GLOBAL STABILITY DETERMINED BY LOCAL PROPERTIES AND THE FIRST VARIATION

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1. In this note we consider a system of autonomous differential equations

$$(1.1) \qquad \dot{x} = f(x) \quad (\cdot = \frac{d}{dt}),$$

where $f: E^n \to E^n$ is a continuously differentiable mapping for $n \geq 2$. We shall assume that $f(0) = 0$ and that the origin is locally asymptotically stable.

Suppose f satisfies the above conditions. Under what additional assumptions on f is it possible to infer global stability?

In [1] Hartman and Olech show that if f also satisfies the additional conditions:

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) the symmetric part of the Jacobian $\frac{1}{2}\left(\frac{\partial f(x)}{\partial x} + \frac{\partial f(x)}{\partial x}^T\right) = H(x)$ (^{T} denotes transpose) is such that the sum of any two eigenvalues of $H(x)$ is ≤ 0 for all x,

(iii) $\int^{\infty} \min\limits_{\|x\|=\rho} \|f(x)\| d\rho = +\infty$ ($\|.\|$ denotes the Euclidean norm), then the origin is globally stable.

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Hartman and Olech also consider the problem in a more general Riemannian space than E^n, but the method of proof is not essentially different from that in E^n with a flat metric.

In section 2 we consider a variant of this problem which, so far as we know, was first considered by Krasovskii [2]. Markus and Yamabe [3] have also considered the same problem and they infer the global stability of (1.1) under weaker assumptions than Krasovskii; the most general result along these lines is that of Hartman and Olech. We show that if the local stability of (1.1) is known, then we can infer global stability with weaker assumptions than Hartman and Olech allow on the number of possible stationary points of (1.1). However our conditions on the matrix $\frac{\partial f}{\partial x}$ are stronger than the ones they assume.

2. Let

$$\dot{x} = f(x) \tag{2.1}$$

be defined for all x in E^n and satisfy the conditions:

(2.2)

(i) $f: E^n \to E^n$ is in C^1 for all x in E^n;

(ii) $f(0) = 0$;

(iii) the origin is locally asymptotically stable;

(iv) there exists a positive definite matrix $A = (a_{ij})$ such that $\frac{\partial f}{\partial x}^T A + A\frac{\partial f}{\partial x}$ is negative semi-definite for all x in E^n.

THEOREM 1. If (2.1) satisfies (2.2) then the origin is globally stable.

Proof: Let α denote the set of points attracted to the equilibrium point 0. Because of the conditions (2.2), α is open and connected. We shall suppose the origin is not globally stable. Then the boundary $\partial(\alpha)$ of (α) is not empty. Since 0

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is closed and compact and $\partial(\alpha)$ is closed, it follows that there exists a point x_o in $\partial(\alpha)$ which is a minimal distance from the origin. We consider the straight line L joining x_o to 0. It is evident that the relative interior of L is in α.

Every point on L has the representation $u\underline{e}$, $0 \leq u \leq \|x_o\|$, where $\underline{e} = \frac{x_o}{\|x_o\|}$.

On some arbitrary segment S of L which contains x_o we consider the one parameter family of solutions $x(t, u)$ of (2.1) such that $x(o, u) = \underline{e}u$ for each u. As t evolves in time the segment S is transformed into a curve S(t). For each fixed $t \geq 0$ for which the solution $x(t, \|x_o\|)$ is defined, we consider the arc length $s_o(t)$ of S(t) which is given by

$$(2.3) \qquad s_o(t) = \int_{u_o}^{\|x_o\|} \left[\frac{\partial x_i^2}{\partial u}(t, u) \right]^{1/2} du .$$

Next consider the function

$$(2.4) \qquad s_1(t) = \int_{u_o}^{\|x_o\|} \left[\frac{\partial x}{\partial u}^T (t, u)\, A \,\frac{\partial x}{\partial u}(t, u) \right]^{1/2} du .$$

Since A is positive definite, there exists a constant $m^2 > 0$ such that

$$\frac{\partial x}{\partial u}^T A \frac{\partial x}{\partial u} \geq m^2 \left\|\frac{\partial x}{\partial u}\right\|^2 .$$

Thus $s_1(t) \geq m s_o(t)$ for all t for which s_1 and s_2 are defined. Moreover, since f is in C^1, it follows that s_o and s_1 are continuously differentiable with respect to t (e.g., see [6] chapter 1). The derivative of $s_1(t)$ is given by

(2.5) $$\frac{ds_1}{dt} = \frac{1}{2}\int_{u_o}^{\|x_o\|} \frac{\left[\frac{\partial x}{\partial u}^T (t,u)\, \frac{\partial f}{\partial x}(t,u)A + A\,\frac{\partial f}{\partial x}(t,u)\,\frac{\partial x}{\partial u}(t,u)\right] du}{[\quad]^{1/2}} ,$$

where $[\quad]^{1/2}$ is the integrand in (2.4) and is never zero, since $\frac{\partial x}{\partial u}$ is the solution of a linear differential equation whose initial value is $\frac{x_o}{\|x_o\|}$.

Because of (2.2 iv), $\frac{ds_1}{dt} \le 0$ for all t for which (2.4) is defined. Thus we can write the following inequality

(2.6) $$\frac{s_1(o)}{m} \ge \frac{s_1(t)}{m} \ge s_o(t) \ge \|x(t, \|x_o\|) - x(t, u_o)\| .$$

Let $\varepsilon > 0$ be arbitrary except that $\varepsilon < \frac{\|x_o\|}{2}$. We select u_o such that

(2.7) $$\frac{s_1(o)}{m} < \frac{\varepsilon}{2} .$$

For example, a u_o satisfying

(2.8) $$(\|x_o\|^2 n^2 \sup_{i,j} |a_{ij}|)^{1/2} (\|x_o\| - u_o) < 2\varepsilon m$$

would be satisfactory.

Using (2.7) in (2.6) we obtain

(2.9) $$\|x(t, \|x_o\|)\| \le \|x(t, u_o)\| + \frac{\varepsilon}{2} .$$

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Since $x(t, u_o)$ is defined for all t, so is $x(t, \|x_o\|)$ and hence $s_o(t)$ and $s_1(t)$ are. Also, since $x(o, u_o)$ is in α it follows that there is a $t_1 > 0$ such that $\|x(t, u_o)\| < \frac{\varepsilon}{2}$ for all $t \geq t_1$. Hence from (2.9) $\|x(t, \|x_o\|)\| \leq \varepsilon < \|x_o\|$ for $t \geq t_1$, which is impossible since, by our selection of x_o, it must be true that $\|x(t, \|x_o\|)\| \geq \|x_o\|$ for all $t \geq 0$. Thus $\partial(\alpha) = \emptyset$ and hence $\alpha = E^n$ which proves the theorem.

COROLLARY 1. If f satisfies the conditions of theorem 1 then $f(x) = 0$ if and only if $x = 0$.

Remark. In effect, the proof of theorem 1 consists in showing that the existence of the Liapunov function

$$V(t, \frac{\partial x}{\partial u}) = \frac{\partial x}{\partial u}^T A \frac{\partial x}{\partial u}$$

implies the stability of the variational system

$$\frac{d}{dt}(\frac{\partial x}{\partial u}) = \frac{\partial f}{\partial x}(x(t, u)) \frac{\partial x}{\partial u} \tag{2.10}$$

(i.e. the orbital stability of 2.1) which then implies the global stability of (2.1), since it is locally asymptotically stable.

With this in mind we obtain the following corollary of theorem 1.

COROLLARY 2. If (2.1) satisfies (2.2i), (2.2ii), (2.2iv) and the condition

$$x^T \left[\frac{\partial f}{\partial x}^T A + A \frac{\partial f}{\partial x} \right] x < 0 \tag{2.11}$$

in some neighborhood U of the origin, then the origin is globally stable.

Proof: Let $V(x) = x^T Ax$. Then along solutions of (2.1)

$$\frac{dV}{dt} = f^T A x + x^T A f = \int_0^1 x^T \left[\frac{\partial f}{\partial x}^T (\varepsilon x) A + A \frac{\partial f}{\partial x} (\varepsilon x) \right] x \, d\varepsilon < 0 .$$

Hence V is a Liapunov function for (2.1) in the neighborhood U. Thus the origin is locally stable and theorem 1 applies.

Corollary 2 generalizes a result of Hartman [4] for Euclidean spaces with flat metrics in that Hartman demands that (2.11) hold everywhere in E^n.

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