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MERCERIAN CONDITIONS FOR THE METHOD (F, d_n)

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1. Introduction. This paper sets forth conditions sufficient that the generalized Lototsky method (F, d_n) be regular and Mercerian. If the d_n's are real and of constant sign, then the conditions are also necessary. Moreover, it follows that if f is a polynomial, then under the same conditions the method (f, d_n) is equivalent to the Sonnenschein method generated by f. Various related results are also given.

2. Definitions and preliminaries.

Definition 2.1. Let f be a nonconstant function holomorphic on the closed unit disk and let $\{d_n\}_1^\infty$ be a complex sequence with $f(1) + d_n \neq 0$. Suppose

$$\prod_1^n \frac{f(z) + d_i}{f(1) + d_i} = \sum_{k=0}^\infty a_{nk} z^k, \quad n \geqq 1.$$

Then the generalized Lototsky method (f, d_n) is defined by the matrix $A = (a_{nk})$, where $a_{00} = 1$, $a_{0k} = 0$, for $k > 0$, and a_{nk} is as above, for $n \geqq 1$. The method (F, d_n) is the special case in which $f(z) = z$. If $f(1) = 1$ and $d_n \equiv 0$, the (f, d_n) method reduces to the Sonnenschein method $Z(f)$.

For a discussion of these methods see [**3**; **4**; **10**], and the literature cited therein.

We shall use $(1 + d_n)!$ for $\Pi_1^n(1 + d_i)$, c_A for the convergence field of A, s for the space of all complex sequences, and m and c for the subspaces of bounded and convergent sequences, respectively.

A matrix A is called Mercerian if $c_A = c$; this does not imply that A is regular, i.e., that A is consistent with the identity matrix I.

Suppose that $B = (b_{nk})$ is the inverse matrix to the (F, d_n) matrix. In [**4**], Jakimovski found a formula for b_{nk} in the event that the d_n's are real and distinct; we derive it without such restrictions. We shall use the notation of [**4**] for divided differences. If f is a polynomial, the discussion [**7**, p. 45] shows that its divided differences are representable in the form

$$[f(x_0), \ldots, f(x_m)] = \frac{1}{m!} f^{(m)}(\xi), \tag{2.2}$$

where, by definition,

$$[f(x_0), \ldots, f(x_m)] = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - x_0) \ldots (z - x_m)} \tag{2.3}$$

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(See [**7**, p. 44, (4)]). It follows that if $\deg f = n$, then $m > n$ implies that $[f(x_0), \ldots, f(x_m)] = 0$. Now set $f(z) = z^n$, and effect the notational change $x_i = d_{i+1}$. Define

$$q_{nm} = (-1)^{n-m}[d_1^{\,n}, \ldots, d_{m+1}^{\,n}],\ m, n \geqq 0.$$

We claim that if $d_0 = 0$, then

$$(2.4) \qquad b_{nm} = (1 + d_m)!q_{nm}, \quad m, n \geqq 0.$$

We have seen that (2.4) is valid whenever $m > n$, for then $q_{nm} = 0$. It is also readily verified for $n = 0, 1$. Thus, assume that for $0 \leqq k \leqq n$, we have $b_{kj} = (1 + d_j)!q_{kj}$, for $j \geqq 0$. By [**8**, (2.11)],

$$(2.5) \qquad b_{n+1,m} = b_{n,m-1}(1 + d_m) - d_{m+1}b_{nm}, 0 \leqq m \leqq n + 1, n \geqq 0.$$

Thus, if $0 \leqq m \leqq n + 1$, we have

$$\begin{aligned} b_{n+1,m} &= (1 + d_m) \cdot (1 + d_{m-1})!q_{n,m-1} - d_{m+1}(1 + d_m)!q_{nm} \\ &= (1 + d_m)!(q_{n,m-1} - d_{m+1}q_{nm}), \end{aligned}$$

so it remains only to show that $q_{n+1,m} = q_{n,m-1} - d_{m+1}q_{nm}$, i.e., that

$$(2.6) \qquad [d_1^{\,n+1}, \ldots, d_{m+1}^{\,n+1}] = [d_1^{\,n}, \ldots, d_m^{\,n}] + d_{m+1}[d_1^{\,n}, \ldots, d_{m+1}^{\,n}].$$

From (2.3), the right side of (2.6) is

$$\frac{1}{2\pi i}\int_C \frac{z^n dz}{(2 - d_m)!} + \frac{d_{m+1}}{2\pi i}\int_C \frac{z^n dz}{(z - d_{m+1})!} = \frac{1}{2\pi i}\int_C \frac{z^{n+1} dz}{(z - d_{m+1})!}$$
$$= [d_1^{\,n+1}, \ldots, d_{m+1}^{\,n+1}].$$

It follows by induction that (2.4) is valid in general.

3. The main results. There are a number of conditions which are sufficient for a triangle to be Mercerian. For example, if Γ is the Banach algebra of conservative matrices, there are

(1) $A \in \Gamma, ||A^{-1}|| < \infty$,
(2) the principal diagonal condition (see [**2**]),
(3) $A \in \Gamma, ||A - I|| < 1$,
(4) $A \in \Gamma$, A has a right inverse in Γ,
(5) $A \in \Gamma$, A has the AB condition, $|a_{nn}| \geqq \epsilon > 0$ (see [**11**]).

However, (2) and (3) require that $\limsup|1 + d_n|! < 2$ and $\sup|1 + d_n|! < 2$, respectively, and these conditions turn out to be too strong. (4) is clearly deficient for computational reasons, and (5) involves showing that A has the AB condition. In general, showing that a matrix has this condition may be very difficult. There are two well-known criteria which suffice, Bosanquet's and Riesz's (see [**11**]), but neither is helpful even in the case in which $d_n \geqq 0$, for each n. Thus, we use (1), which is also a necessary condition for A to be Mercerian.

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In the sequel, let $A = (a_{nk})$ be the (F, d_n) matrix and let $B = (b_{nk}) = A^{-1}$. We now define

(3.1) $$S(j, n) = \sum d_1^{i_1} \ldots d_j^{i_j},\ i_1 + \ldots + i_j = n,$$

where

$$d_0 = 0 \text{ and } S(j, n) = \begin{cases} 1, & n = 0, j > 0 \\ 0, & n < 0 \\ 0, & j \leqq 0. \end{cases}$$

From (2.5), it follows by induction that

(3.2) $$b_{nm} = (1 + d_m)!(-1)^{n-m} S(m + 1, n - m).$$

We remark here that in the case in which the d_n's are distinct, (3.1) is an immediate consequence of (2.4) and the formula

(3.3) $$[d_1^n, \ldots, d_{m+1}^n] = S(m + 1, n - m),$$

which appears on [**9**, p. 8]. Conversely, this formula follows from (2.4) and (3.2) even when the d_n's are not distinct.

From (3.2), it is clear that, if each $d_n \geqq 0$, then

(3.4) $$||B|| = \sup \sum_{m=0}^{n} |b_{nm}| = \sup \sum_{m=0}^{n} (1 + d_m)! S(m + 1, n - m).$$

LEMMA 3.5. *Suppose that each* $d_n \geqq 0$. *Then* $||B|| < \infty$ *if and only if*

$$L = \limsup \sum_{m=0}^{n} S(m + 1, n - m) < \infty.$$

Proof. If $||B|| < \infty$, then $L < ||B|| < \infty$, by (3.4). If $L < \infty$, then

$$\limsup S(n, 1) = \limsup \sum_{1}^{n} d_j = \sum_{1}^{\infty} d_j \leqq L < \infty,$$

so $(1 + d_m)! = O(1)$. Then (3.4) shows that

$$||B|| \leqq O(1) \sup \sum_{m=0}^{n} S(m + 1, n - m) < \infty.$$

COROLLARY 3.6. *Suppose that each* $d_n \geqq 0$. *Then* $||B|| < \infty \Rightarrow \sum_1^{\infty} d_n < \infty$, *and each* $d_n < 1$.

Proof. If $d_N \geqq 1$, then $m \geqq N - 1$ implies that

$$S(m + 1, n - m) \geqq d_N^{n-m} \geqq 1.$$

Hence,

$$\liminf \sum_{m=0}^{n} S(m + 1, n - m) \geqq \liminf \sum_{m=N-1}^{n} S(m + 1, n - m) \geqq \liminf (n - N + 2) = \infty,$$

so $L = \infty$.

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LEMMA 3.7. *If each* $d_n \geqq 0$ *and* $s = \sum_1^\infty d_n < 1$, *then* $||B|| < \infty$.

Proof. From its definition, it is clear that $S(m+1, n-m) \leqq s^{n-m}$, so

$$L \leqq \limsup \sum_{m=0}^{n} s^{n-m} = 1/(1-s).$$

LEMMA 3.8. *Let each* $d_n \geqq 0$. *In order that* $||B|| < \infty$, *it is necessary and sufficient that each* $d_n < 1$ *and* $\sum_1^\infty d_n < \infty$.

Proof. Corollary 3.6 gives the necessity. Now suppose that $q_n \geqq 0$, for $n > N$, and $q_n = 0$, for $1 \leqq n \leqq N$; suppose also that $\sum_{N+1}^\infty q_n < 1$. Define $d_n = q_n$, if $n \neq N$, and let $d_N = p_N$, with $0 \leqq p_N < 1$. Then

$$(3.9) \qquad S(j, n) = \sum p_N{}^{i_N} \cdot q_{N+1}{}^{i_{N+1}} \ldots q_j{}^{i_j},\ i_N + \ldots + i_j = n.$$

Now define $B(j, n)$ and $C(j, n)$ so that $S(j, n) = B(j, n) + C(j, n)$ and $C(j, n)$ is composed of all summands in $S(j, n)$ in which p_N has a positive exponent, i.e.,

$$B(j, n) = \sum q_{N+1}{}^{i_{N+1}} \ldots q_j{}^{i_j},\ i_{N+1} + \ldots + i_j = n,$$

and

$$C(j, n) = \sum p_N{}^{i_N} q_{N+1}{}^{i_{N+1}} \ldots q_j{}^{i_j},\ i_N + \ldots + i_j = n,\ i_N \neq 0.$$

We bear in mind that $C(j, n) = 0$, if $j < N$ or $n \leqq 0$, $B(j, n) = 0$, if $j \leqq 0$ or $n < 0$ and if $0 < j \leqq N$ with $n > 0$, and $B(j, 0) = 1$, if $j > 0$. It is clear from its definition that

$$C(j, n) = \sum_{k=1}^{n} p_N{}^k B(j, n-k), \quad j \geqq N,$$

whence follows

$$(3.10)\ C(m+1, n-m) = \sum_{k=1}^{n-m} p_N{}^k B(m+1, n-m-k), \quad N-1 \leqq m.$$

From (3.10) we get

$$\begin{aligned}\sum_{m=0}^{n} C(m+1, n-m) &= \sum_{m=N-1}^{n-1} C(m+1, n-m) \\ &= \sum_{m=N-1}^{n-1} \sum_{k=1}^{n-m} p_N{}^k B(m+1, n-m-k) \\ &= \sum_{k=1}^{n-N+1} p_N{}^k \sum_{m=0}^{n-k} B(m+1, n-m-k).\end{aligned}$$

If $S_q(m+1, n-m)$ is $S(m+1, n-m)$ with $d = \{d_n\}_1^\infty$ replaced by $q = \{q_n\}_1^\infty$, then $B(m+1, n-m) = S_q(m+1, n-m)$, so the proof of Lemma 3.7 shows that $\limsup \sum_{m=0}^{n} B(m+1, n-m) = b < \infty$. Then

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there is an M with $\sum_{m=0}^{n} B(m+1, n-m) < M$, for each n. Thus,

$$(3.11)\quad L \leqq \limsup \sum_{m=0}^{n} B(m+1, n-m) + \limsup \sum_{m=0}^{n} C(m+1, n-m)$$
$$= b + \limsup \sum_{k=1}^{n-N+1} p_N^{\,k} \sum_{m=0}^{n-k} B(m+1, n-m-k)$$
$$< b + M \limsup \sum_{k=1}^{n-N+1} p_N^{\,k} < \infty.$$

Now, assuming that $N > 1$, let $\bar{q}_n = q_n$, if $n \neq N$, and let $\bar{q}_N = p_N$. Then define $\bar{d}_n = \bar{q}_n$, if $n \neq N-1$, and define $\bar{d}_{N-1} = p_{N-1}$, with $0 \leqq p_{N-1} < 1$. Thus, $\bar{d} = \{\bar{d}_n\}_1^\infty = (0, \ldots, 0, p_{N-1}, p_N, q_{N+1}, q_{N+2}, \ldots)$. Let $\bar{S}(j, n)$ be defined as $S(j, n)$ with $\bar{d}$ substituted for d. We now define $\bar{B}(j, n)$ and $\bar{C}(j, n)$ similarly, so that

$$\bar{S}(j, n) = \sum p_{N-1}{}^{i_{N-1}} \bar{q}_N{}^{i_N} \ldots \bar{q}_j{}^{i_j},\ i_{N-1} + \ldots + i_j = n,$$
$$\bar{B}(j, n) = \sum \bar{q}_N{}^{i_N} \ldots \bar{q}_j{}^{i_j},\ i_N + \ldots + i_j = n,$$
$$\bar{C}(j, n) = \sum p_{N-1}{}^{i_{N-1}} \bar{q}_N{}^{i_N} \ldots \bar{q}_j{}^{i_j},\ i_{N-1} + \ldots + i_j = n,\ i_{N-1} \neq 0,$$

and

$$\bar{S}(j, n) = \bar{B}(j, n) + \bar{C}(j, n),$$

with $\bar{C}(j, n) = 0$, if $j < N-1$ or $n \leqq 0$, $\bar{B}(j, n) = 0$, if $j \leqq 0$ or $n < 0$ and if $0 < j \leqq N-1$ with $n > 0$, and $\bar{B}(j, 0) = 1$, if $j > 0$. As before,

$$\sum_{m=0}^{n} \bar{C}(m+1, n-m) = \sum_{k=1}^{n-N+2} p_{N-1}{}^{k} \sum_{m=0}^{n-k} \bar{B}(m+1, n-k-m).$$

From (3.9), $\bar{B}(m+1, n-m) = S(m+1, n-m)$, so it follows by (3.11) that

$$\bar{L} = \limsup \sum_{m=0}^{n} \bar{S}(m+1, n-m)$$
$$\leqq \limsup \sum_{m=0}^{n} \bar{B}(m+1, n-m) + \limsup \sum_{m=0}^{n} \bar{C}(m+1, n-m)$$
$$= L + \limsup \sum_{k=1}^{n-N+2} p_{N-1}{}^{k} \sum_{m=0}^{n-k} \bar{B}(m+1, n-k-m)$$
$$\leqq L + \bar{M} \limsup \sum_{k=1}^{n-N+2} p_{N-1}{}^{k} < \infty.$$

By induction, it is clear that, if $d_n = p_n$, $1 \leqq n \leqq N$, and $d_n = q_n$, $n > N$, with $0 \leqq p_n < 1$, and if $S(j, n)$ is given by (3.1), then

$$L = \limsup \sum_{m=0}^{n} S(m+1, n-m) < \infty.$$

But any sequence $\{d_n\}_1^\infty$ satisfying the hypotheses may be written as $(p_1, \ldots, p_N, q_{N+1}, \ldots)$, with $\sum q_n < 1$.

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THEOREM 3.12. *Let each* $d_n \geqq 0$. *Then the* (F, d_n) *matrix is regular and Mercerian if and only if each* $d_n < 1$ *and* $\sum d_n < \infty$.

Proof. Lemma 3.8 above and [**10**, Lemma 2.2] show the conditions to be sufficient. Conversely, if A is regular and Mercerian, then $I \supseteq A$, and it follows that $B \in \Gamma$, whence $||B|| < \infty$. Lemma 3.8 gives the necessity.

We remark here that Lemmas 3.8 and 3.5 together with formula (3.3) prove

LEMMA 3.13. *If each* $d_n \geqq 0$, *then* $\limsup \sum_{m=0}^{n} [d_1{}^n, \ldots, d_{m+1}{}^n] < \infty$ *if and only if each* $d_n < 1$ *and* $\sum d_n < \infty$.

LEMMA 3.14. *Let each* $d_n \leqq 0$. *Then, in order that* $||B|| < \infty$, *it is necessary and sufficient that each* $d_n > -1$.

Proof. We observe first that if each x_j is real and in $[a, b]$, then in (2.2) we may assume that $\xi \in [a, b]$, in accordance with [**7**, p. 45]. In particular, if each $x_j \leqq 0$, then so is ξ. By (2.4) and (2.2),

$$b_{nm} = (1 + d_m)!(-1)^{n-m}\binom{n}{m}\xi^{n-m}, \quad \xi \leqq 0.$$

It follows that $\operatorname{sgn} b_{nm} = \operatorname{sgn}(1 + d_m)!$, or else $b_{nm} = 0$.

To prove the necessity, suppose that $d_n < -1$, for some n, and let N be the smallest such n. Define $d_0 = 0$. Then $b_{N-1,N-1} = (1 + d_{N-1})! = \epsilon > 0$. By (2.5), if $\rho = |d_N|$, we have

$$b_{N,N-1} = b_{N-1,N-2}(1 + d_{N-1}) - d_N b_{N-1,N-1} \geqq \rho\epsilon.$$

It easily follows by induction that $b_{N+k,N-1} \geqq \rho^{k+1}\epsilon$, whence $||B|| = \infty$.

On the other hand, [**8**, Theorem 2B] shows that, if $-1 < d_n \leqq 0$, then the identity matrix $I \supseteq A$ (with consistency), so $B \in \Gamma$ and $||B|| < \infty$.

THEOREM 3.15. *Let each* $d_n \leqq 0$. *Then the* (F, d_n) *matrix is regular and Mercerian if and only if each* $d_n > -1$ *and* $\sum d_n$ *converges.*

Proof. [**5**, Theorem 3.12] and Lemma 3.14 show that the conditions are sufficient. Conversely, if A is regular and Mercerian, then $I \supseteq A$, so $||B|| < \infty$ and, thus, $d_n > -1$. Moreover, if $\sum d_n$ diverges, then $a_{nn} = 1/(1 + d_n)! \to \infty$, so $||A|| = \infty$.

The last theorem allows us to generate many matrices which are consistent with I but have strictly smaller convergence fields.

COROLLARY 3.16. *Let* $-1 < d_n \leqq 0$, *for each* n, *and let* $\sum d_n$ *diverge. Then the* (F, d_n) *matrix* A *is consistent with* I *on* c_A, *and* c_A *is a proper subset of* c.

Proof. As in the proof of sufficiency for Lemma 3.14,

$$-1 < d_n \leqq 0 \Rightarrow I \supseteq A \text{ (with consistency)}.$$

The divergence of the series implies that A is not regular and Mercerian, by

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Theorem 3.15. Since I and A are consistent, $c_A = c$ would imply that A is regular and Mercerian.

We now consider the case in which $\{d_n\}$ is a complex sequence. Let $\rho_n = |d_n|$ and $T(m + 1, n - m) = [\rho_1^n, \ldots, \rho_{m+1}^n]$. By (2.4) and (3.3) we have

$$||B|| = \sup \sum_{m=0}^{n} |b_{nm}| \leqq \sup \sum_{m=0}^{n} |1 + d_m|!T(m + 1, n - m).$$

If $\sum \rho_n < \infty$, then $|1 + d_m|! = O(1)$, so

$$||B|| \leqq O(1) \sup \sum_{m=0}^{n} T(m + 1, n - m).$$

If also $\rho_n < 1$, for each n, then Lemma 3.13 implies that $||B|| < \infty$. We have proved

LEMMA 3.17. *If $|d_n| < 1$, for each n, and $\sum |d_n| < \infty$, then $||B|| < \infty$.*

THEOREM 3.18. *$|d_n| < 1$, for each n, and $\sum |d_n| < \infty$ are sufficient conditions for the (F, d_n) matrix to be regular and Mercerian. If each $d_n \geqq 0$, or each $d_n \leqq 0$, then these conditions are also necessary.*

Proof. This follows from [**5**, Theorem 3.12] and Lemma 3.17.

Now let C be the (f, d_n) matrix, A the (F, d_n) matrix, and $Z = Z(f)$ the Sonnenschein matrix generated by f (assuming that $f(1) = 1$). Koch [**6**] has observed that $C = AZ$, and it is easily seen that if $d(Z)$ is the domain of the linear transformation $Z:s \to s$, then $C = A \circ Z$ on $d(Z)$, i.e., if $x \in d(Z)$, then $Cx = (AZ)x = A(Zx)$. In particular, if f is a polynomial, whence $d(Z) = s$, it follows that $C = A \circ Z$ on s. The above theorem now gives

THEOREM 3.19. *Let $|d_n| < 1$, for each n, and $\sum |d_n| < \infty$. Then the (f, d_n) matrix is equivalent to $Z(f)$ on $d(Z)$, and, if f is a polynomial, on all of s.*

This theorem is of interest in part because, while useful, necessary and sufficient conditions for the regularity of the (f, d_n) method are not known, such conditions are known [**3**] for $Z(f)$.

We close with an application of a theorem of Agnew [**1**, Theorem 7.4]. We reproduce a preliminary definition and the theorem below.

Definition. The sequence $x = \{x_n\}$ lies in an angle less than π if there exist z_0, θ_0, and φ such that $0 < \varphi < \pi/2$, and for each n we have

$$x_n = z_0 + r_n \exp\{i(\theta_0 + \theta_n)\},$$

with $r_n \geqq 0$ and $|\theta_n| \leqq \varphi$.

THEOREM (Agnew). *If C and D are positive regular matrices ($c_{nk} \geqq 0$, $d_{nk} \geqq 0$), then every sequence which lies in an angle less than π and is summable to σ by either of the methods CD or $C \circ D$, is summable to σ by the other.*

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THEOREM 3.20. *Let f have real nonnegative Maclaurin coefficients and let $0 \leqq d_n < 1$, for each n, and $\sum d_n < \infty$. Then the methods (f, d_n) and $Z(f)$ are equivalent on the set $\mathscr{A}$ of all sequences each of which lies in some angle less than π. In particular, they are equivalent on m.*

Proof. [**10**, Lemma 2.2] shows that $Z(f)$ is regular, and A is regular and Mercerian. Thus, $(f, d_n) = AZ \sim A \circ Z \sim Z$ on $\mathscr{A}$.

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