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AN APPLICATION OF DIFFERENTIAL SUBORDINATIONS AND SOME CRITERIA FOR UNIVALENCY

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By using the method of differential subordinations, we derive, among other results, some criteria for univalency in the unit disc.

1. Introduction and preliminaries

Let A denote the class of functions $f(z)$ which are analytic in the unit disc $U = \{z\colon |z| < 1\}$ with $f(0) = f'(0) - 1 = 0$.

Let $f(z)$ be analytic in the unit disc U. Then the function $f(z)$ with $f(0) = 0$ and $f'(0) \neq 0$ is said to be *starlike (univalent)* if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \qquad (z \in U). \tag{1.1}$$

The function $f(z)$ with $f'(0) \neq 0$ is said to be *convex (univalent)* if it satisfies

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \qquad (z \in U). \tag{1.2}$$

We note that $f(z)$ is convex in U if and only if $zf'(z)$ is starlike in U. Further, we denote by S^* and K the subclasses of A consisting of functions $f(z)$ which are starlike and convex in U, respectively.

Let $f(z)$ and $F(z)$ be analytic in the unit disc U. Then the function $f(z)$ is said to be *subordinate* to $F(z)$, written $f(z) \prec F(z)$, if $F(z)$ is univalent in U, $f(0) = F(0)$ and $f(U) \subseteq F(U)$.

The general theory of differential subordinations was introduced by Miller and Mocanu [**1**]. The theory of first–order differential subordinations, with many interesting applications, especially in the theory of univalent functions, was considered by Miller and Mocanu [**2**]. Namely, if $\phi\colon \mathrm{C}^2 \longrightarrow C$ (where C is the complex plane) is analytic in a

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domain D, if $h(z)$ is univalent in U, and if $p(z)$ is analytic in U with $(p(z), zp'(z)) \in D$ when $z \in U$, then $p(z)$ is said to satisfy a first-order differential subordination if

$$\phi(p(z), zp'(z)) \prec h(z). \tag{1.3}$$

The univalent function $q(z)$ is said to be a *dominant* of the differential subordination (1.3) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.3). If $\widetilde{q}(z)$ is a dominant of (1.3) and $\widetilde{q}(z) \prec q(z)$ for all dominants of (1.3), then $\widetilde{q}(z)$ is said to be the *best dominant* of the differential subordination (1.3).

By using the method of differential subordinations, we obtain a result which gives some criteria for univalency in the unit disc U. We note that we use methods similar to those used in [4].

The following lemmas are needed for the results in the next section.

LEMMA 1. ([2]). *Let $q(z)$ be univalent in the unit disc U, and let $\theta(w)$ and $\phi(w)$ be analytic in a domain D containing $q(U)$, with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$, and suppose that*

(i) *$Q(z)$ is starlike in the unit disc U,*

and

(ii) $\operatorname{Re}\{(zh'(z))/(Q(z))\} = \operatorname{Re}\{(\theta'(q(z)))/(\phi(q(z))) + (zQ'(z))/(Q(z))\} > 0 \quad (z \in U)$.

If $p(z)$ is analytic in U, with $p(0) = q(0)$, $p(U) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z), \tag{1.4}$$

then $p(z) \prec q(z)$, and $q(z)$ is the best dominant of the differential subordination (1.4).

LEMMA 2. ([3]). *If $f(z) \in A$ and*

$$|\arg(f'(z))| < \frac{\gamma_0 \pi}{2} = 0.968\ldots \qquad (z \in U), \tag{1.5}$$

where $\gamma_0 = 0.6165\ldots$ is the unique root of the equation

$$2\arctan(1-\gamma) + \pi(1-2\gamma) = 0,$$

then $f(z) \in S^*$.

2. Differential subordination and some criteria for univalency

We first prove:

Theorem 1. *Let $p(z)$ be analytic in U, with $p(0)=1$, and let $0<\lambda\leqslant 1$. If*

$$(1-\lambda)p(z)+\lambda zp'(z)\prec\left(\frac{1+z}{1-z}\right)^{\gamma}\left(1-\lambda+\lambda\gamma\frac{2z}{1-z^2}\right)=h(z), \tag{2.1}$$

$0<\gamma\leqslant 1$, *then*

$$p(z)\prec\left(\frac{1+z}{1-z}\right)^{\gamma}$$

and this is the best dominant of (2.1).

Proof: We choose $q(z)=((1+z)/(1-z))^{\gamma}$, $0<\gamma\leqslant 1$, $\phi(w)=\lambda$ and $\theta(w)=(1-\lambda)w$ in Lemma 1. Then the function $q(z)$ is convex in U and $q(0)=1$. Further

$$Q(z)=zq'(z)\phi(q(z))=\lambda zq'(z)$$

is starlike, and for the function

$$h(z)=\theta(q(z))+Q(z)=(1-\lambda)q(z)+\lambda zq'(z),$$

we have

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\}=\operatorname{Re}\left\{\frac{1-\lambda}{\lambda}+\frac{zQ'(z)}{Q(z)}\right\}>0 \qquad (z\in U).$$

Therefore the conditions (i) and (ii) in Lemma 1 are satisfied. By using Lemma 1, we obtain that if $p(z)$ is analytic in the unit disc U with $p(0)=1$ and

$$(1-\lambda)p(z)+\lambda zp'(z)\prec h(z),$$

where $h(z)$ is defined in (2.1), and $0<\lambda\leqslant 1$, then

$$p(z)\prec q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma},$$

and this is the best dominant of the differential subordination (2.1). ∎

In the case $\gamma=1$, Theorem 1 yields:

Theorem 2. *Let $p(z)$ be analytic in U with $p(0)=1$, and $0<\lambda\leqslant 1$. If*

$$\operatorname{Re}\{(1-\lambda)p(z)+\lambda zp'(z)\}>-\frac{\lambda}{2} \qquad (z\in U), \tag{2.2}$$

then

$$\operatorname{Re}\{p(z)\}>0 \qquad (z\in U).$$

Proof: Taking $\gamma=1$ in Theorem 1, we have $q(z)=(1+z)/(1-z)$ and

$$h(z)=(1-\lambda)\frac{1+z}{1-z}+\lambda\frac{2z}{(1-z)^2}. \tag{2.3}$$

It follows from the above that

$$\operatorname{Re}\{h(e^{i\phi})\} = -\frac{\lambda}{2\sin^2(\phi/2)} \leqslant -\frac{\lambda}{2} \qquad (-\pi < \phi \leqslant \pi).$$

Also we have

$$h(0) = 1 - \lambda \geqslant 0 > -\frac{\lambda}{2}.$$

Therefore (2.2) implies that the function $(1-\lambda)p(z) + \lambda z p'(z)$ is subordinate to the function $h(z)$ defined by (2.3). By applying Theorem 1, we conclude that

$$p(z) \prec \frac{1+z}{1-z},$$

that is, that $\operatorname{Re}\{p(z)\} > 0$, which completes the proof of Theorem 2. □

Putting $f'(z)$ with $f(z) \in A$ instead of $p(z)$ in Theorem 2, we have:

COROLLARY 1. *Let* $f(z) \in A$ *and* $0 < \lambda \leqslant 1$. *If*

$$\operatorname{Re}\{(1-\lambda)f'(z) + \lambda z f''(z)\} > -\frac{\lambda}{2} \qquad (z \in U),$$

then

$$\operatorname{Re}\{f'(z)\} > 0 \qquad (z \in U).$$

If we take $\lambda = 1$ and $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ instead of $p(z)$ in Theorem 2, we obtain:

COROLLARY 2. *If* $f(z) \in A$ *and*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right\} > -\frac{1}{2} \quad (z \in U),$$

then $f(z) \in S^*$.

COROLLARY 3. *If* $f(z) \in A$ *and*

$$\operatorname{Re}\left\{z^2\{f(z), z\} + \frac{1}{2}\left(1 + \frac{zf''(z)}{f'(z)}\right)^2\right\} > 0 \qquad (z \in U),$$

where $\{f(z), z\}$ *denotes the Schwarzian derivative defined by*

$$\{f(z), z\} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2,$$

then $f(z) \in K$.

By a similar method to that used in Theorem 2, we may obtain the answer for the case $0 < \gamma < 1$, but this is more complicated than the case $\gamma = 1$. Namely, we have:

Theorem 3. *Let $p(z)$ be analytic in U with $p(0) = 1$, and let $0 < \lambda \leqslant 1$ and $0 < \gamma < 1$. If*

$$\mathrm{Re}\{(1-\lambda)p(z) + \lambda z p'(z)\} > C(\lambda, \gamma) \qquad (z \in U), \tag{2.4}$$

where

$$\begin{cases} C(\lambda, \gamma) = \dfrac{\lambda}{1+\gamma} t_0^{\gamma-1}(a t_0 - \gamma)\sin(\gamma\pi/2), \\ a = \dfrac{1-\lambda}{\lambda}\cot(\gamma\pi/2), \\ t_0 = \dfrac{a + \sqrt{a^2 + 1 - \gamma^2}}{1+\gamma}, \end{cases} \tag{2.5}$$

then
$$|\arg(p(z))| < \frac{\gamma\pi}{2} \qquad (z \in U).$$

Proof: First we consider the function $h(z)$ defined in (2.1) for $0 < \lambda \leqslant 1$ and $0 < \gamma < 1$. Noting that

$$h(e^{i\phi}) = \left(i\cot\frac{\phi}{2}\right)^{\gamma}\left(1 - \lambda + i\frac{\lambda\gamma}{\sin\phi}\right) \qquad (0 < |\phi| < \pi) \tag{2.6}$$

and
$$i\cot\frac{\phi}{2} = \begin{cases} e^{i\pi/2}\cot(\phi/2) & (0 < \phi < \pi), \\ -e^{-i\pi/2}\cot(\phi/2) & (-\pi < \phi < 0), \end{cases}$$

we obtain that

$$\mathrm{Re}\{h(e^{i\phi})\} = \left(\pm\cot\frac{\phi}{2}\right)^{\gamma}\left((1-\lambda)\cos\left(\pm\frac{\gamma\pi}{2}\right) - \lambda\gamma\frac{\sin(\pm\gamma\pi/2)}{\sin\phi}\right), \quad (0 < |\phi| < \pi), \tag{2.7}$$

where we take "+" in the case $0 < \phi < \pi$, and "−" in the case $-\pi < \phi < 0$. It follows from (2.7) that $\mathrm{Re}\{h(e^{i\phi})\}$ is an odd function. Thus we consider only the case

$$\mathrm{Re}\{h(e^{i\phi})\} = \left(\cot\frac{\phi}{2}\right)^{\gamma}\left((1-\lambda)\cos\left(\frac{\gamma\pi}{2}\right) - \lambda\gamma\frac{\sin(\gamma\pi/2)}{\sin\phi}\right) \quad (0 < \phi < \pi). \tag{2.8}$$

We shall show that
$$\mathrm{Re}\{h(e^{i\phi})\} \leqslant C(\lambda, \gamma) \qquad (0 < \phi < \pi),$$

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where $C(\lambda, \gamma)$ is defined by (2.5). Hence we put

$$\cot\frac{\phi}{2} = t \quad (0 < \phi < \pi) \text{ and } a = \frac{1-\lambda}{\lambda}\cot\left(\frac{\gamma\pi}{2}\right).$$

Then (2.8) yields

$$\text{(2.9)} \qquad \operatorname{Re}\{h(e^{i\phi})\} = g(t) = \frac{\lambda\gamma\sin(\gamma\pi/2)}{2}g_1(t),$$

where

$$\text{(2.10)} \qquad g_1(t) = \frac{2a}{\gamma}t^{\gamma} - t^{\gamma-1} - t^{\gamma+1} \qquad (0 < t < +\infty).$$

It is easy to see that the function $g_1(t)$ defined by (2.10) has the maximum value

$$\text{(2.11)} \qquad g_1(t_0) = \frac{2t_0^{\gamma-1}}{1+\gamma}\left(\frac{a}{\gamma}t_0 - 1\right)$$

at the point

$$t_0 = \frac{a+\sqrt{a^2+1-\gamma^2}}{1+\gamma}.$$

Therefore, from (2.9), (2.11) and the previous remark concerning ϕ, we conclude that

$$\text{(2.12)} \qquad \operatorname{Re}\{h(e^{i\phi})\} \leqslant C(\lambda, \gamma) \qquad (-\pi < \phi \leqslant \pi),$$

where, in the cases $\phi = 0$ and $\phi = \pi$, we have

$$\operatorname{Re}\{h(e^{i\phi})\} \to -\infty.$$

Also we may conclude that $C(\lambda, \gamma)$ is an increasing function of t_0, and that

$$C(\lambda, \gamma) \to 1 - \lambda$$

when $t_0 \to +\infty$ (equivalently, $\lambda \to 0$). This implies that

$$\operatorname{Re}\{h(e^{i\phi})\} \leqslant C(\lambda, \gamma) < 1 - \lambda = h(0).$$

Further, if $p(z)$ is analytic in U with $p(0) = 1$, and if

$$\operatorname{Re}\{(1-\lambda)p(z) + \lambda zp'(z)\} > C(\lambda, \gamma) \qquad (z \in U),$$

where $C(\lambda, \gamma)$ is defined by (2.5), then, from the previous facts, we have that

$$(1-\lambda)p(z) + \lambda zp'(z) \prec h(z).$$

Finally, with the aid of Theorem 1, we obtain

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^{\gamma}$$

and

$$|\arg(p(z))| < \frac{\gamma\pi}{2}.$$

□

EXAMPLE 1. Letting $\gamma = 1/2$ and $\lambda = 1/2$ in Theorem 3, we have $a = 1$ and $t_0 = (2+\sqrt{7})/3$. Then

$$2C(1/2, 1/2) = \frac{\sqrt{6}(1+2\sqrt{7})}{18\sqrt{2+\sqrt{7}}} = 0.3972\dots.$$

Therefore we have that if

$$\operatorname{Re}\{p(z) + zp'(z)\} > 2C(1/2, 1/2) = 0.3972\dots,$$

then

$$|\arg(p(z))| < \frac{\pi}{4} \qquad (z \in U).$$

Combining Theorem 3 and Lemma 2, we obtain the following criterion for starlikeness.

COROLLARY 4. *Let $f(z) \in A$, and let $0 < \lambda \leqslant 1$ and $0 < \gamma < 1$. If*

$$\operatorname{Re}\{(1-\lambda)f'(z) + \lambda z f''(z)\} > C(\lambda, \gamma_0) \qquad (z \in U), \tag{2.13}$$

where γ_0 is as in Lemma 2 and $C(\lambda, \gamma)$ is defined by (2.5), then $f(z) \in S^*$.

EXAMPLE 2. For $\lambda = 1$ in Theorem 3, we have that if $p(z)$ is analytic in U with $p(0) = 1$ and $0 < \gamma < 1$, then the following implication

$$\operatorname{Re}\{zp'(z)\} > C(1, \gamma) \Longrightarrow |\arg(p(z))| < \frac{\gamma\pi}{2},$$

where

$$C(1,\gamma) = -\frac{\gamma}{1+\gamma}\left(\frac{1-\gamma}{1+\gamma}\right)^{(\gamma-1)/2} \sin\left(\frac{\gamma\pi}{2}\right),$$

is true. Therefore, from Corollary 4, we obtain

$$\operatorname{Re}\{zf''(z)\} > C(1, \gamma_0) = -0.414076\dots \Longrightarrow f(z) \in S^*,$$

where γ_0 is as in Lemma 2.

There remains the problem of finding the appropriate subset E of the righthand halfplane such that $f(z) \in S^*$, whenever $f'(z) \in E$ for all $z \in U$. For example, in Lemma 2, this type of problem was treated by Mocanu [3]. By using the result of Theorem 3, we may find other subsets which imply starlikeness, whenever $f'(z)$ belong to them for all $z \in U$.

THEOREM 4. *Let $f(z) \in A$, and let $f'(z)$ satisfy*

$$|\arg(f'(z))| < \frac{\gamma\pi}{2} \qquad (\gamma_0 \leqslant \gamma < 1;\ z \in U), \tag{2.14}$$

and

$$\operatorname{Re}\{f'(z)\} > 2C(1/2, 1-\gamma) \qquad (z \in Y), \tag{2.15}$$

where γ_0 is as in Lemma 2 and $C(\lambda, \gamma)$ is defined by (2.5). Then $f(z) \in S^*$.

PROOF: Letting $\lambda = 1/2$, $p(z) = f(z)/z$ and $1-\gamma$ instead of γ in Theorem 3, we have that (2.15) implies

$$\left|\arg \frac{f(z)}{z}\right| < \frac{(1-\gamma)\pi}{2} \qquad (z \in U). \tag{2.16}$$

Therefore, using (2.14) and (2.16), we obtain

$$\left|\arg \frac{zf'(z)}{f(z)}\right| \leqslant |\arg(f'(z))| + \left|\arg \frac{f(z)}{z}\right| < \frac{\pi}{2},$$

which shows that $f(z) \in S^*$.

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