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ON γ-VECTORS AND THE DERIVATIVES OF THE TANGENT AND SECANT FUNCTIONS

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Abstract

In this paper, we show that the γ-vectors of Coxeter complexes (of types A and B) and associahedrons (of types A and B) can be obtained by using derivative polynomials of the tangent and secant functions. We provide a unified grammatical approach to generate these γ-vectors and the coefficient arrays of Narayana polynomials, Legendre polynomials and Chebyshev polynomials of both kinds.

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1. Introduction

Set $y = \tan(x)$ and $z = \sec(x)$. Denote by D the differential operator d/dx. Clearly, we have $D(y) = 1 + y^2$ and $D(z) = yz$. In 1995, Hoffman [8] considered two sequences of *derivative polynomials* defined respectively by $D^n(y) = P_n(y)$ and $D^n(z) = zQ_n(y)$. From the chain rule it follows that the polynomials $P_n(u)$ satisfy $P_0(u) = u$ and $P_{n+1}(u) = (1 + u^2)P'_n(u)$, and similarly $Q_0(u) = 1$ and $Q_{n+1}(u) = (1 + u^2)Q'_n(u) + uQ_n(u)$. The theory of derivative polynomials is an important part of combinatorial trigonometry (see [1, 6, 8–12], for instance).

Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $[n]$, where $[n] = \{1, 2, \ldots, n\}$. The *hyperoctahedral group* B_n is the group of signed permutations of the set $\pm[n]$ such that $\pi(-i) = -\pi(i)$ for all i, where $\pm[n] = \{\pm1, \pm2, \ldots, \pm n\}$. A permutation $\pi = \pi(1)\pi(2)\cdots\pi(n)$, signed or not, is *alternating* if $\pi(1) > \pi(2) < \pi(3) > \cdots \pi(n)$. In other words, $\pi(i) < \pi(i + 1)$ if i is even and $\pi(i) > \pi(i + 1)$ if i is odd. Denote by E_n and E_n^B the number of alternating elements in $\mathfrak{S}_n$ and B_n, respectively. It is well known (see [3, 21]) that

$$\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \tan x + \sec x, \qquad \sum_{n=0}^{\infty} E_n^B \frac{x^n}{n!} = \tan 2x + \sec 2x.$$

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Define functions

$$f = \sec(2x), \quad g = 2\tan(2x).$$

It is natural to consider the following differential system:

$$D(f) = fg, \quad D(g) = 4f^2. \tag{1.1}$$

Define the function $h = \tan(2x)$. Note that $f^2 = 1 + h^2$ and $g = 2h$. So the following result is immediate.

Proposition 1.1. *For $n \geq 0$, we have $D^n(f) = 2^n f Q_n(h)$, $D^n(g) = 2^{n+1} P_n(h)$.*

This paper is a continuation of [13]. In [13], we showed that the coefficient array of the Bessel polynomials can be generated by context-free grammars. In this paper, we show that the coefficient arrays of Narayana polynomials, Legendre polynomials and Chebyshev polynomials of both kinds can also be generated by context-free grammars.

The organisation of this paper is as follows. In the next section we gather together some notation and definitions that will be needed in the rest of the paper. In Section 3 we show that the γ-vectors of Coxeter complexes (of types A and B) and associahedrons (of types A and B) can be obtained by using (1.1). In Section 4 we restate our main result via context-free grammars.

2. Notation, definitions and preliminaries

Recall that a *descent* of a permutation $\pi \in \mathfrak{S}_n$ is a position i such that $\pi(i) > \pi(i+1)$, where $1 \leq i \leq n-1$. Denote by $\operatorname{des}(\pi)$ the number of descents of π. Then the equations

$$A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\operatorname{des}(\pi)} = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle x^k,$$

define the *Eulerian polynomial* $A_n(x)$ and the *Eulerian number* $\left\langle {n \atop k} \right\rangle$ (see [20, A008292]). For each $\pi \in B_n$, we define

$$\operatorname{des}_B(\pi) = \#\{i \in \{0, 1, 2, \ldots, n-1\} | \pi(i) > \pi(i+1)\},$$

where $\pi(0) = 0$. Let

$$B_n(x) = \sum_{\pi \in B_n} x^{\operatorname{des}_B(\pi)} = \sum_{k=0}^{n} B(n,k) x^k.$$

The polynomial $B_n(x)$ is called an *Eulerian polynomial of type B*, while $B(n,k)$ is called an *Eulerian number of type B* (see [20, A060187]).

The *h-polynomial* of a $(d-1)$-dimensional simplicial complex Δ is the generating function $h(\Delta; x) = \sum_{i=0}^{d} h_i(\Delta) x^i$ defined by the identity

$$\sum_{i=0}^{d} h_i(\Delta) x^i (1+x)^{d-i} = \sum_{i=0}^{d} f_{i-1}(\Delta) x^i,$$

where $f_i(\Delta)$ is the number of faces of Δ of dimension i. There is a large literature devoted to h-polynomials of the form

$$h(\Delta; x) = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i x^i (1+x)^{d-2i},$$

where the coefficients γ_i are nonnegative. Following Gal [7], we call $(\gamma_0, \gamma_1, \ldots)$ the γ-vector of Δ, and the corresponding generating function $\gamma(\Delta; x) = \sum_{i\geq 0} \gamma_i x^i$ is the γ-*polynomial*. In particular, the Eulerian polynomials $A_n(x)$ and $B_n(x)$ are respectively known as the h-polynomials of Coxeter complexes of types A and B.

Let us now recall two classical results.

Theorem 2.1 [5, 18]. *For* $n \geq 1$,

$$A_n(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a(n,k) x^k (1+x)^{n-1-2k}.$$

Theorem 2.2 [4, 16]. *For* $n \geq 1$,

$$B_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} b(n,k) x^k (1+x)^{n-2k}.$$

It is well known that the numbers $a(n,k)$ satisfy the recurrence

$$a(n,k) = (k+1)a(n-1,k) + (2n-4k)a(n-1,k-1),$$

with the initial conditions $a(1,0) = 1$ and $a(1,k) = 0$ for $k \geq 1$ (see [20, A101280]), and the numbers $b(n,k)$ satisfy the recurrence

$$b(n,k) = (2k+1)b(n-1,k) + 4(n+1-2k)b(n-1,k-1), \tag{2.1}$$

with the initial conditions $b(1,0) = 1$ and $b(1,k) = 0$ for $k \geq 1$ (see [4, Section 4]).

The h-polynomials of the type A and type B associahedrons are respectively given as follows (see [14, 15, 17, 19], for instance):

$$h(\Delta_{FZ}(A_{n-1}), x) = \frac{1}{n} \sum_{k=0}^{n-1} \binom{n}{k}\binom{n}{k+1} x^k = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} C_k \binom{n-1}{2k} x^k (1+x)^{n-1-2k}, \tag{2.2}$$

$$h(\Delta_{FZ}(B_n), x) = \sum_{k=0}^{n} \binom{n}{k}^2 x^k = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2k}{k}\binom{n}{2k} x^k (1+x)^{n-2k}, \tag{2.3}$$

where $C_k = \binom{2k}{k}/(k+1)$ is the kth *Catalan number* and the coefficient of x^k in (2.2) is the *Narayana number* $N(n, k+1)$.

Define

$$F(n,k) = C_k \binom{n-1}{2k}, \quad H(n,k) = \binom{2k}{k}\binom{n}{2k}.$$

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There are many combinatorial interpretations of the number $F(n,k)$; for example, $F(n,k)$ is number of *Motzkin paths* of length $n-1$ with k up steps (see [20, A055151]). It is easy to verify that the numbers $F(n,k)$ satisfy the recurrence relation

$$(n+1)F(n,k) = (n+2k+1)F(n-1,k) + 4(n-2k)F(n-1,k-1),$$

with initial conditions $F(1,0) = 1$ and $F(1,k) = 0$ for $k \geq 1$, and the numbers $H(n,k)$ satisfy the recurrence relation

$$nH(n,k) = (n+2k)H(n-1,k) + 4(n-2k+1)H(n-1,k-1),$$

with initial conditions $H(1,0) = 1$ and $H(1,k) = 0$ for $k \geq 1$ (see [20, A089627]).

3. Main results

Define the generating functions

$$a_n(x) = \sum_{k\geq 0} a(n,k)x^k, \quad b_n(x) = \sum_{k\geq 0} b(n,k)x^k.$$

The first few $a_n(x)$ and $b_n(x)$ are respectively given as follows:

$$a_1(x) = 1,\ a_2(x) = 1,\ a_3(x) = 1+2x,\ a_4(x) = 1+8x;$$
$$b_1(x) = 1,\ b_2(x) = 1+4x,\ b_3(x) = 1+20x,\ b_4(x) = 1+72x+80x^2.$$

As shown in [8], the exponential generating functions

$$P(u,t) = \sum_{n=0}^{\infty} P_n(u)\frac{t^n}{n!} \quad \text{and} \quad Q(u,t) = \sum_{n=0}^{\infty} Q_n(u)\frac{t^n}{n!}$$

are given by the explicit formulas

$$P(u,t) = \frac{u+\tan(t)}{1-u\tan(t)} \quad \text{and} \quad Q(u,t) = \frac{\sec(t)}{1-u\tan(t)}. \tag{3.1}$$

Combining (3.1) and [4, Prop. 3.5, Prop. 4.10], we immediately get the following result.

THEOREM 3.1. *For* $n \geq 1$,

$$a_n(x) = \frac{1}{x}\left(\frac{\sqrt{4x-1}}{2}\right)^{n+1} P_n\left(\frac{1}{\sqrt{4x-1}}\right), \quad b_n(x) = (4x-1)^{\frac{n}{2}} Q_n\left(\frac{1}{\sqrt{4x-1}}\right).$$

Assume that

$$(fD)^{n+1}(f) = (fD)(fD)^n(f) = fD((fD)^n(f)),$$
$$(fD)^{n+1}(g) = (fD)(fD)^n(g) = fD((fD)^n(g)).$$

We can now present the main result of this paper.

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Theorem 3.2. *For $n \geq 1$,*

$$D^n(f) = \sum_{k=0}^{\lfloor n/2 \rfloor} b(n,k) f^{2k+1} g^{n-2k},$$

$$D^n(g) = 2^{n+1} \sum_{k=0}^{\lfloor n-1/2 \rfloor} a(n,k) f^{2k+2} g^{n-1-2k},$$

$$(fD)^n(f) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} H(n,k) f^{n+1+2k} g^{n-2k},$$

$$(fD)^n(g) = 2(n+1)! \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} F(n,k) f^{n+2+2k} g^{n-1-2k}.$$

Proof. We only prove the assertion for $D^n(f)$, as the others can be proved similarly. It follows from (1.1) that $D(f) = fg$ and $D^2(f) = fg^2 + 4f^3$. For $n \geq 0$, we define $\widetilde{b}(n,k)$ by

$$D^n(f) = \sum_{k=0}^{\lfloor n/2 \rfloor} \widetilde{b}(n,k) f^{2k+1} g^{n-2k}. \tag{3.2}$$

Then $\widetilde{b}(1,0) = 1$ and $\widetilde{b}(1,k) = 0$ for $k \geq 1$. It follows from (3.2) that

$$D(D^n(f)) = \sum_{k=0}^{\lfloor n/2 \rfloor} (2k+1)\widetilde{b}(n,k) f^{2k+1} g^{n-2k+1} + 4 \sum_{k=0}^{\lfloor n/2 \rfloor} (n-2k)\widetilde{b}(n,k) f^{2k+3} g^{n-2k-1}.$$

We therefore conclude that $\widetilde{b}(n+1,k) = (2k+1)\widetilde{b}(n,k) + 4(n+2-2k)\widetilde{b}(n,k-1)$ and complete the proof by comparing it with (2.1). □

Define the generating functions

$$N_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} \binom{n}{k} \binom{n}{k+1} (x+1)^k (x-1)^{n-1-k},$$

$$L_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 (x+1)^k (x-1)^{n-k}.$$

Taking $f^2 = 1 + h^2$ and $g = 2h$ in Theorem 3.2 leads to the following result, the proof of which we omit since it is a straightforward application of (2.2) and (2.3).

Corollary 3.3. *Let $\imath = \sqrt{-1}$. For $n \geq 1$,*

$$(fD)^n(f) = n! f^{n+1} (-\imath)^n L_n(\imath h),$$

$$(fD)^n(g) = 2(n+1)! f^{n+2} (-\imath)^{n-1} N_n(\imath h).$$

It should be noted that the polynomial $L_n(x)/2^n$ is the famous *Legendre polynomial* [20, A100258]. Therefore, from Corollary 3.3, we see that the Legendre polynomial can be generated by $(fD)^n(f)$.

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4. Context-free grammars

Many combinatorial objects permit grammatical interpretations (see [2, 13], for instance). The grammatical method was systematically introduced by Chen [2] in the study of exponential structures in combinatorics. Let A be an alphabet whose letters are regarded as independent commutative indeterminates. A *context-free grammar* G over A is defined as a set of substitution rules that replace a letter in A by a formal function over A. The formal derivative D is a linear operator defined with respect to a context-free grammar G. For example, if $G = \{u \to uv, v \to v\}$, then $D(u) = uv, D(v) = v, D^2(u) = u(v + v^2)$.

It follows from Theorem 3.2 that the γ-vectors of Coxeter complexes (of types A and B) and associahedrons (of types A and B) can be respectively generated by the grammars

$$G_1 = \{u \to uv, v \to 4u^2\}$$

and

$$G_2 = \{u \to u^2 v, v \to 4u^3\}. \tag{4.1}$$

There are many combinatorial sequences that can be generated by using the grammar (4.1). The following proposition is a special result.

Proposition 4.1. *Let G be the same as in (4.1). Then*

$$D^n(uv) = n! \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} 4^k \binom{n+1}{2k} u^{n+1+2k} v^{n+1-2k}.$$

Let $T_n(x)$ and $U_n(x)$ be the *Chebyshev polynomials of the first and second kind* of order n, respectively. We can now conclude the following result, which is based on Proposition 4.1. The proof runs along the same lines as that of Theorem 3.2.

Theorem 4.2. *If* $G = \{u \to u^2 v, v \to u^3\}$, *then*

$$D^n(uv) = n! \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \binom{n+1}{2k} u^{n+1+2k} v^{n+1-2k},$$

$$D^n(u^2) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2k+1} u^{n+2+2k} v^{n-2k}.$$

In particular,

$$D^n(uv)\,|_{u^2=x^2-1, v=x} = n!(x^2 - 1)^{(n+1)/2} T_{n+1}(x),$$
$$D^n(u^2)\,|_{u^2=x^2-1, v=x} = n!(x^2 - 1)^{(n+2)/2} U_n(x).$$

As an extension of the grammar (4.1), we find the following result.

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Theorem 4.3. *If* $G = \{t \to tu^2, u \to u^2v, v \to 4u^3\}$, *then*

$$D^n(t^2u^2) = (n+1)!t^2 \sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{\lfloor \frac{n-k}{2} \rfloor} u^{2n+2-k}v^k,$$

$$D^n(t^2u) = n!t^2 \sum_{k=0}^{n} \binom{n}{k} 2^{n-k}u^{2n+1-k}v^k.$$

Proof. We only prove the assertion for $D^n(t^2u^2)$, since the corresponding assertion for $D^n(t^2u)$ can be proved similarly. Let

$$T(n,k) = \binom{n}{k}\binom{n-k}{\lfloor \frac{n-k}{2} \rfloor}.$$

It is easy to verify that

$$(n+1)T(n,k) = (2n+1-k)T(n-1,k-1) + 2T(n-1,k) + 4(k+1)T(n-1,k+1). \tag{4.2}$$

Clearly, $D(t^2u^2) = 2t^2(u^4 + u^3v)$ and $D^2(t^2u^2) = 3!t^2(2u^6 + 2u^5v + u^4v^2)$. For $n \geq 0$, we define

$$D^n(t^2u^2) = (n+1)!t^2 \sum_{k=0}^{n} \widetilde{T}(n,k)u^{2n+2-k}v^k.$$

Note that

$$\frac{D^{n+1}(t^2u^2)}{(n+1)!t^2} = \sum_k (2n+2-k)\widetilde{T}(n,k)u^{2n+3-k}v^{k+1} + 2\sum_k \widetilde{T}(n,k)u^{2n+4-k}v^k + 4\sum_k k\widetilde{T}(n,k)u^{2n+5-k}v^{k-1}.$$

Thus, we get

$$(n+2)\widetilde{T}(n+1,k) = (2n+3-k)\widetilde{T}(n,k-1) + 2\widetilde{T}(n,k) + 4(k+1)\widetilde{T}(n,k+1).$$

Comparing this with (4.2), we see that the coefficients $\widetilde{T}(n,k)$ satisfy the same recurrence relation and initial conditions as $T(n,k)$, so they agree. □

It is well known that $\binom{n}{k}\binom{n-k}{\lfloor (n-k)/2 \rfloor}$ is the number of paths of length n with steps $U = (1,1), D = (1,-1)$ and $H = (1,0)$, starting at $(0,0)$, staying weakly above the x-axis (that is, left factors of *Motzkin paths*) and having k H-steps (see [20, A107230]). It should be noted that the numbers $\binom{n}{k}2^{n-k}$ are elements of the f-vector for the n-dimensional cubes (see [20, A038207]).

Taking $u = \sec^2(x)$ and $v = 2\tan(x)$, it is clear that $D(u) = uv$ and $D(v) = 2u$. We can easily verify another grammatical description of the γ-vectors of the type A Coxeter complex.

Theorem 4.4. *If* $G = \{u \to uv, v \to 2u\}$, *then*

$$D^n(u) = \sum_{k=0}^{\lfloor n/2 \rfloor} a(n+1,k)u^{k+1}v^{n-2k}.$$

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5. Concluding remarks

Using *Leibniz's formula* for differentiating products, one may easily deduce various convolution formulas for the polynomials considered in this paper. For example, note that

$$D^n(f^2) = \sum_{k=0}^{n} \binom{n}{k} D^k(f) D^{n-k}(f).$$

Then we immediately get the following convolution formula, which has also been obtained by Chow [4, Corollary 5.2]:

$$2^n a_{n+1}(x) = \sum_{k=0}^{n} \binom{n}{k} b_k(x) b_{n-k}(x), \quad n \geqslant 1.$$

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