] & \hookleftarrow & B_I \end{array}$$

The hook arrows are the natural inclusions and, for a power series $\xi\in K[\![x_1,\dots,x_{n+r}]\!]$,

$$\widetilde{\alpha}_1(\xi)=\bigg\{\prod_u \widehat{\theta}(A_u x^u)\bigg\}^{-1}\cdot\tau^{-1}\bigg(\psi\bigg(\prod_u \widehat{\theta}(A_u x^u)\cdot\xi\bigg)\bigg),$$
$$\widetilde{\beta}_1(\xi)=\bigg\{\prod_u \widehat{\theta}_\infty(A_u x^u)\bigg\}^{-1}\cdot\tau^{-1}\bigg(\psi\bigg(\prod_u \widehat{\theta}_\infty(A_u x^u)\cdot\xi\bigg)\bigg),$$
$$\rho(\xi)=\xi\cdot\frac{\prod_u \widehat{\theta}(A_u x^u)}{\prod_u \widehat{\theta}_\infty(A_u x^u)}.$$

It suffices to show that the ratio of twisting factors $\prod_u \widehat{\theta}(A_u x^u)/\widehat{\theta}_\infty(A_u x^u)$, and its reciprocal, are overconvergent, i.e., are elements of B. This will imply that ρ and ρ^{-1} take B_I into B_I, as

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B_I is an ideal of B. By transporting the structures, the ath iterations α_a and β_a of α_1 and β_1 will correspond under $\rho\colon B_I \xrightarrow{\sim} B_I$, and they will have the same Fredholm determinant.

To prove the overconvergence of $\prod_u \widehat{\theta}(A_u x^u)/\widehat{\theta}_\infty(A_u x^u)$, it, in turn, suffices to prove that

$$\widehat{\theta}(t)/\widehat{\theta}_\infty(t) \quad \text{and} \quad \widehat{\theta}_\infty(t)/\widehat{\theta}(t)$$

are overconvergent, i.e., are elements of $K\langle t\rangle^\dagger$. For a documented proof of this basic fact, we refer the reader to Peigen Li's article [Li22, Proposition 2.1(i)], where Li also works out the radius of convergence of the ratio. □

Remark 4.8. The mappings

$$\varphi_\infty\colon \xi \mapsto \xi^\sigma \cdot \left(\prod_u \widehat{\theta}_\infty(A_u x^u)\right)^{-1} \quad \text{and} \quad \varphi\colon \xi \mapsto \xi^\sigma \cdot \left(\prod_u \widehat{\theta}(A_u x^u)\right)^{-1},$$

(see Construction 2.6) together with the connections

$$\nabla = \widehat{\theta}^{-1} \circ d \circ \widehat{\theta} \quad \text{and} \quad \nabla_\infty = \widehat{\theta}_\infty^{-1} \circ d \circ \widehat{\theta}_\infty,$$

define two overconvergent unit-root F-isocrystal structures on

$$B = K\langle x_1, \ldots, x_{n+r}\rangle^\dagger.$$

The above argument shows that the mapping ρ induces an isomorphism between $(B, \nabla, \varphi) = g^*\mathcal{L}_\pi$ and $(B, \nabla_\infty, \varphi_\infty)$. What we need is the slightly stronger result, namely, that β_1 also preserves the subspaces B_I of B: this is to facilitate the chain-level argument.

The plan now is to examine traces and Fredholm determinants of the operator $\beta_a\colon B_I \to B_I$ for subsets I of $\{1, 2, \ldots, n+r\}$, under the 'basis'

$$\{x_1^{u_1} \cdots x_{n+r}^{u_{n+r}} : u_j \geqslant 0 \quad \text{and} \quad u_i \geqslant 1 \text{ if } i \in I\}$$

of B_I. Adolphson and Sperber's idea is to use some combinatorial quantity to measure the absolute value of the entries of the matrix representation of β_a, which we introduce below.

Definition 4.9. Let Δ be the Newton polyhedron of g at infinity, that is, the convex closure of $0 \in \mathbf{R}^{n+r}$ and $\{u \in \mathbf{N}^{n+r} : \text{the coefficient of } x^u \text{ in } g \text{ is nonzero}\}$. Let $C(\Delta)$ be the smallest conical region spanned by Δ.

Define the weight function $w\colon \mathbf{R}^{n+r} \to \mathbf{R}$ by

$$w(y_1, \ldots, y_{n+r}) = y_{n+1} + \cdots + y_{n+r},$$

and define

$$w_I = \min\{w(y) : y \in C(\Delta) \cap \mathbf{N}^{n+r}, y_i > 0, \forall i \in I\}.$$

By construction, if $g(x) = \sum_{u \in \mathbf{N}^{n+r}} a_u x^u$, then $a_u \neq 0$ implies that $u \in \Delta \cap \mathbf{N}^{n+r}$. Since $\Phi(x) = \prod_u \theta_\infty(A_u x^u)$ (see Construction 4.6), the subscripts of the nonzero coefficients of Φ in the power-series expansion are all non-negative integral linear combinations of $u \in \Delta$. Hence, if we write $\Phi(x) = \sum_{u \in \mathbf{N}^{n+r}} \Phi_u x^u$, then $\Phi_u \neq 0$ implies that $u \in C(\Delta)$.

The following theorem is due to Adolphson–Sperber. We have formulated only a weaker version on estimating the first slope, which is sufficient for the purpose of this paper.

Theorem 4.10 [AS87b, Proposition 4.2]. *For any reciprocal root λ of the Fredholm determinant* $\det(1 - t\beta_a | B_I)$,

$$\mathrm{ord}_q(\lambda) \geqslant w_I.$$

Proof. Recall that $\mathbf{F}_q = \mathbf{F}_{p^a}$. By a standard argument (see [Dwo64, Lemma 7.1] or [Bom66, Equation (46)]), a point $(x, y) \in \mathbf{R}^2$ is a vertex of the Newton polygon of $\det(1 - t\beta_a|B_I)$ computed with respect to ord_q if and only if (ax, ay) is a vertex of the Newton polygon of $\det(1 - t\beta_1|B_I)$ (view β_1 as a $\mathbf{Q}_p(\zeta_p)$-linear operator) with respect to the valuation ord_p. Hence, it suffices to estimate the smallest slope of the p-adic Newton polygon of $\det(1 - t\beta_1|B_I)$.

Recall that $\beta_1 = \tau^{-1} \circ \psi \circ \Phi$. We write $\Phi(x) = \sum_{u\in \mathbf{N}^{n+r}} \Phi_u x^u$. Then, by [AS87a, Equation (2.12)],

$$\mathrm{ord}_p(\Phi_u) \geqslant \frac{1}{p-1} w(u). \tag{4.4}$$

(If we use F instead of Φ, we would get a worse bound. So it is crucial to work with the Artin–Hasse representation here.) A $\mathbf{Q}_p(\zeta_p)$-'basis' of B_I is of the form

$$\lambda_l \cdot x^u, \quad l = 1, \ldots, a, u \in \mathbf{N}^{n+r}, u_i \geqslant 1 \text{ if } i \in I,$$

where $\{\lambda_l : l = 1, \ldots, a\}$ is an integral basis of $\mathcal{O}_K$ over $\mathbf{Z}_p[\zeta_p]$. When $I = \emptyset$, the operator β_1 is represented by an infinite matrix

$$C = \begin{bmatrix} C_{00} & C_{01} & \cdots & C_{0i} & \cdots \\ C_{10} & C_{11} & \cdots & C_{1i} & \cdots \\ \vdots & \vdots & & \vdots & \\ C_{i0} & C_{i1} & \cdots & C_{ii} & \cdots \\ \vdots & \vdots & & \vdots & \end{bmatrix},$$

where each C_{ij} is a finite block matrix, corresponding to the components of $\beta_1(\lambda_l x^v)$ with respect to $\lambda_m x^u$, with $w(v) = i$, $w(u) = j$. Any entry in this block matrix is of the form $\lambda_m \lambda_l^{-1} \Phi_{pu-v}$. When $I \neq \emptyset$, the matrix representation of $\alpha_1|_{B_I}$ is a suitable submatrix C_I of C.

An entry of C_I on the diagonal is of the form $\Phi_{(p-1)u}$, where $u_i \geqslant 1$, for all $i \in I$. By (4.4), we find that the trace of $\beta_1|_{B_I}$ satisfies the estimate

$$\mathrm{ord}_p \operatorname{Tr}(\beta_1|B_I) \geqslant w_I.$$

This shows that, in the power-series expansion of $\det(1 - t\beta_1|B_I)$, the p-order of the coefficient of t is at least w_I. When it comes to the coefficient of t^m in the Fredholm determinant, where $m \geqslant 2$, these are computed as sums of $m \times m$ principal minors of the matrix C_I. By employing the determinant formula in terms of the matrix entries, one can readily observe that this coefficient has p-order $\geqslant m w_I$. The theorem can be deduced by examining the Newton polygon of $\det(1 - t\beta_1|B_I)$. □

End of proof of Lemma 4.2. Since the problem only concerns the Fredholm determinant, we can replace α_a by β_a by virtue of Lemma 4.7. By Theorem 4.10, it suffices to show that

$$w_I \geqslant \frac{1}{d}\left(|I'| + \sum_{i\in I''} (d - d_i)\right).$$

This is a problem of linear programming. We are dealing with non-negative integers $y_1, \ldots, y_{n+r}$ subject to the constraints

$$\begin{cases} y_i \geqslant 1, \forall i \in I, \\ y_1 + \cdots + y_n \leqslant d_1 y_{n+1} + \cdots + d_r y_{n+r}, \end{cases} \tag{4.5}$$

and we want to control the minimum of $y_{n+1} + \cdots + y_{n+r}$. Write

$$\xi_i = \begin{cases} y_i - 1, & i \in I; \\ y_i, & i \notin I. \end{cases}$$

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Then (4.5) is equivalent to

$$\begin{cases} \xi_i \geqslant 0 \forall i = 1, 2, \ldots, n+r; \\ \sum_{i=1}^{r} d_i \xi_i \geqslant \sum_{i=1}^{n} \xi_i + |I'| - \sum_{i \in I''} d_i. \end{cases} \tag{4.5$'$}$$

It follows that

$$\begin{aligned} y_{n+1} + \cdots + y_{n+r} &= \sum_{i=1}^{r} \xi_{i+r} + |I''| \\ &= \frac{1}{d_1}\left(\sum_{i=1}^{r}(d_1 - d_i)\xi_{i+r} + \sum_{i=1}^{r} d_i \xi_{i+r}\right) + |I''| \\ \text{[apply (4.5$'$)]} \quad &\geqslant \frac{1}{d_1}\left(|I'| - \sum_{i \in I''} d_i\right) + |I''| = \frac{1}{d_1}\left(|I'| + \sum_{i \in I''}(d_1 - d_i)\right). \end{aligned}$$

This completes the proof. □

5. Two lemmas

The goal of this section is to prove two lemmas, Lemma 5.2 and Lemma 5.4 which will be used in the proofs of Theorem 1.3 and Theorem 1.5.

5.1 An algebraic lemma

Throughout this subsection, we assume that k is an algebraically closed field. The common zero locus, in $\mathbf{A}_k^n$, of a collection of polynomials $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ is be denoted by $Z(f_1, \ldots, f_r)$.

Lemma 5.1. *Let $g_1, \ldots, g_m$ and $f_{m+1}, \ldots, f_r$ be elements of $k[x_1, \ldots, x_n]$. Assume that*

$$\dim Z(g_1, \ldots, g_m) = n - m, \ \dim Z(g_1, \ldots, g_m) > \dim Z(g_1, \ldots, g_m, f_{m+1}, \ldots, f_r). \tag{5.1}$$

Then there are constants $c_{m+1}, \ldots, c_r$ in k, not all zero, such that:

- *if f_{m+1} does not vanish on $Z(g_1, \ldots, g_m)$, $c_{m+1} = 1$;*
- *if f_{m+1} vanishes identically on $Z(g_1, \ldots, g_m)$, $c_{m+1} = 0$; and*
- $\dim Z(g_1, \ldots, g_m, c_{m+1} f_{m+1} + \cdots + c_r f_r) = n - m - 1$.

Proof. Denote the irreducible components of the variety $Z(g_1, ..., g_m)$ by $D_1, \ldots, D_h$. These components have the same dimension by the unmixedness theorem. If f_{m+1} vanishes (identically) on all components $D_1, \ldots, D_h$, then $\dim Z(g_1, ..., g_m, f_{m+1}) = \dim Z(g_1, ..., g_m)$. In this case, we set $c_{m+1} = 0$, drop f_{m+1} from our list and consider the shorter list $\{g_1, \ldots, g_m, f_{m+2}, \ldots, f_r\}$, which still satisfies the condition (5.1).

Hence, without loss of generality, we may assume that f_{m+1} does not identically vanish on $D_1, \ldots, D_{h_1}$ with $h_1 > 0$ but vanishes on $D_{h_1+1}, \ldots, D_h$. If $h_1 = h$, then

$$\dim Z(g_1, \ldots, g_m, f_{m+1}) < \dim Z(g_1, \ldots, g_m).$$

Since the variety $Z(g_1, \ldots, , g_m, f_{m+1})$ is nonempty, this forces that

$$\dim Z(g_1, \ldots, g_m, f_{m+1}) = \dim Z(g_1, \ldots, g_m) - 1,$$

that is, $\dim Z(g_1, ..., g_m, f_{m+1}) = n - m - 1$.

Now assume that $h_1 < h$. By condition (5.1), there is another polynomial among $\{f_{m+2}, \ldots, f_r\}$, say, f_{m+2}, which does not vanish identically on all of $D_{h_1+1}, \ldots, D_h$. Without loss of generality, we may assume that f_{m+2} does not vanish on D_{h_1+1}, …, $D_{h_1+h_2}$ with $h_2 > 0$ but vanishes on $D_{h_1+h_2+1}, \ldots, D_h$.

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CLAIM. There is a nonzero constant c in k such that $f_{m+1}+cf_{m+2}$ does not vanish on the irreducible components $D_1,\dots,D_{h_1},\dots,D_{h_1+h_2}$.

Proof of Claim. Because f_{m+1} vanishes on $D_{h_1+1},\dots,D_{h_1+h_2}$ and f_{m+2} does not, for any nonzero constant c in k, the polynomial $f_{m+1}+cf_{m+2}$ does not vanish on $D_{h_1+1},\dots,D_{h_1+h_2}$. For each $i=1,\dots,h_1$, we can choose x_i in D_i such that $f_{m+1}(x_i)$ is nonzero as f_{m+1} is not identically zero on D_i. Choose nonzero constant c in k such that none of the h_1 numbers

$$f_{m+1}(x_i)+cf_{m+2}(x_i), i=1,\dots,h_1,$$

is zero: one simply chooses any nonzero c in k such that c is not among the h_1 numbers

$$\{-f_{m+1}(x_i)/f_{m+2}(x_i), i=1,\dots,h_1\},$$

which is possible since k is an infinite field. The claim is proved. □

Repeating the above procedure, we see that there are constants $c_{m+1},\dots,c_r$ in k such that the linear combination

$$g_{m+1}=c_{m+1}f_{m+1}+c_{m+2}f_{m+2}+\cdots+c_rf_r$$

does not vanish identically on the component D_i for $i=1,\dots,h$. It follows that

$$\dim Z(g_1,\dots,g_m,g_{m+1})=n-m-1.$$ □

LEMMA 5.2. *Let $f_1,\dots,f_r\in k[x_1,\dots,x_n]$ be a collection of polynomials. Set $d_i=\deg f_i$. Assume that $d_1\geqslant d_2\geqslant\cdots\geqslant d_r$. Let $Z=Z(f_1,\dots,f_r)$. Then there exists a new sequence of polynomials $g_1,\dots,g_r\in k[x_1,\dots,x_n]$ such that:*

(1) $Z(g_1,\dots,g_r)=Z$;
(2) $\deg g_i\leqslant d_i$; *and*
(3) $\dim Z(g_1,\dots,g_{n-\dim Z})=\dim Z$.

Proof. Applying Lemma 5.1 repeatedly gives rise to a new sequence of polynomials $g_1,\dots,g_r\in k[x_1,\dots,x_n]$, which satisfies the following.

– $g_1=f_1$, $g_m=f_m$ if $m>n-\dim Z$.
– There exists an upper-triangular square matrix $B=(b_{\alpha\beta})_{1\leqslant\alpha,\beta\leqslant r}$ with entries in k, whose diagonal entries are either 0 or 1, such that

$$\begin{bmatrix} g_1\\ \vdots\\ g_r\end{bmatrix}=B\cdot\begin{bmatrix} f_1\\ \vdots\\ f_r\end{bmatrix}. \tag{5.2}$$

– $\dim Z(g_1,\dots,g_{n-\dim Z})=\dim Z$.

Thus, the condition (3) is ensured. By construction, $\deg g_i\leqslant d_i$ for any $i=1,2,\dots,r$. The condition (2) is checked.

Proof of (1). Since $g_1,\dots,g_r$ are k-linear combinations of $f_1,\dots,f_r$, Z is contained in the variety $Z(g_1,\dots,g_r)$. We prove that $Z(g_1,\dots,g_r)\subset Z$.

If the jth diagonal entry of B is zero, we say that j is a 'jumping' index. By Lemma 5.1, for each jumping j, f_j vanishes identically on $Z(g_1,\dots,g_{j-1})$; hence, f_j vanishes identically on $Z(g_1,\dots,g_r)$ as well.

It remains to show that if β is not a jumping index, then $f_\beta(Q)=0$ for any $Q\in Z(g_1,\dots,g_r)$. For each jumping j, remove the jth row and jth column from the matrix B. The resulting matrix

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C is upper triangular, and its diagonal entries are all 1. In particular, C is invertible. Evaluating (5.2) at Q, using the vanishing of jumping f_j at Q, we see that, for any nonjumping index α,

$$0 = g_\alpha(Q) = \sum_{\beta \text{ nonjumping}} b_{\alpha\beta} f_\beta(Q),$$

The matrix associated with the above system of linear equations is the invertible matrix C. Thus, $f_\beta(Q) = 0$ for any nonjumping β. This concludes the proof. □

5.2 A lemma on local cohomology

In this subsection, we prove a lemma (Lemma 5.4) on local cohomology in the theory of arithmetic $\mathscr{D}$-modules. It is the analogue of the fact that, in ℓ-adic sheaf theory, the Verdier dual of the shifted constant sheaf $\mathbf{Q}_\ell[\dim Z]$ (hence, $\mathbf{Q}_\ell[\dim Z]$ itself) is a perverse sheaf if Z is a local complete intersection (cf. [KW01, Lemma III 6.5]).

We review some basic concepts about arithmetic $\mathscr{D}$-modules. The reader is referred to Abe and Caro [AC18, § 1] or Abe [Abe18, § 1] for up-to-date surveys.

Let k be a perfect field (for us, $k = \mathbf{F}_q$). Let $\mathcal{O}_K$ be a complete discrete valuation ring with residue field k and field of fractions K. Assume that K has characteristic 0. Let $S = \mathrm{Spf}(\mathcal{O}_K)$.

Let $\mathcal{P}$ be a smooth formal scheme over S. Berthelot [Ber96, § 2.4] introduced a sheaf $\mathscr{D}^\dagger_{\mathcal{P},\mathbf{Q}}$ on $\mathcal{P}$ whose sections are infinite-order differential operators on $\mathcal{P}$ of *finite level*. Caro introduced several finiteness conditions on coherent $\mathscr{D}^\dagger_{\mathcal{P},\mathbf{Q}}$-modules: overcoherence [Car04, Définition 3.1.1], overholonomicity [Car09, Définition 3.1] and 'devissability' by overconvergent F-isocrystals [Car07, Définition 3.2.5 and CT12, Definition 2.3.1]. Finally, in [CT12, Theorem 2.3.16], Caro and Tsuzuki proved that with the presence of Frobenius structures, these finiteness notions are equivalent.

Let $\mathrm{Hol}(\mathcal{P})$ denote the strictly full, thick subcategory of $\mathscr{D}^\dagger_{\mathcal{P},\mathbf{Q}}$-modules generated by overholonomic $\mathscr{D}^\dagger_{\mathcal{P},\mathbf{Q}}$-modules that can be endowed with q^s-power Frobenius structures for some s, although the Frobenius structure is not part of the defining data. Let $D^b_{\mathrm{hol}}(\mathcal{P})$ be the strictly full subcategory of $D^b(\mathscr{D}^\dagger_{\mathcal{P},\mathbf{Q}})$ consisting of complexes with cohomology lying in $\mathrm{Hol}(\mathcal{P})$.

Since $D^b_{\mathrm{hol}}(\mathcal{P})$ is stable under the usual and extraordinary direct image functors, the usual and extraordinary inverse image functors, and duality functors, as shown in [Car09], one can use these categories to canonically associate to each realizable k-variety (defined below) V a coefficient category $D^b_{\mathrm{hol}}(V/K)$ for p-adic cohomology theory, amenable to the Grothendieck six-functor formalism (see [Car12, AC18]).

A *realizable variety* V over k is a variety that admits an immersion $V \to \mathcal{P}$, where $\mathcal{P}$ is a smooth, *proper* formal scheme over S. For each realizable variety V over k, and any immersion $V \to \mathcal{P}$ as above, we have a functor

$$D^b_{\mathrm{hol}}(V/K) \to D^b_{\mathrm{hol}}(\mathcal{P}), \tag{5.3}$$

and this functor induces an equivalence between $D^b_{\mathrm{hol}}(V/K)$ and the strictly full subcategory of $D^b_{\mathrm{hol}}(\mathcal{P})$ consisting of objects that are supported on V, in the sense that the following natural arrows are isomorphisms: i.e.,

$$\mathbf{R}\underline{\Gamma}^\dagger_{\overline{V}}(\mathcal{M}) \xrightarrow{\sim} \mathcal{M} \xrightarrow{\sim} \mathcal{M}({}^\dagger(\overline{V} - V)),$$

where $\overline{V}$ is the Zariski closure of V in $\mathcal{P} \otimes_{\mathcal{O}_K} k$. For the definition of the local cohomology functor $\mathbf{R}\underline{\Gamma}^\dagger_Z$, see [Car04, 2.1.3 (divsior case) and Définition 2.2.6 (general case)]. For the definition of the functor $\mathcal{M} \mapsto \mathcal{M}({}^\dagger Z)$, see [Car04, Définition 2.2.6].

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For any realizable variety V over k, $D^b_{\mathrm{hol}}(V/K)$ has a standard t-structure [AC18, § 1.2]. The objects in the heart of $D^b_{\mathrm{hol}}(V/K)$ are analogues to perverse sheaves on V in the ℓ-adic theory.

For a morphism $f\colon V\to V'$ of realizable varieties, we have the ordinary and extraordinary inverse image functors

$$f^+, f^!\colon D^b_{\mathrm{hol}}(V'/K)\to D^b_{\mathrm{hol}}(V/K),$$

and the ordinary and extraordinary direct image functors

$$f_+, f_!\colon D^b_{\mathrm{hol}}(V/K)\to D^b_{\mathrm{hol}}(V'/K).$$

They satisfy the usual adjunction properties. For each V, there is a duality functor

$$\mathbf{D}_V\colon D^b_{\mathrm{hol}}(V/K)^{\mathrm{op}}\to D^b_{\mathrm{hol}}(V/K),$$

which is a t-exact anti-equivalence satisfying $\mathbf{D}_V^2=\mathrm{Id}$. The duality functors swap ordinary and extraordinary direct and inverse images: i.e.,

$$\mathbf{D}_{V'}f_! = f_+\mathbf{D}_V,\quad \mathbf{D}_{V'}f_+ = f_!\mathbf{D}_V,$$
$$f^!\mathbf{D}_{V'} = \mathbf{D}_V f^+,\quad f^+\mathbf{D}_{V'} = \mathbf{D}_V f^!.$$

Example 5.3. Suppose that $\mathcal{P}$ is a purely n-dimensional, proper smooth formal scheme over S. Let H be a divisor of $P=\mathcal{P}\otimes_{\mathcal{O}_K} k$ and $U=P-H$. Then we have an equivalence

$$D^b_{\mathrm{hol}}(U/K)\simeq\{\mathcal{M}\in D^b_{\mathrm{hol}}(\mathcal{P}):\mathcal{M}\xrightarrow{\sim}\mathcal{M}(^\dagger H)\}.$$

Under this equivalence, the complex $K_U[n]=a^+K[n]$, where $a\colon U\to\operatorname{Spec}k$ is the canonical morphism, is represented by $\mathcal{O}_{\mathcal{P},\mathbf{Q}}(^\dagger H)$, the (specialization of the) sheaf of function on the rigid analytic space $\mathcal{P}_K$ overconvergent along H. Moreover, $\mathcal{O}_{\mathcal{P},\mathbf{Q}}(^\dagger H)$ is self-dual, that is, $\mathbf{D}_U(K_U[n])\simeq K_U[n]$ (see [Abe18, § 1.5.6]). Lastly, we mention that the objects in the heart of the t-structure of $D^b_{\mathrm{hol}}(U/K)$ are represented by actual overholonomic modules with overconvergent singularities along H: i.e.,

$$\{\mathcal{M}\in\mathrm{Hol}(\mathcal{P}):\mathcal{M}\xrightarrow{\sim}\mathcal{M}(^\dagger H)\}.$$

LEMMA 5.4. *Regard $\mathbf{A}^N_k$ as a locally closed subscheme of the formal projective space $\widehat{\mathbf{P}}^N$ over* $\mathrm{Spf}(\mathcal{O}_K)$. *Set $H=\mathbf{P}^N_k-\mathbf{A}^N_k$ to be the hyperplane at infinity. Let $f_1,\dots,f_r$ be regular functions on $\mathbf{A}^N_k$, defining a closed subscheme Z of $\mathbf{A}^N_k$. Then*

$$\mathcal{H}^m\{\mathbf{R}\underline{\Gamma}^\dagger_Z(\mathcal{O}_{\widehat{\mathbf{P}}^N,\mathbf{Q}}(^\dagger H))[r]\}=0 \textit{ unless } -[\dim Z-(N-r)]\leqslant m\leqslant 0.$$

In particular, if $\dim Z=N-r$, *then* $\mathcal{H}^m\{\mathbf{R}\underline{\Gamma}^\dagger_Z(\mathcal{O}_{\widehat{\mathbf{P}}^N,\mathbf{Q}}(^\dagger H))[r]\}$ *is zero unless* $m=0$.

Proof. First, we prove that

$$\mathbf{R}\underline{\Gamma}^\dagger_Z(\mathcal{O}_{\widehat{\mathbf{P}}^N,\mathbf{Q}}(^\dagger H))[r]\in D^{\leqslant 0}_{\mathrm{hol}}(\widehat{\mathbf{P}}^N). \tag{5.4}$$

Consider the function $g\colon\mathbf{A}^N_k\times\mathbf{A}^r_k\to\mathbf{A}^1_k$ defined by $g=\sum x_{N+i}f_i$. Let $\mathcal{L}$ be the $\mathscr{D}^\dagger$-module on $\mathbf{A}^{N+r}_k$ obtained by regarding the Dwork isocrystal as a $\mathscr{D}^\dagger$-module: i.e., $\mathcal{L}=\mathrm{sp}_+(g^*\mathcal{L}_\pi)$ (cf. [AC18, § 1.2.14]). Let $\varpi\colon\mathbf{A}^{N+r}_k\to\mathbf{A}^N_k$ be the projection. Then, by the theorem of Baldassarri and Berthelot (Theorem 2.11),

$$\varpi_+(\mathcal{L})\simeq\mathbf{R}\underline{\Gamma}^\dagger_Z(\mathcal{O}_{\widehat{\mathbf{P}}^N,\mathbf{Q}}(^\dagger H))[r].$$

The inclusion (5.4) follows from the Artin vanishing theorem for arithmetic $\mathscr{D}$-modules (see [AC18, Proposition 1.3.3]).

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Next, we show that

$$\mathbf{R}\underline{\Gamma}^{\dagger}_{Z}(\mathcal{O}_{\widehat{\mathbf{P}}^N,\mathbf{Q}}(^{\dagger}H))[r] \in D^{\geqslant N-r-\dim Z}_{\mathrm{hol}}(\widehat{\mathbf{P}}^N). \tag{5.5}$$

Let $i\colon Z \to \mathbf{A}^N_k$ be the inclusion morphism. Then, by [AC18, § 1.1.7], the local cohomology complex $\mathbf{R}\underline{\Gamma}^{\dagger}_{Z}(\mathcal{O}_{\widehat{\mathbf{P}}^N,\mathbf{Q}}(^{\dagger}H)) \in D^b_{\mathrm{hol}}(\widehat{\mathbf{P}}^N)$ represents $i_+i^!(K_{\mathbf{A}^N_k}[N]) \in D^b_{\mathrm{hol}}(\mathbf{A}^N_k/K)$ via the functor (5.3). Since $\mathbf{D}_{\mathbf{A}^N_k}(K_{\mathbf{A}^N_k}[N]) = K_{\mathbf{A}^N_k}[N]$, (5.5) is equivalent to $i_+i^+(K_{\mathbf{A}^N_k}[N-r]) \in D^{\leqslant \dim Z-(N-r)}_{\mathrm{hol}}(\mathbf{A}^N_k/K)$ by duality. Since i_+ is an exact functor [AC18, Proposition 1.3.2(iii)], it remains to show that

$$i^+K_{\mathbf{A}^N_k} \in D^{\leqslant \dim Z}_{\mathrm{hol}}(Z/K). \tag{5.5bis}$$

To prove this last inclusion, we use some formal properties of the 'constructible t-structure' that introduced in [Abe18, § 1.3]. This is a t-structure $({}^cD^{\leqslant 0}_{\mathrm{hol}}(V/K), {}^cD^{\geqslant 0}_{\mathrm{hol}}(V/K))$ on $D^b_{\mathrm{hol}}(V/K)$ for any variety V. The properties of this t-structure that are relevant to us are the following.

(a) For any closed subvariety W of V, let ι_W denote the inclusion map. Then ${}^cD^{\leqslant 0}_{\mathrm{hol}}(V/K)$ is the full subcategory of $D^b_{\mathrm{hol}}(V/K)$ consisting of $\mathcal{M}$ satisfying the property that, for any closed subvariety W,

$$\mathcal{H}^m\iota^+_W\mathcal{M} = 0 \text{ for any } m > \dim W$$

(see [Abe18, § 1.3.1, second bullet]).

(b) If V is *nonsingular* of pure dimension n, then any overconvergent isocrystal E which admits some q^s-Frobenius structure on V determines a $\mathscr{D}^{\dagger}$-module $\mathcal{E}$. Then $\mathcal{E}[-n]$ (an object in $D^b_{\mathrm{hol}}(V/K)$ of this form is called a *smooth object* in [Abe18]) is in the heart of the constructible t-structure. This is simply because the constructible t-structure is obtained by gluing smooth objects on smooth locally closed subvarieties (see [Abe18, Proposition 1.3.3]).

We apply Property (5.2) with $V = W = Z$. This gives ${}^cD^{\leqslant 0}_{\mathrm{hol}}(Z/K) \subset D^{\leqslant \dim Z}_{\mathrm{hol}}(Z/K)$. So, to prove (5.5bis), we just need to show that $i^+K_{\mathbf{A}^N_k}$ belongs to ${}^cD^{\leqslant 0}_{\mathrm{hol}}(Z/K)$.

We know that $K_{\mathbf{A}^N_k}[N]$ comes from the constant overconvergent isocrystal on $\mathbf{A}^N_k$. By Property (5.2), it lies in the heart of the constructible t-structure, and, in particular, we have that $K_{\mathbf{A}^N_k} \in {}^cD^{\leqslant 0}_{\mathrm{hol}}(\mathbf{A}^N_k/K)$. Since the functor i^+ is exact with respect to the constructible t-structure by [Abe18, Lemma 1.3.2], we have $i^+K_{\mathbf{A}^N_k} \in {}^cD^{\leqslant 0}_{\mathrm{hol}}(Z/K)$. This proves (5.5bis) and completes the proof of the lemma. □

6. Divisibility of Frobenius eigenvalues

We return to the following situation.

Notation 6.1. We are given a collection of polynomials $f_1, \dots, f_r \in \mathbf{F}_q[x_1, \dots, x_n]$, and we denote by

$$Z = \operatorname{Spec} \mathbf{F}_q[x_1, \dots, x_n]/(f_1, \dots, f_r)$$

the vanishing scheme of $f_1, \dots, f_r$. By rearranging the order, we assume that $d_1 \geqslant \cdots \geqslant d_r$, where $d_i = \deg f_i$. The codimension $n - \dim Z$ of Z is denoted by c.

The following easy lemma should be well known. It shows that the vanishing of compactly supported cohomology of Z can be controlled by the number of defining equations of Z. So

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Theorems 1.3 and 1.5 cover all nontrivial cohomology degrees. In its statement, $\mathrm{H}^i_c(Z)$ could either be $\mathrm{H}^i_{\mathrm{rig},c}(Z)$ or $\mathrm{H}^i_c(Z_{\overline{\mathbf{F}}_q}, \mathbf{Q}_\ell)$.

Lemma 6.2. *Let Y be a nonsingular affine variety of dimension n. Let $f_1, \dots, f_r \in \Gamma(Y, \mathcal{O}_Y)$ be regular functions on Y. Let Z be the common zero locus of $f_1, \dots, f_r$ in Y. Then $\mathrm{H}^i_c(Z) = 0$ for $i < n - r$.*

Proof. We have a long exact sequence

$$\cdots \to \mathrm{H}^i_c(Y) \to \mathrm{H}^i_c(Z) \to \mathrm{H}^{i+1}_c(Y - Z) \to \mathrm{H}^{i+1}_c(Y) \to \cdots .$$

If $i \leqslant n-1$, then $\mathrm{H}^i_c(Y) = 0$ by smoothness of Y, Poincaré duality and Artin vanishing. Thus, it suffices to prove that $\mathrm{H}^i_c(Y - Z) = 0$ for $i < n - r + 1$.

Write $Y - Z = \bigcup_{i=r}^r U_i$, where $U_i = Y - \{f_i = 0\}$. Then, for $I \subset \{1, 2, \dots, r\}$, $U_I = \bigcap_{i \in I} U_i$ equals $Y - \{\prod_{i\in I} f_i = 0\}$. We have a Mayer–Vietoris spectral sequence

$$E_1^{-a,b} = \bigoplus_{|I|=a+1} \mathrm{H}^b_c(U_I) \Rightarrow \mathrm{H}^{b-a}_c(Y - Z).$$

Since each U_I is a smooth affine variety of dimension n, by Poincaré duality and Artin vanishing again, $\mathrm{H}^{i+1}_c(U_I) = 0$ if $i < n - 1$. It follows that

$$E_1^{-a,b} \neq 0 \Longrightarrow \begin{cases} b \geqslant n \text{ and} \\ a \leqslant r - 1, \end{cases} \Longrightarrow b - a \geqslant n - r + 1.$$

Therefore, $\mathrm{H}^i_c(Y - Z) = 0$ if $i < n - r + 1$. □

Remark 6.3.

(a) The same argument also works for the Betti cohomology of an algebraic variety Z defined by the vanishing of r regular functions on a smooth affine variety Y over $\mathbf{C}$.

(b) The lemma for rigid cohomology also follows directly from Corollary 2.14.

The remainder of this section is devoted to the proofs of Theorems 1.3 and 1.5.

6.1 Step 1: Reduction

Recall that $n - \dim Z$ is denoted by c. By Lemma 5.2, there exists a finite extension k' of $\mathbf{F}_q$, and a collection of polynomials $g_1, \dots, g_r$, such that:

- $\deg g_1 = d_1$, $\deg g_i \leqslant d_i$;
- Z is the common zero locus of $g_1, \dots, g_r$; and
- $\operatorname{Spec} k'[x_1, \dots, x_n]/(g_1, \dots, g_c)$ has dimension equal to $\dim Z$.

Since the conclusion of Theorem 1.3 is not sensitive to the base field, and since we have

$$\nu_j(n; \deg g_1, \dots, \deg g_r) \geqslant \nu_j(n; d_1, \dots, d_r)$$
$$\epsilon_m(n; \deg g_1, \dots, \deg g_r) \geqslant \epsilon_m(n; d_1, \dots, d_r),$$

it suffices to prove the theorems with $\mathbf{F}_q$ replaced by k' and f_i replaced by g_i. Thus, it suffices to prove Theorems 1.3 and 1.5 under the following additional hypothesis.

$$\textit{The scheme } \operatorname{Spec} \mathbf{F}_q[x_1, \dots, x_n]/(f_1, \dots, f_c) \textit{ has dimension equal to } \dim Z. \tag{6.1}$$

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6.2 Step 2: Slope estimates

First, we prove that the numbers $\nu_j(n; d_1, \ldots, d_r)$ and $\epsilon_m(n; d_1, \ldots, d_r)$ provide lower bounds of the q-order of the Frobenius eigenvalues of $\mathrm{H}^*_{\mathrm{rig},c}(Z)$. Later, we will bootstrap this bound to a bound of q-divisibility of algebraic numbers.

Recall the meaning of g (2.2), G (Construction 2.3), and our convention of the sets I, I' and I'' made in Notation 4.1, as well as the spaces B_I (2.1). Rewrite the overconvergent Dwork complex (2.2) as the total complex of the following double complex (in order to save ink, we have omitted the monomials $\mathrm{d}x^I/x^I$ in the expression).

$$\begin{array}{ccccc}
\bigoplus_{\substack{|I'|=n\\|I''|=0}} B_I & \longrightarrow \cdots \longrightarrow & \bigoplus_{\substack{|I'|=n\\|I''|=c}} B_I & \longrightarrow \cdots \longrightarrow & \bigoplus_{\substack{|I'|=n\\|I''|=r}} B_I \\
\uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow \\
\bigoplus_{\substack{|I'|=0\\|I''|=0}} B_I & \longrightarrow \cdots \longrightarrow & \bigoplus_{\substack{|I'|=0\\|I''|=c}} B_I & \longrightarrow \cdots \longrightarrow & \bigoplus_{\substack{|I'|=0\\|I''|=r}} B_I
\end{array} \tag{6.2}$$

In the diagram, the horizontal differentials are induced by $D'_{n+1}, \ldots, D'_{n+r}$, and the vertical ones are induced by $D'_1, \ldots, D'_n$, where

$$D'_i = x_i \frac{\partial}{\partial x_i} + \pi x_i \frac{\partial G}{\partial x_i} = \exp(-\pi G) \circ x_i \frac{\partial}{\partial x_i} \circ \exp(\pi G), \quad i = 1, 2, \ldots, n+r. \tag{6.3}$$

The following lemma shows that the 0th, 1st, ... and $(c-1)$st columns of the E_1-page of the spectral sequence associated to the double complex (6.2) are all zero.

Lemma 6.4. *For each $0 \leqslant i \leqslant n$, the ith row of (6.2) is exact in cohomology degree $0, 1, \ldots, c-1$.*

Proof. The complex (6.2) uses 'toric' conventions, and it is more suitable for later chain-level manipulations. In the following proof, we use an equivalent 'affine' convention. Write $B = K\langle x_1, \ldots, x_{n+r}\rangle^\dagger$, and let

$$D_j = \frac{\partial}{\partial x_j} + \pi \frac{\partial G}{\partial x_j}, \quad j = 1, \ldots, n+r.$$

Then (6.2) can be written as follows.

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
B^{\oplus n} & \longrightarrow & B^{\oplus nr} & \longrightarrow & B^{\oplus n\binom{r}{2}} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
B & \longrightarrow & B^{\oplus r} & \longrightarrow & B^{\binom{r}{2}} & \longrightarrow & \cdots
\end{array} \tag{6.2$'$}$$

In this double complex, the horizontal arrows are induced by $D_{n+1}, \ldots, D_{n+r}$, and the vertical arrows are induced by $D_1, \ldots, D_n$.

First, we show that the zeroth row of (6.2$'$) is exact in degrees $0, 1, \ldots, c-1$. Since the ith row of (6.2$'$) is an $\binom{n}{i}$-fold direct sum of the zeroth row, the desired exactness, in general, will follow.

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The zeroth row of (6.2′) is itself the total complex of a double complex.

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
B^{\oplus\binom{r-c}{2}} & \longrightarrow & B^{\oplus c\binom{r-c}{2}} & \longrightarrow & B^{\oplus\binom{r-c}{2}\binom{c}{2}} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
B^{\oplus(r-c)} & \longrightarrow & B^{\oplus c(r-c)} & \longrightarrow & B^{\oplus(r-c)\binom{c}{2}} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
B & \longrightarrow & B^{\oplus c} & \longrightarrow & B^{\oplus\binom{c}{2}} & \longrightarrow & \cdots
\end{array}
\tag{6.4}
$$

In the diagram, the horizontal arrows are induced by

$$D_{n+1}, \dots, D_{n+c}$$

and the vertical arrows are induced by

$$D_{n+c+1}, \dots, D_{n+r}.$$

Thus, the ith row is the $\binom{r-c}{i}$-fold direct sum of the zeroth row. If we prove that the zeroth row of (6.4) is acyclic except in top cohomology degree, then every row of (6.4) will be exact except in top cohomology degree. Thus, the total complex of (6.4), which is the zeroth row of the original (6.2′), will have vanishing cohomology in degrees $0, 1, \dots, c-1$.

To prove acyclicity, we make the following auxiliary construction. Consider the projection

$$\varpi' \colon \mathbf{A}^{n+r} \to \mathbf{A}^{n+r-c}, \quad (x_1, \dots, x_{n+r}) \mapsto (x_1, \dots, x_n, x_{n+c+1}, \dots, x_{n+r}).$$

Define $\mathcal{P} = \widehat{\mathbf{P}}^{n+r-c}_{\mathcal{O}_K} \times \widehat{\mathbf{P}}^{c}_{\mathcal{O}_K}$ and $\mathcal{P}' = \widehat{\mathbf{P}}^{n+r-c}_{\mathcal{O}_K}$. The special fiber of $\mathcal{P}$ (respectively, $\mathcal{P}'$) contains $\mathbf{A}^{n+r}$ (respectively, $\mathbf{A}^{n+r-c}$) as a Zariski open subset, with complement H (respectively, H'). Let $\mathcal{L}$ be the overholonomic $\mathscr{D}^{\dagger}_{\mathcal{P},\mathbf{Q}}$-module associated with the Dwork crystal $g'^{*}\mathcal{L}_{\pi}$, where

$$g' = x_{n+1} f_1 + \cdots + x_{n+c} f_c.$$

By construction, $\mathcal{L}$ has an overconvergent singularity along H, i.e., $\mathcal{L} = \mathcal{L}({}^{\dagger}H)$, and is naturally a $\mathscr{D}^{\dagger}_{\mathcal{P},\mathbf{Q}}({}^{\dagger}H)$-module, where $\mathscr{D}^{\dagger}_{\mathcal{P},\mathbf{Q}}({}^{\dagger}H)$ is the sheaf of differential operators on $\mathcal{P}$ of finite level with overconvergent singularities along H (see [Ber96, § 4.2.5] or [BB04, § 2.5]).

Thus, $\mathcal{L}$ represents an object of $D^{b}_{\mathrm{hol}}(\mathbf{A}^{n+r}/K)$. By Theorem 2.11,

$$\varpi'_{+}\mathcal{L} \simeq \mathbf{R}\underline{\Gamma}^{\dagger}_{Z'}(\mathcal{O}_{\mathcal{P}',\mathbf{Q}}({}^{\dagger}H'))[c],$$

where Z' is the zero locus of $f_1, \dots, f_c$ in the affine space $\mathbf{A}^{n} \times \mathbf{A}^{r-c}$: i.e.,

$$Z' = \{x \in \mathbf{A}^{n} : f_1(x) = \cdots = f_c(x) = 0\} \times \mathbf{A}^{r-c}.$$

Given Hypothesis (6.1), we have $\dim Z' = n + r - 2c$, and it follows from Lemma 5.4 that $\mathbf{R}\underline{\Gamma}^{\dagger}_{Z'}(\mathcal{O}_{\mathcal{P}',\mathbf{Q}}({}^{\dagger}H'))[c]$, which is initially a complex of $\mathscr{D}^{\dagger}_{\mathcal{P}',\mathbf{Q}}({}^{\dagger}H')$-modules, is concentrated in degree 0 only. Consequently, by [BB04, Theorem 2.3] (which asserts that the category of $\mathscr{D}^{\dagger}_{\mathcal{P}',\mathbf{Q}}({}^{\dagger}H)$-modules is equivalent to the category of modules over the overconvergent Weyl algebra, the equivalence being induced by the functor $\mathrm{H}^0(\mathcal{P}', -)$), we find that

$$\mathrm{H}^0(\mathcal{P}'; \mathbf{R}\underline{\Gamma}^{\dagger}_{Z'}(\mathcal{O}_{\mathcal{P}',\mathbf{Q}}({}^{\dagger}H'))[c])$$

is a module over the overconvergent Weyl algebra $\mathrm{H}^0(\mathcal{P}'; \mathscr{D}^{\dagger}_{\mathcal{P}',\mathbf{Q}}({}^{\dagger}H'))$, and is thus concentrated in degree 0 only.

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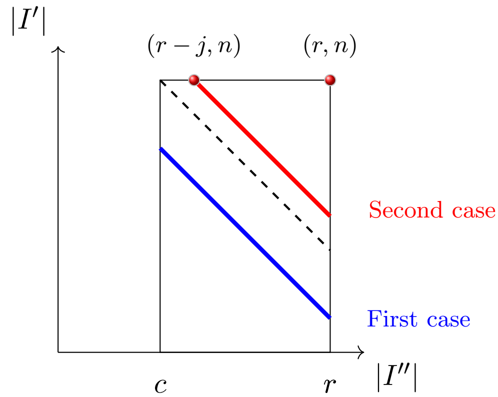

FIGURE 1 (colour online). The double complex.

Now, the zeroth row of (6.4), shifted to the left by r, is

$$B \to B^{\oplus c} \to B^{\oplus \binom{c}{2}} \to \cdots \to B^{\oplus \binom{c}{c-1}} \to \underset{\bullet}{B^{\oplus \binom{c}{c}}}$$

(where the bullet indicates the degree zero item of the chain complex). We want to show that it is acyclic except in degree 0. Since this is a relative de Rham complex with respect to ϖ', according to [BB04, p. 197, Remark], it represents the complex $\mathrm{H}^0(\mathcal{P}'; \varpi'_+\mathcal{L})$, which is isomorphic to

$$\mathrm{H}^0(\mathcal{P}'; \mathbf{R}\underline{\Gamma}^\dagger_{Z'}(\mathcal{O}_{\mathcal{P}',\mathbf{Q}}({}^\dagger H'))[c]),$$

by Theorem 2.11. As discussed in the preceding paragraph, this latter complex is indeed concentrated in degree 0. This completes the proof. □

Since Lemma 6.4 implies that the spectral sequence associated to the double complex (6.2) satisfies

$$E_1^{i,j} = 0, \quad \forall i < c,$$

in Figure 1, only the shaded part contributes to the final abutment of the spectral sequence.

For this reason, $\mathrm{H}^{n+r-j}_{\mathrm{rig}}(\mathbf{A}^{n+r}_{\mathbf{F}_q}, \mathcal{L}_\pi)$ is in fact a subquotient of

$$\bigoplus_{\substack{|I|=n+r-j \\ |I''|\geqslant c}} B_I, \tag{6.5}$$

and Lemma 3.2 can be refined.

LEMMA 6.5. *In Situation 6.1, under Hypothesis 6.1, we have that the Fredholm determinant* $\det(1 - t\cdot F|\mathrm{H}^{n-r+j}_{\mathrm{rig},c}(Z))$ *is a factor of*

$$\prod_{\substack{|I|=n+r-j \\ |I''|\geqslant c}} \det(1 - t\cdot q^{j-r}\alpha_a|B_I).$$

Hence, every Frobenius eigenvalue of $\mathrm{H}^{n-r+j}_{\mathrm{rig},c}(Z)$ is a reciprocal root of $\det(1 - t\cdot q^{j-r}\alpha_a|B_I)$ for some I with $|I| = n+r-j$, $|I''| \geqslant c$.

To proceed, there are two cases.

6.2.1 *First case.* $j \geqslant r-c$. This case corresponds to Theorem 1.3. All the relevant spaces B_I lie on the lower slant line of Figure 1. Let γ be one reciprocal root of $\det(1 - t\cdot q^{j-r}\alpha_a|B_I)$.

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By Lemma 4.2, the q-order of any reciprocal root γ of $\det(1-t\cdot\alpha_a|B_I)$ is at least

$$\frac{1}{d_1}\left(n+r-j+\sum_{i\in I''}(d_1-d_i-1)\right).$$

Since $d_1\geqslant\cdots\geqslant d_r$ and $|I''|\geqslant c$,

$$\begin{aligned}\sum_{i\in I''}(d-d_i-1) &\geqslant \sum_{i=1}^{c}(d-d_i-1)-\sum_{i=c+1}^{|I''|-c}d_i^*\\ &\geqslant \sum_{i=1}^{c}(d-d_i-1)-\sum_{i=c+1}^{r}d_i^*,\end{aligned}$$

where recall that

$$d_i^*=\begin{cases}d_i & \text{if } 1\leqslant i\leqslant c,\\ 1 & \text{if } i>c \text{ and } d_i=d_1,\\ 0 & \text{if } i>c \text{ and } d_i<d_1.\end{cases}$$

Moreover, by [Wan00, Lemma 3.1], the q-order of every reciprocal root of $\det(1-t\cdot\alpha_a|B_I)$ is at least $|I''|\geqslant c$.

Remark 6.6. It should be noted that the cited lemma was stated for a certain Banach space denoted by B^{J_1,J_2} in [Wan00]. In our context, its role is subsumed by the overconvergent space B_I, where $J_1=I'$, $J_2=I''$. The proof of the cited lemma uses only Dwork trace formula, which is applicable to B_I as well.

In fact, the cited lemma actually says that the reciprocal roots of $\det(1-t\cdot\alpha_a|B_I)$ are *algebraic integers* and are divisible by $q^{|I''|}$ in the ring of algebraic integers.

Hence, Lemma 6.5 implies that the q-order of every Frobenius eigenvalue of $\mathrm{H}^{n-r+j}_{\mathrm{rig},c}(Z)$ is at least

$$j-(r-c)+\max\left\{0,\left\lceil\frac{n-j+(r-c)-\sum_{i=1}^{r}d_i^*}{d_1}\right\rceil\right\}. \tag{6.6}$$

Making the change of variable $j-(r-c)\to j$, the above argument implies that the q-order of Frobenius eigenvalues of $\mathrm{H}^{\dim Z+j}_{\mathrm{rig},c}(Z)$ are at least $\nu_j(n;d_1,\ldots,d_r)$.

6.2.2 *Second case.* $0\leqslant j<r-c$, which corresponds to Theorem 1.5. In this case, the spaces B_I appearing in (6.5) all lie on the upper slant line of Figure 1. Thus, $r\geqslant|I''|\geqslant r-j>c$. It follows from Lemma 4.2 that every reciprocal root γ of $\det(1-t\cdot\alpha_a|B_I)$ satisfies $\mathrm{ord}_q\gamma\geqslant(1/d_1)(n+r-j+\sum_{i\in I''}(d_1-d_i-1))$. Since $|I''|\geqslant r-j$, arguing as in the first case,

$$\begin{aligned}&\frac{1}{d_1}\left(n+r-j+\sum_{i\in I''}(d_1-d_i-1)\right)\\ \geqslant&\frac{1}{d_1}\left(n+r-j+\sum_{i=1}^{r-j}(d_1-d_i-1)-\sum_{i=r-j+1}^{r}d_i^*\right)\\ =&\frac{1}{d_1}\left(n-\sum_{i=1}^{r-j}d_i-\sum_{i=r-j+1}^{r}d_i^*\right)+r-j.\end{aligned} \tag{6.7}$$

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Again, by [Wan00, Lemma 3.1], we have $\mathrm{ord}_q\,\gamma \geqslant |I''| \geqslant r-j$. Hence, by Lemma 6.5, the q-order of every Frobenius eigenvalue of $\mathrm{H}^{n-r+j}_{\mathrm{rig},c}(Z)$ is at least

$$\max\left\{0, \left\lceil \frac{n-\sum_{i=1}^{r-j} d_i - \sum_{i=r-j+1}^{r} d_i^*}{d_1} \right\rceil\right\} = \epsilon_j(n; d_1, \ldots, d_r).$$

6.3 Step 3. Bootstrap

To finish the proof of Theorems 1.3 and 1.5, it remains to explain why the *a priori* weaker q-order estimate given above implies the integrality as well as divisibility in the ring of algebraic integers. All we need is the following lemma (see Lemma 6.8 below).

Lemma 6.7. *If γ is a reciprocal root of the Fredholm determinant* $\det(1-t\alpha_a|B_I)$*, then γ is an algebraic integer, and any Galois conjugate of it is still a reciprocal root of* $\det(1-t\alpha_a|B_I)$.

That γ is an algebraic integer had been shown by the first author in [Wan00, Lemma 3.1], but it will naturally come up again in the argument below.

Before proving Lemma 6.7 let us take a tour through the Dwork theory of exponential sums. For $J\subset\{1,2,\ldots,n+r\}$, let $X_{(J)}$ be the linear subspace of $\mathbf{A}^{n+r}$ defined by the vanishing of the variables $(x_j)_{j\in J}$, and let $X^*_{(J)}$ be its standard embedded torus. Let g_J be the restriction of g to $X_{(J)}$. For the nontrivial additive character Ψ (see (3.1)), consider the exponential sum

$$S^*_{(J),m} = \sum_{x\in X^*_{(J)}(\mathbf{F}_{q^m})} (\Psi\circ \mathrm{Tr}_{\mathbf{F}_{q^m}/\mathbf{F}_q})(g_J(x)).$$

Let $B^J = B/\sum_{j\in J} x_j B$. Applying Dwork trace formula (Theorem 3.7) to g_J gives

$$(q^m-1)^{n+r-|J|}\mathrm{Tr}(\alpha_a^m|B^J) = S^*_{(J),m}, \tag{6.8}$$

where, as before, α_a is the nuclear operator defined in § 2.

For $I\subset\{1,2,\ldots,n+r\}$, let $B_I = (\prod_{i\in I} x_i)\cdot B$. Then B/B_I should be thought of as a dagger algebra lifting the divisor $D_I=\{\prod_{i\in I} x_i = 0\}$. The divisor D_I, being of strict normal crossings, has a standard semisimplicial resolution

$$\cdots \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \coprod_{\substack{J\subset I\\|J|=2}} X_{(J)} \;\substack{\longrightarrow\\\longrightarrow}\; \coprod_{\substack{J\subset I\\|J|=1}} X_{(J)} \longrightarrow D_I.$$

By inclusion-exclusion, $\mathrm{Tr}(\alpha_a^m|B_I) = \sum_{J\subset I}(-1)^{|J|}\mathrm{Tr}(\alpha_a^m|B^J)$. Exponentiating, we get

$$\det(1-t\alpha_a|B_I) = \prod_{J\subset I}\det(1-t\alpha_a|B^J)^{(-1)^{|J|}}. \tag{6.9}$$

Proof of Lemma 6.7. The proof of the assertion is based on a similar, but more precise, argument used in the visibility proof.

For each $J\subset\{1,2,\ldots,n+r\}$, by (6.8),

$$\det(1-t\alpha_a|B^J)^{\delta^{n+r-|J|}} = L^*_J(t)^{(-1)^{n+r-|J|-1}},$$

where

$$L^*_J(t) = \exp\left\{\sum_{m=1}^{\infty} S^*_{(J),m}\frac{t^m}{m}\right\}.$$

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But, by Lemma 3.8 (applying to the lower dimensional affine space $X_{(J)}$), L_J^* is an alternating product of zeta functions of $\zeta_{Z^*\cap X_{(E')}}(q^{|E'|}t)$, where E' is the intersection of a subset E of J with $\{1,2,\ldots,r\}$. Thus, by the second equation in Definition 3.4, as well as (6.9), $\det(1-t\alpha_a|B_I)$ is an infinite alternating product of 'shifted' zeta functions $\zeta_{Z^*\cap X_{(E')}}(q^M t)$, where $M\in\mathbf{N}$.

Therefore, any reciprocal root of $\det(1-t\alpha_a|B_I)$, say, γ, is a reciprocal zero or reciprocal pole of some zeta function $\zeta_{Z^*\cap X_{(J)}}(q^M t)$, not being canceled in the infinite product, for some natural number M. Therefore, γ is an algebraic integer. Since such a shifted zeta function is a ratio of integral polynomials of constant term one, the (reciprocal) minimal polynomial of γ must not be canceled either. □

By Lemma 6.7, all the conjugates of γ are still reciprocal roots of $\det(1-t\alpha_a|B_I)$. Thus, the q-orders of the conjugates are bounded by (6.6) or (6.7), depending on $|I|$. The proof of Theorems 1.3 and 1.5 is then concluded thanks to the following elementary lemma.

LEMMA 6.8. *Fix an algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$. Let $\gamma\in\overline{\mathbf{Q}}_p$ be an algebraic integer. Suppose that, for any automorphism σ of $\overline{\mathbf{Q}}_p$, $\mathrm{ord}_q(\sigma(\gamma))\geqslant m$. Then $q^m\mid\gamma$ in the ring of algebraic integers.*

Proof. Let $P(T)=T^e-a_1T^{e-1}+\cdots+(-1)^e a_e$ be the minimal polynomial of γ. Then, for every i, $a_i\in\mathbf{Z}$. Since a_i is an elementary symmetric polynomial of $\sigma(\gamma)$, the hypothesis implies that $\mathrm{ord}_q(a_i)\geqslant m$. Hence, we can write $a_i=q^{im}\cdot b_i$ for some $b_i\in\mathbf{Z}$. The numbers $\sigma(\gamma)\cdot q^{-m}$ all satisfy the polynomial equation $T^e-b_1T^{e-1}+\cdots+(-1)^e b_e=0$, and thus are all algebraic integers. The lemma is proved. □

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Conflicts of interest

None.

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