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INVERSE SEMIGROUPS OF HOMEOMORPHISMS ARE HOPFIAN

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If X is a nonempty topological T_1 space then the set of all homeomorphisms whose domains and ranges are closed subsets of X forms a semigroup under partial composition of functions. We call it $I_F(X)$. If, in a semigroup, every element a is matched with a unique element b such that $aba = a$ and $bab = b$ then the semigroup is an inverse semigroup (b is called the inverse of a and is denoted by a^{-1}). We have that $I_F(X)$ is an inverse semigroup with the algebraic inverse of a map f being just the inverse map f^{-1}. In this paper we examine epimorphisms from $I_F(X)$ onto $I_F(Y)$. The main theorem gives conditions under which an epimorphism must be an isomorphism. A consequence of this theorem is that for many spaces X (including all finite n-dimensional Euclidean cubes I^n, all finite n-dimensional spheres S^n, and the Cantor discontinuum $\mathscr{C}$) every epimorphism from $I_F(X)$ onto $I_F(Y)$ must be an isomorphism (Y is an arbitrary first countable T_1 space). Thus for all of these spaces the semigroup $I_F(X)$ is hopfian (every surjective endomorphism is an isomorphism). Another theorem shows that $I_F(R)$ is also hopfian (R denotes the real line). In [**1**] a research article stated some of these results. The case where X is the unit interval or the Cantor discontinuum was mentioned. The present paper extends those results but uses entirely different techniques.

These inverse semigroups $I_F(X)$ behave nicely in the sense that $I_F(X)$ and $I_F(Y)$ are isomorphic if and only if X and Y are homeomorphic (see [**4**]). In fact, if ϕ is an isomorphism from $I_F(X)$ onto $I_F(Y)$ then there is a homeomorphism h from X onto Y such that $\phi(f) = h \circ f \circ h^{-1}$ for all $f \in I_F(X)$. Idempotents (elements f such that $f \circ f = f$) in $I_F(X)$ are identity maps on closed subsets K of X and will be denoted by $\langle K \rangle$. The identity map on the point y will be denoted by $\langle y \rangle$. The zero of the semigroup $I_F(X)$ is just the empty map and will be denoted by 0. Throughout this paper we shall assume that $|X| > 2$, X is T_2, and Y is nontrivial T_1 (i.e., Y has more than one point). We also assume that ϕ is an epimorphism from $I_F(X)$ onto $I_F(Y)$. Note that we then have that $\phi(0) = 0$ and $(\phi(f))^{-1} = \phi(f^{-1})$. Epimorphisms carry idempotents to idempotents and so if $\langle F \rangle \in I_F(X)$ then $\phi\langle F \rangle = \langle R \rangle$ for some closed subset R of Y. Conversely, if $\langle R \rangle \in I_F(Y)$ then there exists a closed subset F of X such that $\phi\langle F \rangle = \langle R \rangle$ (see [**2**], p. 57). The notation $\langle x, y \rangle$ will denote the homeomorphism whose domain is the point x and whose range is the point y. McAlister [**3**] has shown that if $\phi\langle x, y \rangle \neq 0$ for some $x, y \in X$ then ϕ is an

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isomorphism. If $y \in Y$ let $D_y = \bigcap \{F : \phi\langle F\rangle = \langle y\rangle\}$. The collection $\{F : \phi\langle F\rangle = \langle y\rangle\}$ satisfies the finite intersection property (if $\phi\langle F\rangle = \langle y\rangle = \phi\langle H\rangle$ then $\phi\langle F \cap H\rangle = \phi(\langle F\rangle \circ \langle H\rangle) = \phi\langle F\rangle \circ \phi\langle \mathrm{N}\rangle = \langle y\rangle$). Thus if X is compact then $D_y \neq \emptyset$ for all $y \in Y$.

LEMMA 1. *Suppose ϕ is an epimorphism from $I_F(X)$ onto $I_F(Y)$. (a) Let $k \in I_F(X)$ be such that $K = \operatorname{dom} k$ (domain of k), $H = \operatorname{ran} k$ (range of (k)), $\phi\langle K\rangle = \langle R\rangle$ and $\phi\langle H\rangle = \langle S\rangle$. Then $\phi(k)$ maps R homeomorphically onto S. (b) Suppose $\phi\langle F\rangle = \langle T\rangle$, $R \subseteq T$ and R is homeomorphic to S. Then there exist homeomorphic sets K and H such that $H \subseteq F$, $\phi\langle H\rangle = \langle R\rangle$ and $\phi\langle K\rangle = \langle S\rangle$.*

Proof. (a) We have that

$$\operatorname{dom} \phi(k) = \operatorname{dom} ((\phi(k))^{-1} \circ \phi(k)) = \operatorname{dom} (\phi(k^{-1}) \circ \phi(k)) = \operatorname{dom} \phi(k^{-1} \circ k) = \operatorname{dom} \phi\langle K\rangle = R.$$

Likewise $\operatorname{ran} \phi(k) = S$.

(*b*) Since R is homeomorphic to S there exists $k \in I_F(X)$ such that $\phi(k)$ maps R homeomorphically onto S. Suppose $\operatorname{dom} k = J$ and $\operatorname{ran} k = G$. Then $\phi\langle J\rangle = \phi(k^{-1} \circ k) = \langle R\rangle$ and $\phi\langle G\rangle = \langle S\rangle$. Now

$$\phi\langle J \cap F\rangle = \phi\langle J\rangle \circ \phi\langle F\rangle = \langle R\rangle \circ \langle T\rangle = \langle R \cap T\rangle = \langle R\rangle.$$

Let $H = J \cap F$ and $K = k(H)$. Then $\phi\langle H\rangle = \langle R\rangle$, $H \subseteq F$ and H is homeomorphic to K. We also have that

$$\phi\langle K\rangle = \phi\langle k(H)\rangle = \phi(k \circ \langle H\rangle \circ k^{-1}) = \phi(k) \circ \phi(k)^{-1} = \langle \phi(k)(R)\rangle = \langle S\rangle.$$

LEMMA 2. *Suppose $\phi\langle F\rangle = \langle p\rangle$ for some $p \in Y$ and compact $F \subseteq X$. Then $D_y \neq \emptyset$ for all $y \in Y$ and $D_y = \bigcap \{K : \phi\langle K\rangle = \langle y\rangle, K \text{ compact}\}$.*

Proof. We have that

$$D_p = F \cap \{K : \phi\langle K\rangle = \langle p\rangle\} = \bigcap \{F \cap K : \phi\langle K\rangle = \langle p\rangle\} \subseteq \bigcap \{K : \phi\langle K\rangle = \langle p\rangle, K \text{ compact}\} \subseteq D_p.$$

Thus $D_p = \bigcap \{K : \phi\langle K\rangle = \langle p\rangle, K \text{ compact}\}$ and since the latter collection has the finite intersection property, $D_p \neq \emptyset$. Since F is compact, $\langle F\rangle$ generates an ideal $\mathcal{U}$ whose idempotents are all identities on compact sets. Now $\phi(\mathcal{U})$ is an ideal of $I_F(Y)$ which contains $\langle p\rangle$ and so contains all maps of the form $\langle y\rangle$. Therefore for any $y \in Y$, there is a compact set K such that $\phi\langle K\rangle = \langle y\rangle$. We now apply the first part of the proof to obtain the fact that $D_y \neq \emptyset$ for all y and $D_y = \bigcap \{K : \phi\langle K\rangle = \langle y\rangle, K \text{ compact}\}$.

Definition 3. X will be called *admissible* (respectively *strongly admissible*) if whenever F is a proper compact subset of X (respectively F is a compact subset of X), $x \in F$ and U is any neighborhood of x, then there exists a homeomorphism h from F into U such that $h(x) = x$.

Remark. The space S^1 is admissible but not strongly admissible. All noncompact admissible spaces are strongly admissible. The class of admissible spaces is not productive (e.g., $S^1 \times S^1$) but the class of strongly admissible spaces is productive.

PROPOSITION 4. *The product of strongly admissible spaces is strongly admissible.*

Proof. Let $\prod_{j \in J} X_j$ be a product of strongly admissible spaces, let K be a compact subset of this product, let $q \in K$ and let G be a neighborhood of q. Then there exists a finite set $\{1, 2, \ldots, N\}$ and open sets $G_j \subseteq X_j$ for $j = 1, \ldots, N$ such that

$$q \in p_1^{-1}(G_1) \cap \ldots \cap p_N^{-1}(G_N) \subseteq G$$

where p_j denotes the projection map onto X_j. Since each X_j is strongly admissible there exist homeomorphisms h_j $(j = 1, \ldots, N)$ from $p_j(K)$ into G_j such that $h_j(q_j) = q_j$. Define h from $\prod_{j \in J} p_j(K)$ into $\bigcap_{i=I}^{N} p_j^{-1}(G_j)$ by

$$(h(x))_j = h_j(x_j) \quad \text{for} \quad j = 1, \ldots, N$$

$$(h(x))_j = x_j \quad \text{otherwise.}$$

Then h is a homeomorphism and $h(q) = q$. Since $K \subseteq \prod_{j \in J} p_j(K)$ and $\bigcap_{j=1}^{N} p_j^{-1}(G_j) \subseteq G$ the proof is complete.

PROPOSITION 5. I^n, R^n, I^∞, $\mathscr{C}$ *(the Cantor discontinuum), the space of rational numbers and the space of irrational numbers are all strongly admissible.* S^n *is admissible.*

Proof. It follows from well known results that I, R, $\mathscr{C}$, the rationals and the irrationals are strongly admissible and that S^n is admissible. Now apply Proposition 4.

Remark. Note that the two point discrete space D is not even admissible but the product of D with itself a countable number of times is strongly admissible since it is homeomorphic to $\mathscr{C}$.

LEMMA 6. *Suppose X is admissible and $\phi\langle J\rangle = \langle p\rangle$ for some $p \in Y$ and compact $J \subseteq X$. Let $y, w \in Y$ with $y \neq w$. Then $D_y \cap D_w = \emptyset$.*

Proof. We know that $D_y \neq \emptyset$ for all y by Lemma 2. Now suppose $D_y \cap D_w \neq \emptyset$. Let $x \in D_y \cap D_w$ and let F be such that $\phi\langle F\rangle = \langle y\rangle$. Then there exists $z \in F$ such that $z \notin D_w$ (otherwise if K is such that $\phi\langle K\rangle = \langle w\rangle$ and $F \subseteq K$ then $\phi\langle F\rangle = 0$ which is a contradiction). Now $z \notin D_w$ and so there exists W such that $\phi\langle W\rangle = \langle w\rangle$ but $z \notin W$. The set W is closed and so let U be a neighborhood of z where $W \cap U = \emptyset$. We have that X is admissible, $z \in F$ and $z \in U$ and so U contains a copy of F. Call it Z. Then $\phi\langle Z\rangle = \langle q\rangle$

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for some $q \in Y$ (see Lemma 1). Now $Z \cap W = \emptyset$ and so $Z \cap D_w = \emptyset$. Hence $q \neq w$ and $q \neq y$ ($x \in D_y \cap D_w$). Let $\phi\langle W \cup Z\rangle = \langle R\rangle$. Then $w \in R$ and

$$q \in R(\langle w\rangle = \phi\langle W\rangle = \phi\langle W\rangle \circ \phi\langle W \cup Z\rangle = \langle w\rangle \circ \langle R\rangle = \langle \{w\} \cap R\rangle).$$

By Lemma 1 choose homeomorphic sets K and H where $K \subseteq W \cup Z$, $\phi\langle K\rangle = \langle\{w, q\}\rangle$ and $\phi\langle H\rangle = \langle\{y, w\}\rangle$. Let k map K onto H and let $S = K \cap W$, $Q = K \cap Z$. Then $\phi\langle S\rangle = \langle w\rangle$ and $\phi\langle Q\rangle = \langle q\rangle$. But $S \cap Q = \emptyset$ ($W \cap Z = \emptyset$). Now $k(S) \subseteq H$ and so $\phi\langle k(S)\rangle = \langle y\rangle$ or $\langle w\rangle$. Without loss of generality suppose $\phi\langle k(S)\rangle = \langle y\rangle$. Then $\phi\langle k(Q)\rangle = \langle w\rangle$. Since $S \cap Q = \emptyset$ we have that $k(S) \cap k(Q) = \emptyset$ also. But $x \in k(S) \cap k(Q)$ since $x \in D_y \cap D_w$. This is a contradiction. Thus $D_y \cap D_w = \emptyset$.

LEMMA 7. *Suppose X is admissible and $\phi\langle J\rangle = \langle p\rangle$ for some compact set J. Then $|D_y| = 1$ for all $y \in Y$.*

Proof. Suppose $|D_y| > 1$ for some $y \in Y$. Let $x, z \in D_y$ where $x \neq z$. Let U be a neighborhood of x such that $z \notin U$. Let F be such that $\phi\langle F\rangle = \langle y\rangle$ and since X is admissible let k be a homeomorphism from F into U where $k(x) = x$. Then $\phi\langle k(F)\rangle = \langle w\rangle$ for some $w \in Y$. Now $w \neq y$ since $z \notin k(F)$ and $z \in D_y$. We show that $x \in D_w$. Suppose H is such that $\phi\langle H\rangle = \langle w\rangle$. Then $\phi\langle H \cap k(F)\rangle = \langle w\rangle$. Now $k^{-1}(H \cap k(F)) \subseteq F$ and is homeomorphic to $H \cap k(F)$. Therefore $\phi\langle k^{-1}(H \cap k(F))\rangle = \langle y\rangle$. Thus $x \in k^{-1}(H \cap k(F))$. But since $k(x) = x$ this means that $x \in H \cap k(F)$. Therefore $x \in H$ and so also $x \in D_w$. But then $D_w \cap D_y \neq \emptyset$. This is a contradiction by the last lemma. Thus $|D_y| = 1$ for all $y \in Y$.

Remark. If $\phi\langle J\rangle = \langle p\rangle$ for some $p \in Y$ and compact J then the last lemma says that for each $y \in Y$ there is associated an $x \in X$ such that $D_y = \{x\}$. Define a map h from Y into X by $h(y) = x$. The function h will be one-to-one by Lemma 6.

LEMMA 8. *Suppose X is admissible and $\phi\langle J\rangle = \langle p\rangle$ for some compact J. Then for every $y \in Y$ and every neighborhood U of $h(y)$ there exists a closed set F such that $F \subseteq U$ and $\phi\langle F\rangle = \langle y\rangle$.*

Proof. Suppose not. Let $x = h(y)$ and let U be a neighborhood of x such that for all F with $\phi\langle F\rangle = \langle y\rangle$ there exists $z \in F - U$. We first show that the collection $\{F - U : \phi\langle F\rangle = \langle y\rangle\}$ satisfies the finite intersection property (clearly the sets are nonempty and closed). Consider $\bigcap_{i=1}^n (F_i - U) = (\bigcap_{i=1}^n F_i) - U$ where $\phi\langle F_i\rangle = \langle y\rangle$ for all $i = 1 \ldots n$. We have that $\phi\langle \bigcap_{i=1} F_i\rangle = \langle y\rangle$ also and hence $(\bigcap_{i=1}^n F_i) - U \neq \emptyset$ by assumption. Therefore $\{F - U : \phi\langle F\rangle = \langle y\rangle\}$ satisfies the finite intersection property. Now

$$\bigcap \{F : \phi\langle F\rangle = \langle y\rangle\} = \bigcap \{F : \quad \phi\langle F\rangle = \langle y\rangle,\ F \text{ compact}\}$$

by Lemma 2. Therefore

$$\bigcap \{F - U : \phi\langle F\rangle = \langle y\rangle\} = \bigcap \{F - U : \quad \phi\langle F\rangle = \langle y\rangle,\ F \text{ compact}\} \subseteq K$$

where K is compact and $\phi\langle K\rangle = \langle y\rangle$. Therefore $\bigcap \{F - U : \phi\langle F\rangle = \langle y\rangle\} \neq \emptyset$. But

$$\bigcap \{F - U : \quad \phi\langle F\rangle = \langle y\rangle\} \subseteq \bigcap \{F : \quad \phi\langle F\rangle = \langle y\rangle\} = \{x\}.$$

Thus $\bigcap \{F - U : \quad \phi\langle F\rangle = \langle y\rangle\} = \{x\}$. But $x \in U$. This is a contradiction.

LEMMA 9. *Suppose X is admissible and $\phi\langle J\rangle = \langle p\rangle$ for some $p \in Y$ and compact J. Suppose also that $\phi(k)(y) = z$ for some $k \in I_F(X)$ where $y, z \in Y$. Then $k(h(y)) = h(z)$.*

Proof. Let dom $k = K$ and let F be any closed set where $F \subseteq K$ and $\phi\langle F\rangle = \langle y\rangle$ (see Lemma 1). Then $h(y) \in F$ and

$$\phi(k \circ \langle F\rangle) = \phi(k) \circ \phi\langle F\rangle = \phi(k) \circ \langle y\rangle = \langle y, \phi(k)(y)\rangle = \langle y, z\rangle.$$

Thus the range of $\phi(k \circ \langle F\rangle)$ is z. But by Lemma 1 this means that $\phi\langle k(F)\rangle = \langle z\rangle$ and so $h(z) \in k(F)$. By the last lemma we can take F inside arbitrary neighborhoods U of $h(y)$ and so $k(h(y)) = h(z)$.

Notation. Write $y_\alpha \to y$ if the net $\{y_\alpha\}$ converges to y.

LEMMA 10. *Suppose X is admissible and $\phi\langle J\rangle = \langle p\rangle$ for some $p \in Y$ and compact J. Then Y is not discrete.*

Proof. Suppose Y is discrete. We first show that $h(Y)$ must also be discrete. Hence suppose $h(Y)$ is not discrete and let $h(y_\alpha) \to h(y)$ where $y_\alpha \neq y$ for all α. Since $\{y_\alpha\}$ is homeomorphic to $\{y_\alpha\} \cup \{y\}$ there exists a homeomorphism k such that $\phi(k)$ maps $\{y_\alpha\} \cup \{y\}$ onto $\{y_\alpha\}$. Let $\phi(k)(y) = y_\beta$. By Lemma 9 we have that $k(h(y)) = h(y_\beta)$ and $k(h(y_\alpha)) \in \{h(y_\alpha)\}$ for all α. But $h(y_\alpha) \to h(y)$ and since k is a homeomorphism we have that $k(h(y_\alpha)) \to k(h(y))$. This is a contradiction since $k(h(y)) = h(y_\beta)$. Thus $h(Y)$ is discrete. We may now choose $h(y) \in h(Y)$ and an open neighborhood U of $h(y)$ such that $U \cap h(Y) = \{h(y)\}$. Let $F \subseteq U$ be such that $\phi\langle F\rangle = \langle y\rangle$ and F is compact. If $|F| = 1$ then ϕ is an isomorphism (see [**3**]) and hence X is homeomorphic to Y. But then X is discrete and clearly cannot be admissible (recall that $|X| > 2$). Therefore $|F| \geqq 2$. Let $x \in F$ with $x \neq h(y)$ and let V be a neighborhood of x such that $V \subseteq U$ but $h(y) \notin V$. Then since X is admissible there exists a homeomorphism f from F into V such that $f(x) = x$. The set $f(F)$ is homeomorphic to F and so $\phi\langle f(F)\rangle = \langle z\rangle$ for some z. But

$$f(F) \subseteq V \subseteq X - h(Y)$$

and so $f(F) \cap D_z = \emptyset$. This is a contradiction. Thus Y is not discrete.

Recall that a completely (or hereditarily) normal space X is one where if A and B are subsets of X with $A \cap \bar{B} = \emptyset = \bar{A} \cap B$ then there are disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$. All metric spaces are completely normal.

LEMMA 11. *Let X be completely normal and let F and H be subsets of X where $F \cap H = \emptyset$, $x \notin F \cup H$, $\bar{F} = F \cup \{x\}$ and $\bar{H} = H \cup \{x\}$. Then there exist disjoint open sets U and V such that $F \subseteq U$, $H \subseteq V$ and $\bar{U} \cap \bar{V} = \{x\}$.*

Proof. Let $Z = X - \{x\}$. Then Z is normal, F and H are closed in Z. Thus there are open subsets A and B of Z such that $F \subseteq A$, $H \subseteq B$ and $A \cap B = \emptyset$. Now $A = D \cap Z$ and $B = E \cap Z$ where D and E are open in X. Let $U = D - \{x\}$ and $V = E - \{x\}$.

THEOREM 12. *Suppose X is a completely normal space which is admissible, Y is nontrivial first countable T_1, ϕ is an epimorphism from $I_F(X)$ onto $I_F(Y)$ and $\phi\langle J\rangle = \langle p\rangle$ for some $p \in Y$ and compact set J. Then ϕ is an isomorphism.*

Proof. By Lemma 10 the space Y is not discrete. Let $y_n \to y$ where $y_n \neq y$ for all n. We first show that the sequence $\{h(y_n)\}$ accumulates at $h(y)$. Suppose not. Then there exists an open set U such that $h(y) \in U$ but $h(y_n) \notin U$ for all n. In fact, we can choose U so that $h(y_n) \notin \bar{U}$ for all n. Let $F = X - U$. Then for all n, $h(y_n) \in X - \bar{U} \subseteq F$. The set $X - \bar{U}$ is open and so for all n there exists F_n such that $h(y_n) \in F_n \subseteq X - \bar{U} \subseteq F$ and $\phi\langle F_n\rangle = \langle y_n\rangle$ (see Lemma 8). Therefore if $\phi\langle F\rangle = \langle H\rangle$ we have that $y_n \in H$ for all n. But then $y \in H$ and hence $h(y) \in F$. This is a contradiction. Therefore the sequence $\{h(y_n)\}$ accumulates at $h(y)$. Without loss of generality assume that $h(y_n) \to h(y)$. Now choose distinct subsequences $\{y_n'\}$ and $\{y_n''\}$ of $\{y_n\}$. By the above we may assume that $h(y_n') \to h(y)$ and $h(y_n'') \to h(y)$. Let $F = \{h(y_n')\}$ and $H = \{h(y_n'')\}$. The sets F and H satisfy the conditions of the last lemma and so let U and V be as in the lemma. Then $\bar{U} \cap \bar{V} = \{h(y)\}$. Let $\phi\langle \bar{U}\rangle = \langle R\rangle$ and $\phi\langle \bar{V}\rangle = \langle S\rangle$. Since U is a neighborhood of each $h(y_n')$ there exist closed sets L_n' such that $h(y_n') \in L_n' \subseteq U \subseteq \bar{U}$ and $\phi\langle L_n'\rangle = \langle y_n'\rangle$ (see Lemma 8). Therefore $y_n' \in R$. Likewise each y_n'' belongs to S. But since R and S are closed this means that $y \in R \cap S$. Then $\langle R \cap S\rangle = \langle R\rangle \circ \langle S\rangle = \phi\langle \bar{U}\rangle \circ \phi\langle \bar{V}\rangle = \phi\langle \bar{U} \cap \bar{V}\rangle = \phi\langle h(y)\rangle$. Now since $y \in R \cap S$ we have that $\phi\langle h(y)\rangle \neq 0$. But then ϕ is an isomorphism by [**3**] (and hence X is homeomorphic to Y).

COROLLARY 13. *Suppose X is I^n, S^n, $\mathscr{C}$ (the Cantor discontinuum) or I^∞. Then any epimorphism ϕ from $I_F(X)$ onto $I_F(Y)$ (where Y is any nontrivial first countable T_1 space) must be an isomorphism.*

Proof. Let $y \in Y$. Then since ϕ is an epimorphism there exists a closed set J such that $\phi\langle J\rangle = \langle y\rangle$. But J must be compact since X is compact. Now apply Theorem 12.

COROLLARY 14. *Suppose X is I^n, S^n, $\mathscr{C}$ or I^∞. Then the semigroup $I_F(X)$ is hopfian.*

Although Theorem 12 shows that for many spaces X and Y any epimorphism from $I_F(X)$ onto $I_F(Y)$ must be an isomorphism this is not always the case. If X is any space which does not contain proper closed homeomorphic copies of itself (for instance X could be R^n) and Y is trivial (i.e., $Y = \{y\}$) then the following map ϕ will be an epimorphism from $I_F(X)$ onto $I_F(Y)$:

$$\phi(f) = 0 \quad \text{if} \quad \operatorname{dom} f \neq X$$
$$\phi(f) = \langle y \rangle \quad \text{otherwise.}$$

For another example of an epimorphism which is not an isomorphism let $X = R$ (the reals), $Y = \{y, z\}$ and define an epimorphism ϕ by the following:

$y \in \operatorname{dom} \phi(f)$ if $[a, \infty) \subseteq \operatorname{dom} f$ for some a

$z \in \operatorname{dom} \phi(f)$ if $(-\infty, b] \subseteq \operatorname{dom} f$ for some b

if $y \in \operatorname{dom} \phi(f)$ then $\phi(f)(y) = y$ if $f[a, \infty) = [c, \infty)$ for some c,
$\phi(f)(y) = z$ otherwise

if $z \in \operatorname{dom} \phi(f)$ then $\phi(f)(z) = z$ if $f(-\infty, b] = (-\infty, d]$ for some d,
$\phi(f)(z) = y$ otherwise

$\phi(f) = 0$ for all other maps f.

Although not all epimorphisms from $I_F(R)$ onto $I_F(Y)$ are isomorphisms we do have the result that all epimorphisms from $I_F(R)$ onto $I_F(R)$ are isomorphisms:

THEOREM 15. *$I_F(R)$ is hopfian.*

Proof. Let ϕ be an epimorphism from $I_F(R)$ onto $I_F(R)$. Call a set $W \subseteq R$ *right ended* (respectively *left ended*) if W contains a set of the form $[w, \infty)$ (respectively $(-\infty, w]$). Suppose $a \in R$ and $\phi\langle A \rangle = \langle [a, \infty) \rangle$. Choose B homeomorphic to A such that $\phi\langle B \rangle = \langle (-\infty, b] \rangle$ where $b < a$. Then $\phi\langle A \cap B \rangle = \phi\langle A \rangle \circ \phi\langle B \rangle = 0$. If A is both right and left ended then B must be also and hence $A \cap B$ contains a copy of A $(A \neq R)$. But then $\phi\langle A \cap B \rangle \neq 0$ which is a contradiction. Therefore A cannot be both right and left ended. If A is neither right nor left ended then there exist sets B and C homeomorphic to A where A, B and C are mutually disjoint. Let $\phi\langle B \rangle = \langle S \rangle$ and $\phi\langle C \rangle = \langle T \rangle$. Then S and T are homeomorphic to $[a, \infty)$ and so at least two of the three sets S, T and $[a, \infty)$ have nonempty intersection. But this is impossible since A, B and C are mutually disjoint. Thus if $\phi\langle A \rangle = \langle [a, \infty) \rangle$ then A must be right or left ended but not both (true for arbitrary $a \in R$ and $A \subseteq R$ such that $\phi\langle A \rangle = \langle [a, \infty) \rangle$). Without loss of generality suppose A is right ended (and hence not left ended). Now let $B = k(A)$ where k maps R onto R by $k(x) = -x$. Then $\phi\langle B \rangle = \langle S \rangle$ for some S homeomorphic to $[a, \infty)$. If S is of the form $[s, \infty)$ then

$$\phi\langle A \cap B \rangle = \phi\langle A \rangle \circ \phi\langle B \rangle = \langle [a, \infty) \cap [s, \infty) \rangle.$$

But $A \cap B$ is neither right nor left ended which contradicts the above result. Therefore S is of the form $(-\infty, s]$. Now if A is not contained in $[w, \infty)$ for some w then let C be homeomorphic to A with C right ended but $A \cap C \subseteq$

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$[r, \infty)$ for some r. Then $\phi\langle C\rangle = \langle T\rangle$ with T homeomorphic to $[a, \infty)$. As above T cannot be of the form $(-\infty, t]$ since $B \cap C$ is neither right nor left ended. Let $T \cap [a, \infty) = [e, \infty)$ and let $W = A \cap C$. Then $\phi\langle W\rangle = \langle [e, \infty)\rangle$, W is right ended and $W \subseteq [w, \infty)$ for some w. Choose U homeomorphic to W with $\phi\langle U\rangle = (-\infty, e]$. If U is right ended then $U \cap W$ contains a copy of W but $\phi\langle U \cap W\rangle = \langle e\rangle$ which is a contradiction. Therefore U is left ended and $U \subseteq (-\infty, u]$ for some u. But $U \cap W$ is compact and $\phi\langle U \cap W\rangle = \langle e\rangle$. By Theorem 12, ϕ is an isomorphism.

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