



RESEARCH ARTICLE

Length spectrum of large genus random metric maps

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Abstract

We study the length of short cycles on uniformly random metric maps (also known as ribbon graphs) of large genus using a Teichmüller theory approach. We establish that, as the genus tends to infinity, the length spectrum converges to a Poisson point process with an explicit intensity. This result extends the work of Janson and Louf to the multi-faced case.

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1. Introduction

A *map*, or a *ribbon graph*, is a graph with a cyclic ordering of the edges at each vertex. By substituting edges with ribbons and attaching them at each vertex in accordance with the given cyclic order, we create an oriented surface with boundaries on which the graph is drawn (see Figure 1). Since Tutte’s pioneering work [58], ribbon graphs have been extensively studied, partly due to the increased interest following the realisation of their importance in two-dimensional quantum gravity.

Much attention has been devoted to the study of *metric maps* (i.e., ribbon graphs with the assignment of a positive real number to each edge). Remarkably, the moduli space parametrising metric ribbon graphs of a fixed genus g and n faces of fixed lengths is naturally isomorphic to the moduli space of Riemann surfaces of genus g with n punctures [31, 52, 14]. This fact was employed by Harer and Zagier to compute the Euler characteristic of the moduli space of Riemann surfaces [32] and by Kontsevich in his proof of Witten’s conjecture [59, 37]. The latter is a formula that computes the ‘number’ of metric ribbon graphs recursively on the Euler characteristic: a topological recursion. The same type of recursion applies to the ‘number’ of hyperbolic surfaces, as discovered by Mirzakhani [45].

Recently, intensive research efforts have been centred around the *random large genus regime*, both in the combinatorial and in the hyperbolic contexts. The study of large genus asymptotics holds significance for several reasons. Firstly, the intricate nature of several quantities simplifies enormously in the large genus limit, leading to closed-form asymptotic evaluations. Secondly, many interesting quantities associated to several geometric models appear to be exclusively attainable in the asymptotic regime. A (far from exhaustive) list of examples in the combinatorial setting include the connectivity [54, 41, 15], the local limit [18, 19] and cycle statistics [35, 36]. Analogously, examples in the hyperbolic setting include the connectivity [46, 17, 16], the local limit [49], curve statistics [30, 47, 23, 24, 62, 33] and the Laplacian spectrum [63, 40, 3, 34, 39, 28, 57, 50].

1.1. The results

In the present article, we study short cycles on metric ribbon graphs of large genus. A *cycle* on a metric ribbon graph is a (finite) set of distinct edges, say $\{e_1, \dots, e_k\}$, such that for some k vertices v_1, \dots, v_k , the edge e_i joins v_i to v_{i+1} , where $v_{k+1} = v_1$. The length of a cycle is the sum of the lengths of its edges. For a given metric ribbon graph G , we define its *length spectrum* $\Lambda(G)$ as the multiset of lengths of all cycles in G that do not represent a face.

Denote by $G_{g,L}$ a uniform random metric ribbon graph of genus g with n marked faces of lengths $L = (L_1, \dots, L_n) \in \mathbb{R}_{>0}^n$. See Section 2 for the precise notion of random metric ribbon graph. Our main result, proved in Sections 3 and 4, is an explicit description of $\Lambda(G_{g,L})$ in the large genus limit as a *Poisson point process*.

Theorem A. *Let n be a fixed positive integer, and let $L = L(g) = (L_1(g), \dots, L_n(g))$ be a sequence of vectors of positive real numbers such that, as $g \rightarrow \infty$,*

$$L_1(g) + \dots + L_n(g) \sim 12g - 12 + 6n \sim 12g. \tag{1}$$

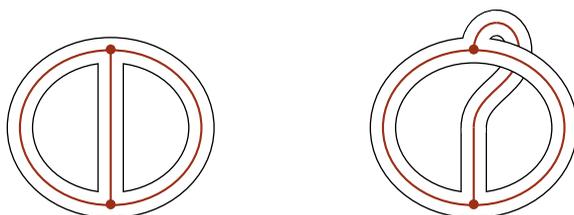


Figure 1. A ribbon graph of genus 0 with 3 faces (left) and a ribbon graph of genus 1 with 1 face (right).

The random multiset $\Lambda(\mathbf{G}_{g,L})$, viewed as a point process on $\mathbb{R}_{\geq 0}$, converges in distribution as $g \rightarrow \infty$ to a Poisson point process with intensity λ defined by

$$\lambda(\ell) := \frac{\cosh(\ell) - 1}{\ell}. \tag{2}$$

As $\mathbf{G}_{g,L}$ has almost surely $6g - 6 + 3n$ edges, and the total length of its faces is twice the total length of all edges, the scaling condition (1) implies that, on average, every edge has length 1 as $g \rightarrow \infty$. In this sense, the scaling condition is a natural assumption in this context. Besides, throughout the paper, L will typically denote a sequence of vectors indexed by the genus g . To simplify the notation, we will just write L instead of $L(g)$, when there is no potential for ambiguity.

Actually, we shall prove the following result which implies Theorem A through the method of moments. For any nonempty interval $[a, b) \subset \mathbb{R}_{\geq 0}$, denote by $N_{[a,b)}(\mathbf{G}_{g,L})$ the number of cycles in $\mathbf{G}_{g,L}$ of length falling within the interval $[a, b)$.

Theorem B. *Let n and L be as in Theorem A. For any disjoint intervals $[a_1, b_1), \dots, [a_p, b_p) \subset \mathbb{R}_{\geq 0}$, the random vector*

$$\left(N_{[a_1, b_1)}(\mathbf{G}_{g,L}), \dots, N_{[a_p, b_p)}(\mathbf{G}_{g,L}) \right) \tag{3}$$

converges in distribution as $g \rightarrow \infty$ to a vector of independent Poisson variables of means

$$\left(\int_{a_1}^{b_1} \lambda(\ell) d\ell, \dots, \int_{a_p}^{b_p} \lambda(\ell) d\ell \right). \tag{4}$$

As an application, we obtain the law of the length of the shortest cycle, known as *girth* or *systole*, on a random metric ribbon graph of large genus.

Corollary C. *Let n and L be as in Theorem A. The girth of $\mathbf{G}_{g,L}$ converges in distribution to a non-homogeneous exponential distribution with rate function λ . In other words, we have*

$$\lim_{g \rightarrow \infty} \mathbb{P}[\text{girth}(\mathbf{G}_{g,L}) \leq t] = 1 - \exp\left(- \int_0^t \lambda(\ell) d\ell\right). \tag{5}$$

We remark that all results presented here hold in the more general setting where the boundary lengths are subjected to the scaling condition $L_1 + \dots + L_n \sim \mu 12g$ for some $\mu > 0$. In this case, the intensity is given by $\lambda_\mu(\ell) := (\cosh(\ell/\mu) - 1)/\ell$. Besides, all results are still valid when replacing ‘cycles’ by ‘closed walks that do not traverse the same edge more than D times’, with D a fixed positive integer.

In comparison to the analogous results for hyperbolic surfaces due to Mirzakhani and Petri [47], a natural comment is due. The length spectrum of a hyperbolic surface (or more generally, any Riemannian manifold) is commonly defined as the multiset of lengths of the shortest primitive closed curve in each free homotopy class. In this combinatorial setting, it would make sense to consider the length spectrum defined analogously, instead of restricting it to only cycles. However, Theorem B fails to hold true when all closed curves are considered, and it fails even when restricted to all simple closed curves. More precisely, let $\bar{N}_{[a,b)}(G)$ denote the number of (free homotopy classes of primitive) closed curves in G with length falling within the interval $[a, b)$, and let $\bar{N}_{[a,b)}^\circ(G)$ denote the number of simple closed curves in G .

Theorem D. *For any fixed (g, n) , boundary lengths $L \in \mathbb{R}_{>0}^n$, and $[a, b) \subset \mathbb{R}_{\geq 0}$, we have the following:*

- $\mathbb{E}[\bar{N}_{[a,b)}(\mathbf{G}_{g,L})] = \infty$,
- $\mathbb{E}[\bar{N}_{[a,b)}^\circ(\mathbf{G}_{g,L})] < \infty$ and $\mathbb{E}[\bar{N}_{[a,b)}^\circ(\mathbf{G}_{g,L})^k] = \infty$ for any $k > 3/2$.

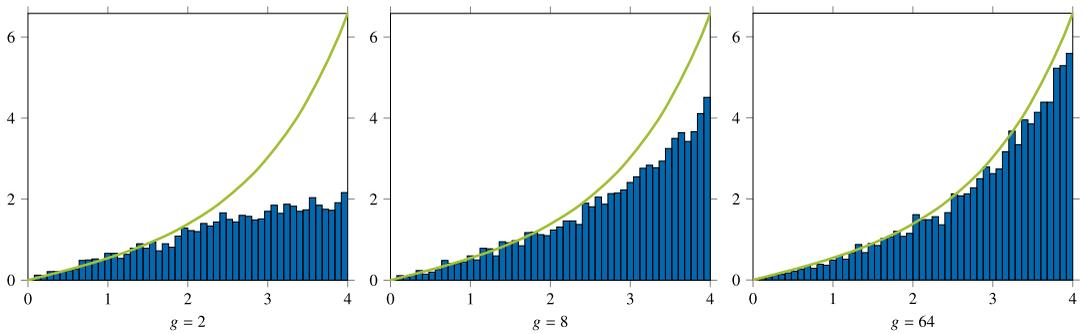


Figure 2. In blue, the cycle length statistics of random unicellular metric maps of genus $g = 2, 8,$ and $64,$ sampled over 10^3 units and properly rescaled. The predicted intensity λ is depicted in lime.

The intuitive reason behind the above failing is the fact that metric ribbon graphs have zero curvature, and as such, they can be scaled. Hence, the number of short closed curves on a metric ribbon graph G can blow up when G approaches the boundary of the moduli space. See Section 5 for a detailed discussion.

Theorem A is supported by evidence from numerical simulations, as discussed in Section 6. Figure 2 illustrates cycle length statistics derived from three samples of 10^3 uniform random one-faced metric ribbon graphs of genera 2, 8 and 64, respectively. The theoretical prediction is depicted in lime.

1.2. Related works and proof strategy

In [47], Mirzakhani and Petri study the length spectrum of a random closed hyperbolic surface S_g of genus $g,$ sampled according to the Weil–Petersson measure. They prove that the length spectrum $\Lambda(S_g)$ converges in distribution as $g \rightarrow \infty$ to a Poisson point process with intensity λ defined as in Equation (2). In [36], Janson and Louf consider uniform random unicellular (i.e., one-faced) maps $U_{v,g}$ of genus g with v vertices, metrised by assigning length 1 to all edges. They prove that, as $g, v \rightarrow \infty$ with $g = o(v),$ the normalised length spectrum $\sqrt{12g/v} \cdot \Lambda(U_{v,g})$ converges in distribution to a Poisson point process with exactly the same intensity $\lambda.$ The convergence of the length spectrum of large random hyperbolic surfaces or random maps to a Poisson point process is not entirely unexpected (such events follow the *Poisson paradigm*; see [61, 43, 53, 56] for results along these lines). However, the precise matching of intensity functions is somehow miraculous.

While both [47] and [36] employ the moment method in their proofs, their approaches are of completely different natures. On the one hand, Mirzakhani and Petri compute moments by means of an integration formula developed by Mirzakhani in her thesis [45], and the large genus asymptotic analysis relies on the work of Mirzakhani and Zograf on Weil–Petersson volumes [46, 48]. On the other hand, the approach adopted by Janson and Louf is entirely combinatorial. In their proof, a bijection due to Chapuy, Féray and Fusy [20] between unicellular maps and trees decorated by permutations plays a crucial role, enabling them to proceed using results on random trees and random permutations, both extensively explored subjects. We emphasise that, as the Chapuy–Féray–Fusy bijection is tied to the unicellular case, the method of Janson and Louf does not extend to the multi-faced case.

The current paper follows a Teichmüller theory approach, and the proof strategy is similar to that of [47]. More precisely, our proof makes use of the framework established in [5], which brings several tools from hyperbolic geometry into combinatorics, as well as the recent result by Aggarwal [1] on the large genus asymptotics of ψ -class intersection numbers. This combination of techniques allows us to extend the results of [36] to the multi-faced cases.

Let us briefly mention why the model considered in [36] coincides with the one discussed in the present work for $n = 1.$ Curien, Janson, Kortchemski, Louf and Marzouk proposed an alternative model to random metric unicellular maps which behave as $U_{v,g}$ (see [42]). Let V_g be a uniform random

unicellular trivalent map of genus g , metrised by assigning to all $6g - 3$ edges i.i.d. $\text{Exp}(1)$ random lengths ℓ_i (exponential distribution of parameter 1). Denote by $L := 2(\ell_1 + \dots + \ell_{6g-3})$ the length of the unique face. It is a standard fact that if $\ell_1, \dots, \ell_{6g-3}$ are i.i.d. $\text{Exp}(1)$ variables, then the random vector $X := (2\ell_1/L, \dots, 2\ell_{6g-3}/L)$ is $\text{Dir}(1^{6g-3})$ distributed (Dirichlet distribution of order $6g - 3$ of parameters $(1, 1, \dots, 1)$), and L and X are independent thanks to Lukács's proportion-sum independence theorem. Such a model, conditioned on L being fixed, is nothing but $\mathbf{G}_{g,L}$.

1.3. Outlook

As highlighted in [36, Section 1.4], random metric ribbon graphs and random hyperbolic surfaces exhibit remarkably similar behaviours. The former's combinatorial and topological nature allows for the application of combinatorics and graph theory to 2D geometry, as well as facilitating numerical tests (see Figure 2). This advantage is evident in the computation of combinatorial versions of Mirzakhani's kernels \mathcal{R} and \mathcal{D} , which simplifies to a straightforward combinatorial check [5]. A plausible explanation for these similarities is the spine construction from [14], explored in various contexts such as symplectic structures [25] and shapes of complementary subsurfaces [8].

With this perspective in mind, it is then natural to consider the question of the Laplacian spectrum of $\mathbf{G}_{g,L}$ as $g \rightarrow \infty$ and its relation to that of random hyperbolic surfaces (see [38] for a definition of the Laplace operator on a metric graph). To address this, identifying the *local limit*, also known as the *Benjamini–Schramm limit* [11], of $\mathbf{G}_{g,L}$ as $g \rightarrow \infty$ is crucial. The Benjamini–Schramm convergence implies the convergence of spectral measures, a principle recently extended to metric graphs [4, 2].

It is conjectured that the local limit of a random trivalent unicellular map is the random infinite trivalent metric tree \mathbf{T} with i.i.d. $\text{Exp}(1)$ distributed edge lengths, suggesting $\mathbf{G}_{g,L}$ converges to \mathbf{T} . Consequently, exploring the Laplacian spectrum of \mathbf{T} and comparing it to that of the hyperbolic plane is of interest. The local limit of unicellular maps, particularly when the genus grows proportionally to the number of edges, has been identified in [6] as a supercritical Galton–Watson tree conditioned to be infinite. More recently, moderate genus growth has been studied in [21], where the mesoscopic scaling limit of the core of such maps was proved to be the infinite trivalent tree with i.i.d. exponential edge lengths.

2. Background

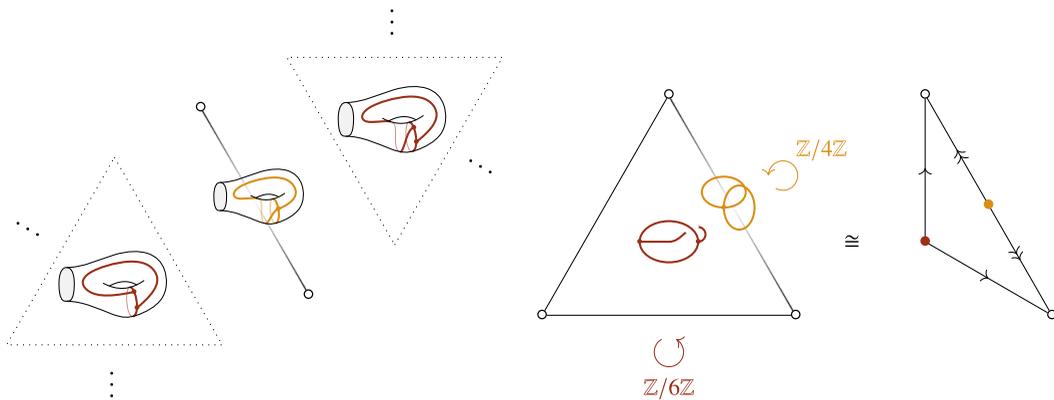
In this section, we recall some background material about the geometry of the combinatorial Teichmüller and moduli spaces (see [5] for more details), as well as some probabilistic tools that will be used in order to prove the main result of the paper.

2.1. Combinatorial Teichmüller and moduli spaces

A ribbon graph is a finite graph G together with a cyclic order of the edges at each vertex. By replacing each edge by a closed ribbon and glueing them at each vertex according to the cyclic order, we obtain a topological, oriented, compact surface called the geometric realisation of G . Notice that the graph is a deformation retract of its geometric realisation. We will assume that G is connected and all vertices have valency ≥ 3 .

The geometric realisation of a ribbon graph G will have $n \geq 1$ boundary components, also called faces, and we always assume they are labelled as $\partial_1 G, \dots, \partial_n G$. Denote by $V(G)$, $E(G)$, $F(G)$ the set of vertices, edges and faces, respectively. We define the genus $g \geq 0$ to be the genus of the geometric realisation. Thus, $|V(G)| - |E(G)| + |F(G)| = 2 - 2g$. The datum (g, n) is called the type of G .

A metric ribbon graph is the data (G, ℓ) of a ribbon graph G together with the assignment of a positive real number for each edge – that is, $\ell: E(G) \rightarrow \mathbb{R}_{>0}$. In the following, we will omit the map ℓ from the notation and simply denote it by ℓ_G when needed. For a given metric ribbon graph G and a nontrivial edge-path γ , we can define the length $\ell_G(\gamma)$ as the sum of the length of edges (with multiplicity) visited by γ . In particular, we can talk about length of the boundary components $\ell_G(\partial_i G)$.



(a) The combinatorial Teichmüller space of a one-holed torus. It has infinitely many cells 2D cells and 1D cells, corresponding to the different embeddings of the two ribbon graphs of type (1, 1).

(b) The combinatorial moduli space of type (1, 1). It has two orbicells, corresponding to the two ribbon graphs of type (1, 1). Every point has stabiliser $\mathbb{Z}/2\mathbb{Z}$ given by the elliptic involution, except for the 3-valent metric ribbon graph with all edge lengths being equal (with $\mathbb{Z}/6\mathbb{Z}$ -stabiliser), and the 4-valent metric ribbon graph with all edge lengths being equal (with $\mathbb{Z}/4\mathbb{Z}$ -stabiliser).

Figure 3. *The combinatorial Teichmüller space of a one-holed torus (left), and the corresponding moduli space.*

Fix now a connected, compact, oriented surface Σ of genus $g \geq 0$ with $n \geq 1$ labelled boundaries, denoted $\partial_1 \Sigma, \dots, \partial_n \Sigma$. Fix $L \in \mathbb{R}_{>0}^n$. Define the combinatorial Teichmüller space as the space parametrising metric ribbon graphs of type (g, n) with fixed boundary lengths L embedded into Σ , up to isotopy:

$$\mathcal{T}_\Sigma^{\text{comb}}(L) := \left\{ (G, f) \left| \begin{array}{l} G \text{ is a metric ribbon graph} \\ \text{with boundary lengths } \ell_G(\partial_i G) = L_i \\ f: G \hookrightarrow \Sigma \text{ is a retract} \end{array} \right. \right\} / \sim \tag{6}$$

where the equivalence relation is given by

$$(G, f) \sim (G', f') \quad \text{if and only if} \quad \exists \varphi: G \rightarrow G' \text{ an isometry such that } f' \circ \varphi = f \text{ and } f \text{ and } f' \text{ are isotopic.} \tag{7}$$

Notice that, as G is a retract of Σ , it has the same genus and number of boundary components as Σ . Often, we will denote elements of $\mathcal{T}_\Sigma^{\text{comb}}(L)$ by \mathbb{G} .

It can be shown that $\mathcal{T}_\Sigma^{\text{comb}}(L)$ is a real polytopal complex of dimension $6g - 6 + 2n$. The cells are labelled by embedded ribbon graphs, and they parametrise all possible metrics with fixed boundary lengths on the corresponding embedded graph; the cells are glued together via edge degeneration (see Figure 3a for an example). The pure mapping class group MCG_Σ of isotopy classes of orientation preserving homeomorphisms of Σ preserving the boundary components naturally acts on $\mathcal{T}_\Sigma^{\text{comb}}(L)$. The quotient space

$$\mathcal{M}_{g,n}^{\text{comb}}(L) := \mathcal{T}_\Sigma^{\text{comb}}(L) / \text{MCG}_\Sigma \tag{8}$$

is called the combinatorial moduli space. It parametrises metric ribbon graphs of type (g, n) with fixed boundary lengths. It is a real polytopal orbicomplex of dimension $6g - 6 + 2n$. The orbicells are labelled by ribbon graphs, and they parametrise all possible metrics with fixed boundary lengths on the corresponding ribbon graph, up to automorphism (see Figure 3ba for an example).

2.2. The symplectic structure, the length function, and the integration formula

As showed by Kontsevich [37], the moduli space $\mathcal{M}_{g,n}^{\text{comb}}(L)$ carries a natural symplectic form ω that we call Kontsevich form. The associated cohomology class has deep connections with the moduli space of Riemann surfaces, as recalled in Subsection 2.3.

Denote by $dG := \frac{\omega^{3g-3+n}}{(3g-3+n)!}$ the volume form associated to ω . The symplectic volumes

$$V_{g,n}(L) := \int_{\mathcal{M}_{g,n}^{\text{comb}}(L)} dG \tag{9}$$

are finite, and they have been computed recursively by Kontsevich using matrix model techniques. It is worth mentioning that the symplectic volume form is proportional to the Lebesgue measure defined on each top-dimensional cell forming the combinatorial moduli space, and as such, the symplectic volumes are proportional to the asymptotic counting of metric ribbon graphs with edge lengths in $\frac{1}{k}\mathbb{Z}$ as $k \rightarrow \infty$. A geometric proof of Kontsevich’s recursion, based on the existence of Fenchel–Nielsen coordinates and a Mirzakhani-type recursion for the constant function 1 on the combinatorial Teichmüller space, can be found in [5].

Let us recall the notion of combinatorial Fenchel–Nielsen coordinates. Fix an embedded metric ribbon graph $\mathbb{G} \in \mathcal{T}_{\Sigma}^{\text{comb}}(L)$ and a free homotopy class γ of a (non-null) simple closed curve in Σ . Consider the unique representative of γ that has been homotoped to the embedded graph as a non-backtracking edge-path. We will refer to it as the geodesic representative. The geodesic length $\ell_{\mathbb{G}}(\gamma)$ is defined by adding up the lengths of the edges visited by its geodesic representative. Thus, every free homotopy class γ of nontrivial simple closed curves on Σ defines a function $\ell(\gamma): \mathcal{T}_{\Sigma}^{\text{comb}}(L) \rightarrow \mathbb{R}_{>0}$ that assigns to \mathbb{G} the geodesic length $\ell_{\mathbb{G}}(\gamma)$.

Consider now a pants decomposition \mathcal{P} of Σ – that is, a collection $(\gamma_m)_{m=1}^{3g-3+n}$ of simple closed curves that cut Σ into a disjoint union of pairs of pants. For a given embedded metric ribbon graph $\mathbb{G} \in \mathcal{T}_{\Sigma}^{\text{comb}}(L)$, we can assign to each curve in \mathcal{P} the data of two real numbers in $\mathbb{R}_{>0} \times \mathbb{R}$: the length $\ell_{\mathbb{G}}(\gamma_m)$ and the twist $\tau_{\mathbb{G}}(\gamma_m)$ of the gluing. Thus, we get a map

$$\mathcal{T}_{\Sigma}^{\text{comb}}(L) \longrightarrow (\mathbb{R}_{>0} \times \mathbb{R})^{\mathcal{P}}, \quad \mathbb{G} \longmapsto (\ell_{\mathbb{G}}(\gamma_m), \tau_{\mathbb{G}}(\gamma_m))_{m=1}^{3g-3+n} \tag{10}$$

called the combinatorial Fenchel–Nielsen coordinates. They are the analogue of Fenchel–Nielsen coordinates in hyperbolic geometry, and Dehn–Thurston coordinates in the theory of measured foliations. Lengths and twists form a global coordinate system on the combinatorial Teichmüller space that is Darboux for the Kontsevich symplectic form (canonically lifted to a mapping class group invariant form). This is the combinatorial analogue of Wolpert’s magic formula [60] in the hyperbolic setting.

Theorem 2.1 (Combinatorial Wolpert’s formula [5]). *For every pants decomposition, the Kontsevich form on $\mathcal{T}_{\Sigma}^{\text{comb}}(L)$ is canonically given by*

$$\omega = \sum_{m=1}^{3g-3+n} d\ell_m \wedge d\tau_m. \tag{11}$$

The above formula allows for the integration of natural geometric functions defined on the combinatorial moduli space. This fact, which is the combinatorial analogue of Mirzakhani’s integration formula [45], is one of the main ingredients in the geometric proof of the volume recursion. In order to state the formula, let us introduce some notation.

A stable graph consists of the data $\Gamma = (V(\Gamma), H(\Gamma), (g_v)_{v \in V(\Gamma)}, \nu, \iota)$ satisfying the following properties.

1. $V(\Gamma)$ is the set of vertices, equipped with the assignment of nonnegative integers $(g_v)_{v \in V(\Gamma)}$ called the genus decoration.
2. $H(\Gamma)$ is the set of half-edges, the map $\nu: H(\Gamma) \rightarrow V(\Gamma)$ associates to each half-edge the vertex it is incident to, and $\iota: H(\Gamma) \rightarrow H(\Gamma)$ is an involution that pairs half-edges together.

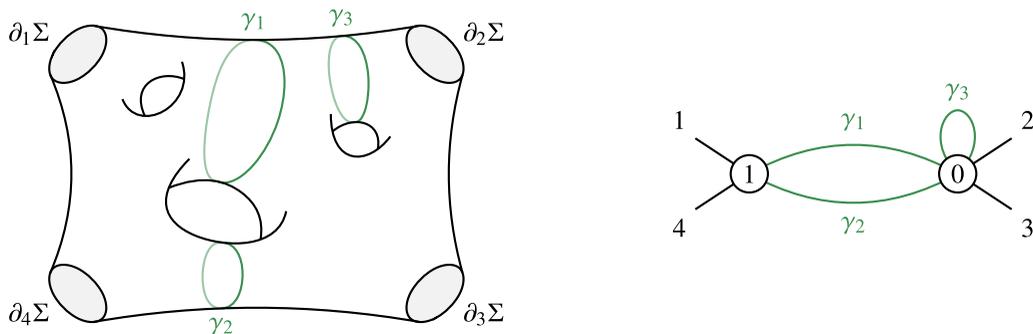


Figure 4. The stable graph corresponding to an ordered tuple of curves.

3. The set of 2-cycles of ι is the set of edges, denoted $E(\Gamma)$ (self-loops are permitted).
4. The set of 1-cycles (i.e., fixed points) of ι is the set of leaves, denoted $\Lambda(\Gamma)$. We require that leaves are labelled: there is a bijection $\ell: \Lambda(\Gamma) \rightarrow \{1, \dots, n\}$, where $n := |\Lambda(\Gamma)|$.
5. The pair $(V(\Gamma), E(\Gamma))$ defines a connected graph.
6. If v is a vertex, denote by $n_v := |v^{-1}(v)|$ its valency. We require that for each vertex v , the stability condition $2g_v - 2 + n_v > 0$ holds.

An isomorphism of stable graphs $\varphi: \Gamma \rightarrow \Gamma'$ is a pair of bijective maps

$$\varphi_V: V(\Gamma) \longrightarrow V(\Gamma'), \quad \varphi_H: H(\Gamma) \longrightarrow H(\Gamma'), \tag{12}$$

of their sets of vertices and half-edges which are compatible with the functions g, ν, ι , and the leaf labelling. That is,

$$\begin{aligned} g_{\varphi_V(v)} &= g_v & \forall v \in V(\Gamma), \\ \nu(\varphi_H(h)) &= \nu(h) \quad \text{and} \quad \iota(\varphi_H(h)) = \iota(h) & \forall h \in H(\Gamma), \\ \ell(\varphi_H(\lambda)) &= \ell(\lambda) & \forall \lambda \in \Lambda(\Gamma). \end{aligned} \tag{13}$$

We denote by $\text{Aut}(\Gamma)$ the automorphism group of Γ .

For a given stable graph Γ , define its genus as

$$g(\Gamma) := \sum_{v \in V(\Gamma)} g(v) + h^1(\Gamma), \tag{14}$$

where $h^1(\Gamma)$ is the first Betti number of Γ . We denote by $G_{g,n}$ the (finite) set of isomorphism classes of stable graphs of genus g with n leaves. Abusing notation, we denote the isomorphism class of a stable graph Γ with the same symbol.

Stable graphs naturally appear as mapping class group orbits of primitive multicurves. Let $\gamma = (\gamma_1, \dots, \gamma_r)$ be an ordered primitive multicurve – that is, an r -tuple of free homotopic classes of simple closed curves on Σ that are non-null, non-peripheral and distinct. Denote by Γ the mapping class group orbit $\text{MCG}_\Sigma \cdot \gamma$. Then Γ is identified with (an isomorphism class of) a stable graph with r labelled edges, where the vertices and the genus decoration correspond to the connected components of $\Sigma \setminus \gamma$, the s -th edge corresponds to the curve γ_s , and the i -th leaf corresponds to the boundary component $\partial_i \Sigma$ (see Figure 4 for an example). Notice that, in contrast to the previous definition, stable graphs have now labelled edges.

For a given measurable function $F: \mathbb{R}_{>0}^n \times \mathbb{R}_{>0}^r \rightarrow \mathbb{R}$, define $F_\Gamma: \mathcal{T}_\Sigma^{\text{comb}}(L) \rightarrow \mathbb{R}$ as

$$F_\Gamma(\mathbb{G}) := \sum_{\alpha \in \Gamma} F(L, \ell_{\mathbb{G}}(\alpha)), \tag{15}$$

where $\ell_{\mathbb{G}}(\alpha) = (\ell_{\mathbb{G}}(\alpha_1), \dots, \ell_{\mathbb{G}}(\alpha_r))$. If the above series is absolutely convergent, the function F_{Γ} descends naturally to a function on the moduli space $\mathcal{M}_{g,n}^{\text{comb}}(L)$, that we denote with the same symbol. Its integral is given by the following formula.

Theorem 2.2 (Integration formula [5]). *If the series (15) is absolutely convergent and its projection onto $\mathcal{M}_{g,n}^{\text{comb}}(L)$ is integrable with respect to the Kontsevich measure, then the integral of F_{Γ} over the combinatorial moduli space is given by*

$$\int_{\mathcal{M}_{g,n}^{\text{comb}}(L)} F_{\Gamma} dG = \frac{1}{|\text{Aut}(\Gamma)|} \int_{\mathbb{R}_{>0}^r} F(L, \ell) V_{\Gamma}(L, \ell) \prod_{s=1}^r \ell_s d\ell_s, \tag{16}$$

where $V_{\Gamma}(L, \ell) := \prod_{v \in \mathcal{V}(\Gamma)} V_{g_v, n_v}(L_v, \ell_v)$. Here, L_v (resp. ℓ_v) denotes the tuple of lengths associated to the leaves (resp. half-edges that are not leaves) attached to v .

2.3. Connection with intersection numbers and Aggarwal’s asymptotic formula

A notable aspect of the combinatorial moduli space $\mathcal{M}_{g,n}^{\text{comb}}(L)$ is its homeomorphism to the moduli space of Riemann surfaces $\mathcal{M}_{g,n}$ (both considered as real topological orbifolds). This fact was proved by Harer [31] using meromorphic differentials (based on the seminal works of Jenkins and Strebel and unpublished works of Thurston and Mumford) and by Penner and Bowditch–Epstein [52, 14] using hyperbolic geometry. For instance, the moduli space $\mathcal{M}_{1,1}^{\text{comb}}(L)$ in Figure 3ba is nothing but the moduli of elliptic curves $\mathcal{M}_{1,1} = [\text{SL}(2, \mathbb{Z}) \backslash \mathbb{H}^2]$.

Consequently, any topological invariant of $\mathcal{M}_{g,n}^{\text{comb}}(L)$ can be translated into a corresponding topological invariant of $\mathcal{M}_{g,n}$ and vice versa. As mentioned in the introduction, Harer and Zagier [32] utilise this fact to compute the Euler characteristic of the moduli space of curves, and Kontsevich employs it to prove Witten’s conjecture [59, 37]. The former is a consequence of the following result.

Theorem 2.3 (Symplectic volumes as intersection numbers [37]). *The symplectic volumes satisfy*

$$V_{g,n}(L) = \sum_{\substack{(d_1, \dots, d_n) \in \mathbb{Z}_{\geq 0}^k \\ |d|=3g-3+n}} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n} \prod_{i=1}^n \frac{L_i^{2d_i}}{2^{d_i} d_i!}, \tag{17}$$

where

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n} := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \tag{18}$$

are the Witten–Kontsevich intersection numbers over the Deligne–Mumford compactification of the moduli space of Riemann surfaces. Here, $\psi_i := c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ is the first Chern class of the i -th tautological line bundle \mathcal{L}_i (i.e., the line bundle whose fibre over $[C, p_1, \dots, p_n] \in \overline{\mathcal{M}}_{g,n}$ is the cotangent line at p_i of C).

The main tool to estimate volumes in the large genus limit is an asymptotic formula for the Witten–Kontsevich intersection numbers. The formula was conjectured by Delecroix–Goujard–Zograf–Zorich based on numerical data [22], and proved shortly after by Aggarwal [1] through a combinatorial analysis of the associated Virasoro constraints. An alternative proof was recently given in [29] and in [26] through a combinatorial and resurgent analysis of determinantal formulae, respectively.

Theorem 2.4 (Large genus asymptotic of intersection numbers [1]). *For any stable (g, n) with $n^2 < g/800$ and $(d_1, \dots, d_n) \in \mathbb{Z}_{\geq 0}^n$ with $|d| = 3g - 3 + n$, we have as $g \rightarrow \infty$*

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n} := \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n} \prod_{i=1}^n (2d_i + 1)!! = \frac{(6g - 5 + 2n)!!}{g! 24^g} (1 + o(1)) \tag{19}$$

uniformly in n and (d_1, \dots, d_n) .

Combining (17) and (19), we obtain that for any stable (g, n) with $n^2 < g/800$, and any $L = (L_1, \dots, L_n) \in \mathbb{R}_{>0}^n$,

$$V_{g,n}(L) = (1 + o(1)) \frac{(6g - 5 + 2n)!!}{g! 24^g} [z^{6g-6+3n}] \prod_{i=1}^n \frac{\sinh(L_i z)}{L_i}, \tag{20}$$

uniformly in n and L , where $[z^d] f(z)$ denotes the coefficient of z^d in the Taylor expansion of $f(z)$ around $z = 0$. We will come back to this observation in the next section, when discussing estimates on Kontsevich volumes.

The asymptotic formula (19) becomes false if n grows too rapidly compared to g (for instance, consider the exact formula $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{0,n} = \binom{n-3}{d_1, \dots, d_n}$). However, the following estimate always holds.

Theorem 2.5 (Uniform bound on intersection numbers [1]). *For any stable (g, n) and $(d_1, \dots, d_n) \in \mathbb{Z}_{\geq 0}^n$ with $|d| = 3g - 3 + n$, we have*

$$\langle \langle \tau_{d_1} \cdots \tau_{d_n} \rangle \rangle_{g,n} \leq \left(\frac{3}{2}\right)^{n-1} \frac{(6g - 5 + 2n)!!}{g! 24^g}. \tag{21}$$

2.4. Random metric ribbon graphs and the method of moments

As the combinatorial moduli space has a finite volume, we can turn it into a probability space. More precisely, given a measurable set $A \subseteq \mathcal{M}_{g,n}^{\text{comb}}(L)$, define

$$\mathbb{P}[A] := \frac{1}{V_{g,n}(L)} \int_A dG. \tag{22}$$

Given random variables $X_1, \dots, X_p : \mathcal{M}_{g,n}^{\text{comb}}(L) \rightarrow \mathbb{R}$ and sets $E_1, \dots, E_p \subset \mathbb{R}$, define

$$\mathbb{P}[X_1 \in E_1, \dots, X_p \in E_p] := \mathbb{P}\left[\bigcap_{i=1}^p \{G \in \mathcal{M}_{g,n}^{\text{comb}}(L) \mid X_i(G) \in E_i\}\right]. \tag{23}$$

If $E_i = \{x_i\}$ is a singleton, we simply denote the above quantity as $\mathbb{P}[X_1 = x_1, \dots, X_p = x_p]$. For $p = 1$, we refer to the function $\mathbb{P}[X = x]$ as the probability mass function of X . We also define the expectation value of a random variable X to be

$$\mathbb{E}[X] := \frac{1}{V_{g,n}(L)} \int_{\mathcal{M}_{g,n}^{\text{comb}}(L)} X(G) dG. \tag{24}$$

Often, we write $\mathbb{E}[X(\mathbf{G}_{g,L})]$ where $\mathbf{G}_{g,L}$ is a random element in $\mathcal{M}_{g,n}^{\text{comb}}(L)$ to emphasise the dependence on the genus and the number/lengths of the boundary components.

As in [47, 36], the main probabilistic tool used in this paper is the method of moments, which we now recall. Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space, and let $X : \Omega \rightarrow \mathbb{Z}_{\geq 0}$ be an integer-valued random variable. For any integer r , define

$$(X)_r := X(X - 1) \cdots (X - r + 1). \tag{25}$$

The expectation value $\mathbb{E}[(X)_r]$ is called the r -th factorial moment of X .

A special class of integer-valued random variables is constituted by those following a Poisson distribution. We say that X is Poisson distributed with mean $\lambda \in \mathbb{R}_{>0}$ if its probability mass function satisfies

$$\mathbb{P}[X = k] = e^{-\lambda} \frac{\lambda^k}{k!} \quad \text{for all } k \in \mathbb{Z}_{\geq 0}. \tag{26}$$

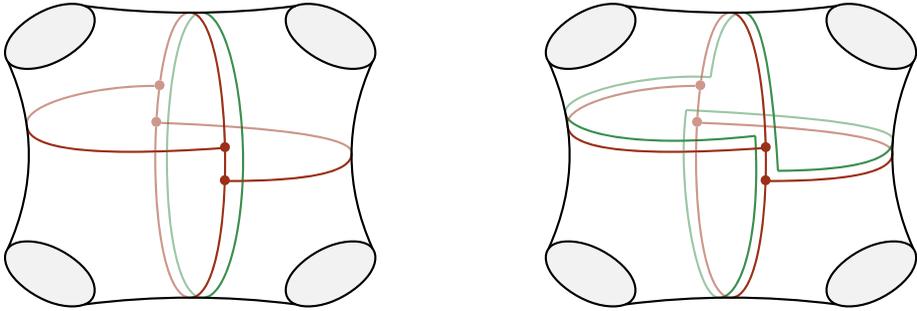


Figure 5. Example (left) and non-example (right) of a cycle on an embedded metric ribbon graph of type $(0, 4)$.

In this case, it can be shown that the r -th factorial moment of X is given by $\mathbb{E}[(X)_r] = \lambda^r$ for all integers $r \in \mathbb{Z}_{\geq 0}$. The method of moments asserts that the converse is also true. In other words, an integer-valued random variable is Poisson distributed with mean λ if and only if its r -th factorial moment is given by λ^r . More precisely, we have the following result (see, for instance, [12]).

Theorem 2.6 (Method of moments). *Let $\{(\Omega_d, \mathcal{B}_d, \mathbb{P}_d)\}_{d \in \mathbb{Z}_{\geq 1}}$ be a sequence of probability spaces, and let $(X_{d,1}, \dots, X_{d,p}) : \Omega_d \rightarrow \mathbb{Z}_{\geq 0}^p$ be an integer-valued random vector. Suppose there exists $(\lambda_1, \dots, \lambda_p) \in \mathbb{R}_{>0}^p$ such that*

$$\lim_{d \rightarrow \infty} \mathbb{E} \left[\prod_{i=1}^p (X_{d,i})_{r_i} \right] = \prod_{i=1}^p \lambda_i^{r_i} \tag{27}$$

for all $(r_1, \dots, r_p) \in \mathbb{Z}_{\geq 0}^p$. Then $(X_{d,1}, \dots, X_{d,p})_{d \in \mathbb{Z}_{\geq 1}}$ converges jointly in distribution to a vector of independent Poisson variables with parameters $(\lambda_1, \dots, \lambda_p)$. In other words, for all $(k_1, \dots, k_p) \in \mathbb{Z}_{\geq 0}^p$:

$$\lim_{d \rightarrow \infty} \mathbb{P}[X_{d,1} = k_1, \dots, X_{d,p} = k_p] = \prod_{i=1}^p e^{-\lambda_i} \frac{\lambda_i^{k_i}}{k_i!}. \tag{28}$$

Poisson distributions appear, for instance, in the context of point processes. To put it concretely, a point process on $\mathbb{R}_{\geq 0}$ is a collection of integer-valued random variables $X_S : \Omega \rightarrow \mathbb{Z}_{\geq 0}$ indexed by Borel sets S on $\mathbb{R}_{\geq 0}$, such that for all countable collections $\{[a_i, b_i]\}_{i=1}^{\infty}$ of disjoint intervals, we have $X_{\cup_i [a_i, b_i]} = \sum_i X_{[a_i, b_i]}$. The value X_S can be thought of as the number of events happening in the set S .

A point process is called Poisson if there exists a locally integrable nonnegative function $\lambda : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, called the intensity, such that $X_{[a,b]}$ is Poisson distributed with mean

$$\int_a^b \lambda(\ell) \, d\ell, \tag{29}$$

and for any disjoint intervals $[a_1, b_1], \dots, [a_p, b_p] \subset \mathbb{R}_{\geq 0}$, $X_{[a_1, b_1]}, \dots, X_{[a_p, b_p]}$ are independent.

3. The combinatorial length spectrum

3.1. Setup

Fix a topological surface Σ of type (g, n) and a cell in $\mathcal{T}_{\Sigma}^{\text{comb}}(L)$ (i.e., an embedded ribbon graph $G \hookrightarrow \Sigma$ that is a retract). A free homotopy class of a non-null, non-peripheral closed curve C is called a cycle if its unique non-backtracking edge-path representative visits each edge at most once. Notice that the notion of cycle depends on the underlying embedded ribbon graph. Moreover, we assume that a cycle is non-peripheral – that is, we exclude face perimeters. See Figure 5 for an example and a non-example.

The main object of study is the bottom part of the length spectrum of cycles on metric ribbon graphs – that is, the function $N_{g,L,[a,b]} : \mathcal{T}_\Sigma^{\text{comb}}(L) \rightarrow \mathbb{Z}_{\geq 0}$ defined as

$$N_{g,L,[a,b]}(\mathbb{G}) := \left| \left\{ \begin{array}{l} \text{C cycle in } \mathbb{G} \\ \text{with length } \ell_{\mathbb{G}}(\text{C}) \in [a, b] \end{array} \right\} \right|. \tag{30}$$

The function $N_{g,L,[a,b]}$ is mapping class group invariant, so it descends to a function on the combinatorial moduli space (denoted with the same symbol) that we can regard as a point process on $\mathbb{R}_{\geq 0}$. As presented in the introduction, our main result (Theorem A) is the fact that $N_{g,L,[a,b]}$ converges in distribution as $g \rightarrow \infty$ to a Poisson point process of intensity given by

$$\lambda_\mu(\ell) := \frac{\cosh(\ell/\mu) - 1}{\ell}, \tag{31}$$

under the following assumption on the scaling of the boundary components.

Boundary scaling assumption. The boundary lengths $L(g) = (L_1(g), \dots, L_n(g))$ are positive real functions of g , and there exists $\mu > 0$ such that, as $g \rightarrow \infty$,

$$|L(g)| := L_1(g) + \dots + L_n(g) \sim \mu 12g. \tag{32}$$

For ease of notation, we often omit the dependence on g and simply write L for $L(g)$.

Remark 3.1. As the homeomorphism $\mathcal{M}_{g,n}^{\text{comb}}(L) \rightarrow \mathcal{M}_{g,n}^{\text{comb}}(\mu L)$, defined by global scaling, preserves the Kontsevich measure and scales the length spectrum globally by μ , many of the results stated below for a general μ follow directly from the case $\mu = 1$.

As outlined in the introduction, the boundary scaling assumption is naturally explained by the fact that, for a random ribbon graph of type (g, n) with constant edge-lengths μ , the total boundary length is given by $|L| = 2\mu(6g - 6 + 3n) \sim \mu 12g$. It is also worth mentioning that the mean $\int_a^b \lambda_\mu(\ell) d\ell$ can be written as

$$\frac{1}{2} \int_{a/\mu}^{b/\mu} \mathcal{S}^2(\ell) \ell d\ell, \quad \text{where} \quad \mathcal{S}(\ell) := \frac{\sinh(\ell/2)}{\ell/2}. \tag{33}$$

The function \mathcal{S} is the ubiquitous function appearing in the enumerative theory of Riemann surfaces (see, for instance, [51]). The square and the symmetry factor $\frac{1}{2}$ naturally appear by assigning the function \mathcal{S} to the two unlabelled branches of the cycle C . Moreover, the measure $\ell d\ell$ is naturally interpreted as a length-twist measure, after integrating out the twist parameter.

Following [47], the strategy consists in employing the method of moments together with the key observation that the function

$$N_{g,L,I} := \prod_{i=1}^p (N_{g,L,[a_i,b_i]})_{r_i} \tag{34}$$

has a simple geometric interpretation: it counts the number of ordered p -tuples where the i -th item is the number of ordered r_i -tuples of cycles with length in $[a_i, b_i]$. Here, we suppose that all intervals $[a_i, b_i]$ are disjoint, and we denoted $I := \prod_{i=1}^p [a_i, b_i]^{r_i}$.

Another key step in the proof is to split $N_{g,L,I}$ as a sum of three terms:

$$N_{g,L,I} = N_{g,L,I}^\circ + N_{g,L,I}^\times + N_{g,L,I}^\times, \tag{35}$$

where we have set

- $N_{g,L,I}^\circ$ for the number of tuples such that all cycles are simple, distinct and non-separating,
- $N_{g,L,I}^\times$ for the number of tuples such that all cycles are simple, distinct and separating,
- $N_{g,L,I}^\times$ for the number of tuples such that at least one cycle is non-simple, or at least one pair of cycles intersect.

We also denote by $\bar{N}_{g,L,I}^*$ the corresponding counting where ‘cycle’ is replaced by ‘closed curve’. Clearly, $N_{g,L,I}^* \leq \bar{N}_{g,L,I}^*$ for $* \in \{\circ, \asymp, \times\}$. The goal of Subsection 3.4 is to prove the following claims: under the boundary scaling assumption (32), as $g \rightarrow \infty$,

- A) $\mathbb{E}[\bar{N}_{g,L,I}^\circ] = (1 + o(1)) \prod_{i=1}^p \left(\int_{a_i}^{b_i} \lambda_\mu(x) dx\right)^{r_i}$,
- B) $\mathbb{E}[N_{g,L,I}^\asymp] = O(g^{-1/2})$,
- C) $\mathbb{E}[N_{g,L,I}^\times] = O(g^{-1/2})$,
- D) $\mathbb{E}[\bar{N}_{g,L,I}^\circ] - \mathbb{E}[N_{g,L,I}^\circ] = O(g^{-1/2})$.

Claims (A)–(D), together with the splitting (35), imply that

$$\lim_{g \rightarrow \infty} \mathbb{E}[N_{g,L,I}] = \prod_{i=1}^p \left(\int_{a_i}^{b_i} \lambda_\mu(x) dx\right)^{r_i}. \tag{36}$$

Together with the method of moments, the main result of the paper follows.

The key ingredients that contribute to the proof of Claims (A)–(D) are an estimate on the Kontsevich volumes, the integration formula, and an estimate on a certain sum over stable graphs.

3.2. Volume estimates

As explained in Subsection 2.3, the large genus asymptotics of the Kontsevich volumes are expressed in terms of coefficients of the form $[z^{6g-6+3n}] \prod_{i=1}^n \sinh(L_i z)$. More generally, in this section, we are going to compute the $g \rightarrow \infty$ asymptotic behaviour of coefficients of the form

$$\mathfrak{F}_g := [z^{6g-6+3n}] F_g(z), \quad \text{where} \quad F_g(z) := \left(\prod_{i=1}^n \sinh(L_i(g)z)\right) \left(\prod_{j=1}^m \sinh(\ell_j(g)z)\right), \tag{37}$$

under the following assumptions:

- $n \geq 1$ and $m \geq 0$ are fixed integers,
- $L(g) = (L_1(g), \dots, L_n(g))$ and $\ell(g) = (\ell_1(g), \dots, \ell_m(g))$ are positive real functions of g , with $|L(g)| \sim \mu 12g$ (the boundary scaling assumption (32)) and $|\ell(g)| = O(1)$, as $g \rightarrow \infty$. To simplify the notation, we often drop the dependence of $L(g)$ and $\ell(g)$ on g and simply write L and ℓ .

Notice that, strictly speaking, both \mathfrak{F}_g and F_g are actually functions of (L, ℓ) , which, in turn, is an $(n+m)$ -tuple of functions of g satisfying the above assumptions. For simplicity, we omit the dependence on (L, ℓ) and emphasise only the dependence on g .

In order to compute the asymptotic behaviour of \mathfrak{F}_g , we apply the saddle-point method (see [27, Chapter VIII]).

Proposition 3.2. *Under the assumptions above, we have*

$$\mathfrak{F}_g \sim \frac{\rho^{-(6g-6+3n)}}{\sqrt{3\pi g}} \left(\prod_{i=1}^n \sinh(L_i \rho)\right) \left(\prod_{j=1}^m \sinh\left(\frac{\ell_j}{2\mu}\right)\right), \tag{38}$$

where $\rho := (6g - 6 + 3n)/|L| \sim 1/(2\mu)$.

Proof. The function $F_g(z)$ is holomorphic on the whole complex z -plane. By Cauchy’s integral formula, for any radius $r \in \mathbb{R}_{>0}$,

$$\mathfrak{F}_g = \frac{1}{2\pi i} \oint_{|z|=r} \frac{F_g(z)}{z^{6g-5+3n}} dz = \frac{r^{-(6g-6+3n)}}{\pi} \int_{-\pi/2}^{\pi/2} F_g(re^{i\theta}) e^{-(6g-6+3n)i\theta} d\theta.$$

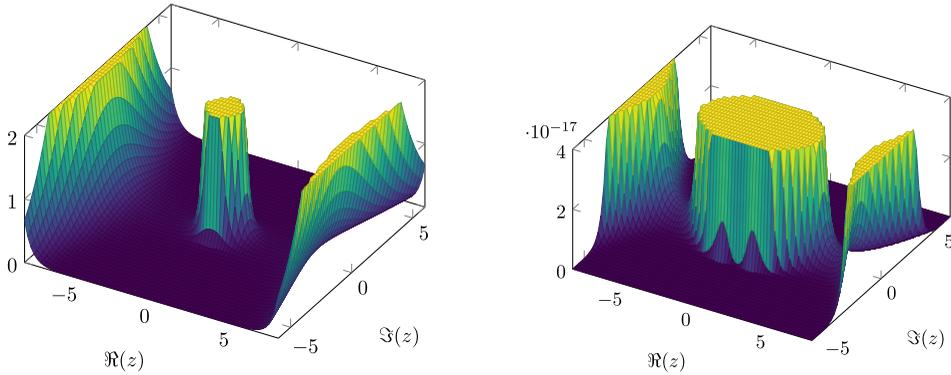


Figure 6. The graph of the absolute value of the function $F_g(z)z^{-(6g-5+3n)}$ for $g = 2$ (left) and $g = 10$ (right), with $n = m = 1$, $L_1 = g$, and $\ell_1 = 1$. Notice the two saddle-points at $z \sim \pm 6$, becoming more and more pronounced as g increases.

Here, we used the parity property of the integrand and reduce the integral over a semicircle. See Figure 6 for a 3D plot of the absolute value of the integrand. Following the terminology of [27, Chapter VIII], in order to apply the saddle-point method, we shall choose

- a radius so that the integration contour is an approximate saddle-point contour,
- a splitting of the approximate saddle-point contour as $C_{\text{centr}} + C_{\text{tail}}$, so that the following holds true.
 - TAIL PRUNING. The integral along C_{tail} is negligible.
 - CENTRAL APPROXIMATION. The integral along C_{centr} is well-approximated by an incomplete Gaussian integral.
 - TAIL COMPLETION. The incomplete Gaussian integral is asymptotically equivalent to the complete Gaussian integral.

We first start with choosing the radius r and the splitting. Let us write the integrand $F_g(re^{i\theta})e^{-(6g-6+3n)i\theta}$ as $e^{f(\theta)}$, where

$$f(\theta) := \sum_{i=1}^n \log \sinh(L_i r e^{i\theta}) + \sum_{j=1}^m \log \sinh(\ell_j r e^{i\theta}) - (6g - 6 + 3n)i\theta.$$

For simplicity, we omit the dependence on r and g (through the functions (L, ℓ)). Furthermore, in what follows, prime notation is used to denote derivatives with respect to θ . The saddle-point equation for the radius r , namely,

$$f'(0) = i \sum_{i=1}^n L_i r \coth(L_i r) + i \sum_{j=1}^m \ell_j r \coth(\ell_j r) - (6g - 6 + 3n)i = 0,$$

has a unique solution, which is asymptotically equivalent to $1/(2\mu)$ as $g \rightarrow \infty$. We use $\rho = \rho(g) := (6g - 6 + 3n)/|L| \sim 1/(2\mu)$ as an approximate saddle-point contour. Near the saddle-point, we have $f'(0) = O(1)$ as $g \rightarrow \infty$. Moreover, we have

$$\begin{aligned} f''(0) &= \sum_{i=1}^n (L_i^2 \rho^2 \operatorname{csch}^2(L_i \rho) - L_i \rho \coth(L_i \rho)) + \sum_{j=1}^m (\ell_j^2 \rho^2 \operatorname{csch}^2(\ell_j \rho) - \ell_j \rho \coth(\ell_j \rho)) \\ &= -6g + O(1), \end{aligned}$$

$$f'''(0) = i \sum_{i=1}^n (3L_i^2 \rho^2 \operatorname{csch}^2(L_i \rho) - 2L_i^3 \rho^3 \operatorname{csch}^2(L_i \rho) \operatorname{coth}(L_i \rho) - L_i \rho \operatorname{coth}(L_i \rho))$$

$$+ i \sum_{j=1}^m (3\ell_j^2 \rho^2 \operatorname{csch}^2(\ell_j \rho) - 2\ell_j^3 \rho^3 \operatorname{csch}^2(\ell_j \rho) \operatorname{coth}(\ell_j \rho) - \ell_j \rho \operatorname{coth}(\ell_j \rho)) = -i6g + O(1).$$

Now we set a cut-off θ_0 such that $f'(0)\theta_0 \rightarrow 0$, $f''(0)\theta_0^2 \rightarrow \infty$, $f'''(0)\theta_0^3 \rightarrow 0$ when $g \rightarrow \infty$. A possible choice is then $\theta_0 = g^{-2/5}$. We thus introduce

$$\mathfrak{F}_{g,\text{centr}} := \frac{\rho^{-(6g-6+3n)}}{\pi} \int_{\mathcal{C}_{\text{centr}}} e^{f(\theta)} d\theta, \quad \mathfrak{F}_{g,\text{tail}} := \frac{\rho^{-(6g-6+3n)}}{\pi} \int_{\mathcal{C}_{\text{tail}}} e^{f(\theta)} d\theta,$$

where $\mathcal{C}_{\text{centr}} := \{\rho e^{i\theta} \in \mathbb{C} : |\theta| \leq \theta_0\}$ and $\mathcal{C}_{\text{tail}} := \{\rho e^{i\theta} \in \mathbb{C} : \theta_0 \leq |\theta| \leq \pi/2\}$. We can now proceed with the three main checks of the saddle-point method: tail pruning, central approximation and tail completion.

TAIL PRUNING. Along the tail contour $\mathcal{C}_{\text{tail}}$, the integrand is negligible as $g \rightarrow \infty$. Indeed, observe that along the semicircle $\{\rho e^{i\theta} \in \mathbb{C} : |\theta| \leq \pi/2\}$, the integrand is strongly peaked at $\theta = 0$. Thus, along the tail $\mathcal{C}_{\text{tail}}$, we have $e^{f(\theta)-f(0)} = O(e^{|L|\rho(\cos(g^{-2/5})-1)}) = O(e^{Cg^{-1/5}})$ for some $C > 0$.

CENTRAL APPROXIMATION. Thanks to the choice $\theta_0 = g^{-2/5}$, we have a quadratic approximation for f along the central contour $\mathcal{C}_{\text{centr}}$ – that is, $e^{f(\theta)} = e^{f(0)-3\theta^2 g(1 + O(g^{-1/5}))}$. Therefore,

$$\begin{aligned} \mathfrak{F}_{g,\text{centr}} &= \frac{\rho^{-(6g-6+3n)}}{\pi} e^{f(0)} \int_{-\theta_0}^{\theta_0} e^{-3\theta^2 g} d\theta (1 + O(g^{-1/5})) \\ &= \frac{\rho^{-(6g-6+3n)}}{\pi\sqrt{3g}} e^{f(0)} \int_{-\sqrt{3}g^{1/10}}^{\sqrt{3}g^{1/10}} e^{-\varphi^2} d\varphi (1 + O(g^{-1/5})). \end{aligned}$$

TAIL COMPLETION. It follows from $\int_a^{+\infty} e^{-\varphi^2} d\varphi = O(e^{-a^2})$ that

$$\begin{aligned} \mathfrak{F}_{g,\text{centr}} &= \frac{\rho^{-(6g-6+3n)}}{\pi\sqrt{3g}} e^{f(0)} \int_{-\infty}^{+\infty} e^{-\varphi^2} d\varphi (1 + O(g^{-1/5})) \\ &= \frac{\rho^{-(6g-6+3n)}}{\sqrt{3\pi g}} \left(\prod_{i=1}^n \sinh(L_i \rho) \right) \left(\prod_{j=1}^m \sinh(\ell_j \rho) \right) (1 + O(g^{-1/5})) \\ &\sim \frac{\rho^{-(6g-6+3n)}}{\sqrt{3\pi g}} \left(\prod_{i=1}^n \sinh(L_i \rho) \right) \left(\prod_{j=1}^m \sinh\left(\frac{\ell_j}{2\mu}\right) \right). \end{aligned}$$

Together with the tail pruning, we have shown that $\mathfrak{F}_g \sim \mathfrak{F}_{g,\text{centr}}$. This completes the proof. □

A direct consequence of the above proposition is an estimate on the symplectic volumes.

Corollary 3.3 (Volume estimates). *For any sequence $L = L(g) = (L_1, \dots, L_n) \in \mathbb{R}_{>0}^n$ satisfying the boundary scaling assumption (32), we have as $g \rightarrow \infty$,*

$$V_{g,n}(L) \sim \frac{(6g - 5 + 2n)!!}{g! 24g} \frac{\rho^{-(6g-6+3n)}}{\sqrt{3\pi g}} \prod_{i=1}^n \frac{\sinh(L_i \rho)}{L_i}, \tag{39}$$

where $\rho := (6g - 6 + 3n)/|L| \sim 1/(2\mu)$. Moreover, for any fixed $\ell = (\ell_1, \dots, \ell_r) \in \mathbb{R}_{>0}^r$, we have as $g \rightarrow \infty$,

$$\frac{V_{g-r,n+2r}(L, \ell, \ell)}{V_{g,n}(L)} \sim 2^r \prod_{s=1}^r \frac{\cosh(\ell_s/\mu) - 1}{\ell_s^2}. \tag{40}$$

Proof. The first estimate is a direct combination of Equation (20) and Proposition 3.2. As for the second estimate, the same results imply

$$\frac{V_{g-r,n+2r}(L, \ell, \ell)}{V_{g,n}(L)} \sim \left(\frac{24g}{6g-5+2n}\right)^r \prod_{s=1}^r \left(\frac{\sinh(\ell_s/(2\mu))}{\ell_s}\right)^2 \sim 2^r \prod_{s=1}^r \frac{\cosh(\ell_s/\mu) - 1}{\ell_s^2}.$$

The first estimate follows from $\frac{g!}{(g-r)!} \sim g^r$, while the last estimate from $\sinh^2(\frac{x}{2}) = \frac{1}{2}(\cosh(x) - 1)$. \square

For a sanity check, consider the one-faced case (i.e., when $n = 1$). The 1-point ψ -class intersection numbers admit the closed-form expression $\langle \tau_{3g-2} \rangle_{g,1} = 1/(g! 24^g)$, and hence, we have

$$V_{g,1}(L) = \frac{1}{g! 24^g} \frac{L^{6g-4}}{2^{3g-2}(3g-2)!}. \tag{41}$$

By Stirling’s formula, the above expression is asymptotically equivalent to (39) with $n = 1$ and $\mu = 1$.

3.3. Simple, distinct, non-separating closed curves

We are now ready to prove Claim (A). Recall the notation $I = \prod_{i=1}^p [a_i, b_i]^{r_i}$ and $\lambda_\mu(\ell) = \frac{\cosh(\ell/\mu)-1}{\ell}$.

Proposition 3.4. Fix $L = (L_1, \dots, L_n) \in \mathbb{R}_{>0}^n$ satisfying the boundary scaling assumption (32). Then

$$\lim_{g \rightarrow \infty} \mathbb{E} \left[\bar{N}_{g,L,I}^\circ \right] = \prod_{i=1}^p \left(\int_{a_i}^{b_i} \lambda_\mu(\ell) d\ell \right)^{r_i}. \tag{42}$$

Proof. Notice that $\mathbb{E}[\bar{N}_{g,L,I}^\circ]$ is exactly in the form of the integration formula (Theorem 2.2), as all curves under consideration are simple and distinct. Besides, since all curves are non-separating, the associated stable graph Γ has $r := r_1 + \dots + r_p$ self-loops attached to a single vertex of genus $g - r$ and n leaves. In this case, $|\text{Aut}(\Gamma)| = 2^r$, corresponding to the swapping of half-edges composing the r loops. Moreover, in the notation of Theorem 2.2, $F = \mathbb{1}_I$ is the indicator function of the set I . Thus, we find

$$\mathbb{E} \left[\bar{N}_{g,L,I}^\circ \right] = \frac{2^{-r}}{V_{g,n}(L)} \int_{\mathbb{R}_{>0}^r} \mathbb{1}_I(\ell) \cdot V_{g-r,n+2r}(L, \ell, \ell) \prod_{s=1}^r \ell_s d\ell_s.$$

The result then follows from the second estimate of Corollary 3.3. \square

3.4. The negligible terms

The goal of this section is to prove Claims (B)–(D). The main difference compared to the previous section is that the integration formula loses its effectiveness when dealing with non-simple curves. To overcome this limitation, we consider subsurfaces where the curves are filling. Similar methods have been applied in, for example, [47, 63, 40, 3]. This strategy is implemented by the introduction of an auxiliary function $M_{A,B;\Gamma}$ depending on parameters A and B and a separating stable graph Γ . The main reason behind the introduction of such a function is that the functions $N_{g,L,I}^\times, N_{g,L,I}^\times$ and $\bar{N}_{g,L,I}^\circ - N_{g,L,I}^\circ$ can all be bounded by the sum over stable graphs involving the auxiliary $M_{A,B;\Gamma}$ (for some specific choices of A and B).

Recall the notation for stable graphs introduced in Subsection 2.2. Denote by $\mathbb{G}_{g,n}^{\text{sep}}$ the set of stable graphs of type (g, n) with at least two vertices (geometrically, they correspond to separating multicurves). Let $A \in \mathbb{R}_{\geq 1}, B \in \mathbb{R}_{>0}$ and $\Gamma \in \mathbb{G}_{g,n}^{\text{sep}}$. Define $M_{A,B;\Gamma} : \mathcal{T}_\Sigma^{\text{comb}}(L) \rightarrow \mathbb{R}$ by setting

$$M_{A,B;\Gamma}(\mathbb{G}) := \left\{ \left\{ (\gamma_i)_{i=1, \dots, |\mathbb{E}(\Gamma)|} \mid \begin{array}{l} \gamma \text{ multicurve in } \Sigma \text{ marked by } \Gamma \\ \text{with length } \ell_{\mathbb{G}}(\gamma) \leq B \end{array} \right\} \right\} \cdot 2^{|\mathbb{V}(\Gamma)|+|\mathbb{E}(\Gamma)|} \prod_{v \in \mathbb{V}_B(\Gamma)} A^{6g_v-6+3n_v}. \tag{43}$$

Here, $V_B(\Gamma)$ denotes the subset of vertices $v \in V(\Gamma)$ such that v is adjacent to no leaf with length larger than B . For ease of notation, we omit the dependence on the boundary lengths $L = (L_1, \dots, L_n)$. The function $M_{A,B;\Gamma}$ is mapping class group invariant, and it descends to a function on the combinatorial moduli space that will be denoted with the same symbol.

The main technical result, whose proof is postponed to Section 4, is an estimate for the sum over stable graphs of the expectation value of $M_{A,B;\Gamma}$.

Theorem 3.5 (Main estimate). *For any $A \geq 1, B > 0$, as $g \rightarrow \infty$,*

$$\sum_{\Gamma \in \mathbb{G}_{g,n}^{sep}} \mathbb{E}[M_{A,B;\Gamma}] = O(g^{-1/2}). \tag{44}$$

Let us start with Claim (B) – that is, an estimate on the combinatorial length spectrum of simple, distinct, separating cycles.

Proposition 3.6 (Estimate on $N_{g,I,L}^\times$). *The following bound holds true:*

$$N_{g,I,L}^\times \leq \sum_{\Gamma \in \mathbb{G}_{g,n}^{sep}} M_{1,br;\Gamma} \tag{45}$$

where $b = \max\{b_1, \dots, b_p\}$ and $r = r_1 + \dots + r_p$. Thus, $\mathbb{E}[N_{g,I,L}^\times] = O(g^{-1/2})$.

Proof. Clearly, $N_{g,I,L}^\times \leq \tilde{N}_{g,I,L}^\times$. However, any multicurve γ counted by $\tilde{N}_{g,I,L}^\times$ will have an associated stable graph with at least two vertices (since γ is separating) and total length bounded by $b_1 r_1 + \dots + b_p r_p \leq br$. Thus,

$$N_{g,I,L}^\times \leq \tilde{N}_{g,I,L}^\times \leq \sum_{\Gamma \in \mathbb{G}_{g,n}^{sep}} M_{1,br;\Gamma}.$$

The value $2^{|\mathbb{V}(\Gamma)|+|\mathbb{E}(\Gamma)|}$ appearing in Equation (43) is an overestimate in this case. The estimate on the expectation value then follows from Theorem 3.5. □

We now proceed with Claim (C) – that is, an estimate on the combinatorial length spectrum of non-simple cycles. Here, it will be crucial to consider cycles rather than curves.

Proposition 3.7 (Estimate on $N_{g,L,I}^\times$). *The following bound holds true:*

$$N_{g,L,I}^\times \leq r! \sum_{\Gamma \in \mathbb{G}_{g,n}^{sep}} M_{2^{r+1},2br;\Gamma}, \tag{46}$$

where $b = \max\{b_1, \dots, b_p\}$ and $r = r_1 + \dots + r_p$. Thus, $\mathbb{E}[N_{g,L,I}^\times] = O(g^{-1/2})$.

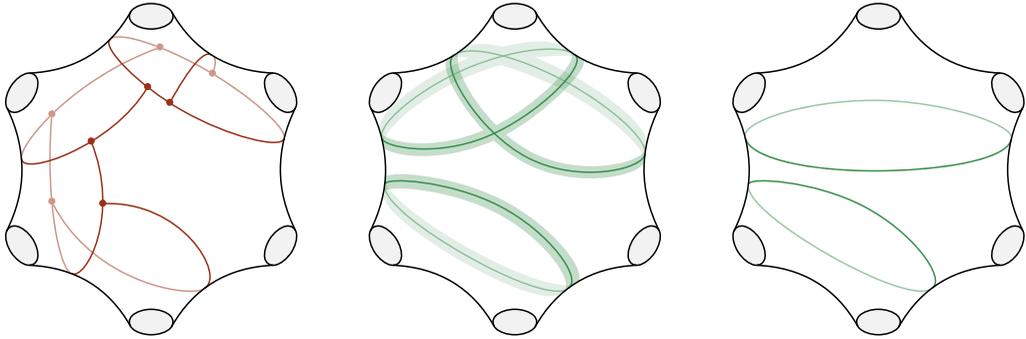
Proof. There are $r!$ ways to arrange r distinct cycles into an ordered r -tuple. Thus, we find

$$N_{g,L,I}^\times \leq r! \left\{ \left\{ \begin{array}{l} \text{unordered collection } C \text{ of distinct cycles} \\ \text{with at least one intersection} \\ \text{and length } \ell(C) \leq br \end{array} \right\} \right\}.$$

We claim that the right-hand side is bounded by

$$r! \sum_{\Gamma \in \mathbb{G}_{g,n}^{sep}} \left\{ \left\{ \begin{array}{l} \gamma \text{ unordered multicurve of type } \Gamma \\ \text{with length } \ell(\gamma) \leq B \end{array} \right\} \right\} \cdot 2^{|\mathbb{V}(\Gamma)|+|\mathbb{E}(\Gamma)|} \prod_{v \in V_B(\Gamma)} A^{6g_v-6+3n_v},$$

where $A = 4^r$ and $B = 2br$ are constants. The claim would complete the proof, since the above quantity is bounded by the same expression with ‘unordered’ replaced by ‘ordered’, which is nothing but (43).



(a) An embedded metric ribbon graph \mathbb{G} on a sphere with 6 holes.

(b) A collection of intersecting cycles C on \mathbb{G} and the associated tubular neighbourhood.

(c) The resulting separating multicurve γ . One can check that $\ell_{\mathbb{G}}(\gamma) \leq 2\ell_{\mathbb{G}}(C)$.

Figure 7. From a collection of intersecting cycles to a separating multicurve.

The strategy is to assign to any given unordered collection of distinct cycles a separating multicurve. We start from an empty multicurve γ . Let $C = C_1 + \dots + C_r$ be the unordered collection of distinct cycles under consideration, and let K be a connected components of $C_1 \cup \dots \cup C_r$, say $K = C_{i_1} \cup \dots \cup C_{i_k}$. Denote by Ξ_K its tubular neighbourhood. Notice that Ξ_K is a subsurface of Σ with boundary, and there are two mutually exclusive situations that can occur.

1. Ξ_K is a cylinder. In this case, $K = C_i$ for some i and C_i is simple. We then add C_i to γ .
2. Ξ_K has negative Euler characteristics. If a boundary component of Ξ_K is peripheral in Σ , then we just ignore it. The remaining boundary components form an unordered multicurve, say $\beta = \beta_1 + \dots + \beta_m$. Then we add β to γ . Note that by construction, the total length of boundary components of Ξ_K is bounded by $2(\ell(C_{i_1}) + \dots + \ell(C_{i_k})) \leq 2br$.

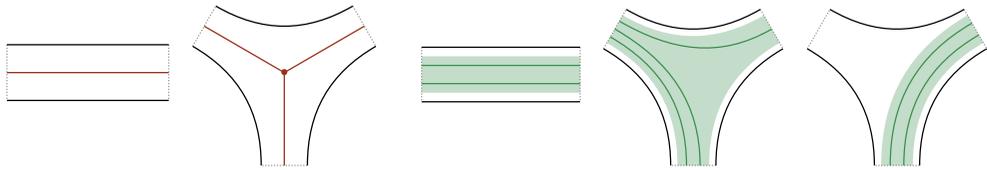
We repeat the same procedure for all connected components of $C_1 \cup \dots \cup C_r$. Note that the same curve may be added several times (at most 3) during the construction. Since we want a primitive multicurve, we will just keep one copy. By construction, all curves added to γ are disjoint, the resulting multicurve γ is separating, and we have $\ell(\gamma) \leq 2\ell(C) \leq 2br = B$. See Figure 7 for an example.

The drawback of this procedure is that different collections of cycles may yield the same multicurve. Given a (resulting) separating multicurve γ of type Γ , we can estimate the number of different collections of cycles that yield the same γ as follows.

First we bound the number of possible tubular neighbourhoods that can give rise to γ . Note that such a tubular neighbourhood is the union of cylinders and stable subsurfaces. By construction, cylinders correspond to a subset of $E(\Gamma)$, while stable subsurfaces correspond to a subset of $V_B(\Gamma)$. Hence, the number of tubular neighbourhoods that give rise to the same Γ is bounded by $2^{|E(\Gamma)|+|V_B(\Gamma)|} \leq 2^{|E(\Gamma)|+|V(\Gamma)|}$.

Next, we bound the number of collections of cycles C that can give rise to the same tubular neighbourhood, say Ξ . Note that a cylinder in Ξ always corresponds to a cycle in C , but a stable subsurface v in Ξ can be induced by different subsets of C . We claim that the number of such subsets is bounded by $A^{6g_v-6+3n_v}$ with $A = 2^{r+1}$. Indeed, the number of cycles in the subsurface v is bounded by $2^{6g_v-6+3n_v}$, as there are at most $6g_v - 6 + 3n_v$ edges in v and each edge can either be part of the cycle or not. Moreover, the size of the subset under consideration can vary from 2 to r . Hence, the number of subsets of C giving rise to v is bounded by $(2^{6g_v-6+3n_v})^2 + \dots + (2^{6g_v-6+3n_v})^r \leq 2^{(r+1)(6g_v-6+3n_v)}$. This gives the desired bound. □

Remark 3.8. The above proof, which is the only one where considering cycles rather than curves actually matters, generalises to the counting of D -cycles. A D -cycle is an edge-path that visits edges at most D times. In particular, cycles are the same as 1-cycles. In this case, the above algorithm still holds,



(a) The local decomposition of a surface into ribbons and discs, according to the embedded ribbon graph. (b) The local representation of the thick neighbourhood of a multicurve.

Figure 8. The decomposition of a surface into ribbons and discs, and the thick neighbourhood of a multicurve.

but we have to modify the argument for the estimate on the number of different collections of cycles that yield the same tubular neighbourhood. In this case, if a stable subsurface Ξ corresponds to a vertex $v \in V_B(\Gamma)$, then the number of collections of D -cycles in Ξ of size r is bounded by

$$\sum_{s=2}^r \left((0! + 1! + \dots + D!)^{6g_v - 6 + 3n_v} \right)^s \leq ((D + 1)!)^{(r+1)(6g_v - 6 + 3n_v)}. \tag{47}$$

Thus, Equation (46) still holds with $A = ((D + 1)!)^{r+1}$ and $B = 2br$.

We conclude with a proof of Claim (D) – that is, an estimate on the difference between the combinatorial length spectrum of simple curves and simple cycles.

Proposition 3.9 (Estimate on $\bar{N}_{g,I,L}^\circ - N_{g,I,L}^\circ$). *The following bound holds true:*

$$\bar{N}_{g,L,I}^\circ - N_{g,L,I}^\circ \leq \sum_{\Gamma \in \mathcal{G}_{g,n}^{ep}} M_{1,3br;\Gamma}, \tag{48}$$

where $b = \max\{b_1, \dots, b_p\}$ and $r = r_1 + \dots + r_p$. Thus, $\mathbb{E}[\bar{N}_{g,L,I}^\circ] - \mathbb{E}[N_{g,L,I}^\circ] = O(g^{-1/2})$.

Proof. Let $\gamma = (\gamma_1, \dots, \gamma_r)$ be an r -tuple of simple closed curves which is counted in $\bar{N}_{g,L,I}^\circ$ but not in $N_{g,L,I}^\circ$. The embedded ribbon graph gives rise to a decomposition of the surface into ribbons (assigned to edges) and discs (assigned to vertices); see Figure 8a. By taking the geodesic representative of γ – that is, the unique non-backtracking edge-path in the homotopy class – we obtain a collection on non-intersecting segments in each ribbon and a collection of non-intersecting switches in each disc; see Figure 8b. Define the thick neighbourhood of γ as the open subset of Σ obtained by taking in each ribbon (resp. disc) the connected open subset that contains all segments (resp. switches); see Figure 8b again. The boundary of the thick neighbourhood of γ is a multicurve β (possibly with peripheral components that we ignore), that we can make into an ordered multicurve in accordance with an arbitrary order of the (countable) set of closed curves in Σ .

We claim that β is nontrivial and separating. Indeed, if β were null-homotopic, then γ would be null-homotopic as well (it would be contained within the subsurface bounded by β). Moreover, if β were non-separating, then the thick neighbourhood of γ would be a collection of cylinders which contain γ , in contradiction with the fact that γ traverses at least one ribbon more than once.

Therefore, β is a nontrivial separating multicurve. We append to the end of γ the curves in β that are not already in γ . Thus, we obtain an ordered separating multicurve, denoted by $\tilde{\gamma}$. Although we have no control over the number of components of $\tilde{\gamma}$, we do know that $\ell(\tilde{\gamma}) \leq 3\ell(\gamma)$. Since the procedure is injective (we simply added curves at the end of the original tuple), we have $\bar{N}_{g,L,I}^\circ - N_{g,L,I}^\circ \leq \bar{N}_{g,L,\Delta \leq 3br}^\infty$, where $\bar{N}_{g,L,\Delta \leq C}^\infty$ counts separating multicurves whose total length is bounded by C . Note that $\bar{N}_{g,L,\Delta \leq C}^\infty$

is slightly differ from the counting $\tilde{N}_{g,L,I}^{\asymp}$, since in the former case, we do not fix the number curves. Nonetheless, the proof-strategy of Proposition 3.6 holds with no modifications, and hence the claimed bound. \square

4. Proof of Theorem 3.5

The goal of this section is to prove the main estimate on the sum over stable graphs of the expectation value of the auxiliary function $M_{A,B,\Gamma}$. We divide the proof in two parts: an estimate on the single expectation value $\mathbb{E}[M_{A,B;\Gamma}]$ and an estimate on its sum over all separating stable graphs.

4.1. Estimates on the auxiliary functions

To prepare for the estimate on the expectation value of the auxiliary function, we start by giving a basic estimate.

Lemma 4.1. *Let $\Gamma \in \mathbb{G}_{g,n}^{sep}$. Then*

$$\left(\frac{(6g - 5 + 2n)!!}{g! 24^g} \right)^{-1} \prod_{v \in V(\Gamma)} \frac{(6g_v + 5 + 2n_v)!!}{g_v! 24^{g_v}} \leq \frac{4^{2-|V(\Gamma)|+|E(\Gamma)|}}{(2g - 2 + n)!} \prod_{v \in V(\Gamma)} (2g_v - 2 + n_v)!. \tag{49}$$

Proof. For ease of notation, write $v = |V(\Gamma)|$ and $e = |E(\Gamma)|$. The relations

$$\sum_{v \in V(\Gamma)} g_v = g - 1 + v - e, \quad \sum_{v \in V(\Gamma)} n_v = n + 2e, \quad \prod_{i=1}^m \binom{A_i}{A_{i,1}, \dots, A_{i,r}} \leq \binom{\sum_i A_i}{\sum_i A_{i,1}, \dots, \sum_i A_{i,r}}$$

(see [1, Lemma 2.3] for the last inequality) imply

$$\begin{aligned} \prod_{v \in V(\Gamma)} \frac{(6g_v - 5 + 2n_v)!!}{g_v! 24^{g_v}} &= \prod_{v \in V(\Gamma)} \binom{6g_v - 5 + 2n_v}{3g_v - 3 + n_v, 2g_v - 2 + n_v, g_v} \frac{(2g_v - 2 + n_v)!}{2^{3g_v-3+n_v} 24^{g_v}} \\ &\leq \frac{1}{2^{3g-3+n-e} 24^{g-1+v-e}} \binom{6g - 6 + 2n + v - 2e}{3g - 3 + n - e, 2g - 2 + n, g - 1 + v - e} \prod_{v \in V(\Gamma)} (2g_v - 2 + n_v)!. \end{aligned}$$

The connectivity of Γ implies $v \leq e + 1$; hence,

$$\begin{aligned} \frac{(6g - 6 + 2n + v - 2e)!}{(6g - 5 + 2n)!} &\leq \frac{1}{(6g - 5 + 2n)^{1-v+2e}}, \\ \frac{(3g - 3 + n)!}{(3g - 3 + n - e)!} &\leq (3g - 3 + n)^e, \quad \frac{g!}{(g - 1 + v - e)!} \leq 2(g - 1)^{1-v+e}. \end{aligned}$$

We then conclude that

$$\begin{aligned} &2^e 24^{1-v+e} \frac{(6g - 6 + 2n + v - 2e)!}{(6g - 5 + 2n)!} \frac{(3g - 3 + n)!}{(3g - 3 + n - e)!} \frac{g!}{(g - 1 + v - e)!} \\ &\leq 2 \cdot 4^{1-v+e} \frac{(6g - 6 + 2n)^e (6g - 6)^{1-v+e}}{(6g - 6 + 2n)^{1-v+2e}} \leq 2 \cdot 4^{1-v+e} \left(\frac{6g - 6}{6g - 6 + 2n} \right)^{1-v+e} \leq 4^{2-v+e}, \end{aligned}$$

which proves the lemma. \square

Proposition 4.2. *For any constants $A \in \mathbb{R}_{\geq 1}$ and $B \in \mathbb{R}_{> 0}$, there exists $C = C(A, B, n) > 0$ such that for all $g \gg 0$ and $\Gamma \in \mathbb{G}_{g,n}^{sep}$, the following estimate holds:*

$$\mathbb{E}[M_{A,B;\Gamma}] \leq \frac{\sqrt{g}}{|\text{Aut}(\Gamma)|} \frac{C^{|E(\Gamma)|}}{(2|E(\Gamma)|)!} \frac{\prod_{v \in V(\Gamma)} (2g_v - 2 + n_v)!}{(2g - 2 + n)!}. \tag{50}$$

Proof. For ease of notation, write $V = V(\Gamma)$, $V_B = V_B(\Gamma)$, $E = E(\Gamma)$, $\chi = 2g - 2 + n$, and $\chi_v = 2g_v - 2 + n_v$ for every $v \in V$. Applying the integration formula (16), we deduce that

$$\mathbb{E}[M_{A,B;\Gamma}] = \frac{1}{V_{g,n}(L)} \frac{2^{|V|+|E|}}{|\text{Aut}(\Gamma)|} \left(\prod_{v \in V_B} A^{3\chi_v} \right) \int_{\Delta_{\leq B}^E} V_\Gamma(L, \ell) \prod_{e \in E} \ell_e \, d\ell_e,$$

where $\Delta_{\leq B}^E = \{ \ell \in \mathbb{R}_{>0}^E : \sum_e \ell_e \leq B \}$. Applying Theorem 2.5 to bound the intersection numbers in the integrand, we obtain

$$\begin{aligned} \mathbb{E}[M_{A,B;\Gamma}] &\leq \frac{1}{V_{g,n}(L)} \frac{2^{|V|+|E|}}{|\text{Aut}(\Gamma)|} \left(\frac{3}{2} \right)^{n+2|E|-|V|} \left(\prod_{v \in V_B} A^{3\chi_v} \right) \\ &\times \int_{\Delta_{\leq B}^E} \prod_{v \in V} \frac{(6g_v - 5 + 2n_v)!!}{g_v! 24^{g_v}} [z^{3\chi}] \prod_{i=1}^n \frac{\sinh(L_i z)}{L_i} \prod_{e \in E} \left(\frac{\sinh(\ell_e z)}{\ell_e} \right)^2 \ell_e \, d\ell_e. \end{aligned}$$

We can bound part of the right-hand side with the first estimate from Corollary 3.3 and with Lemma 4.1: setting $\rho := (6g - 6 + 3n)/|L| \sim 1/(2\mu)$, we have

$$\frac{1}{V_{g,n}(L)} \frac{2^{|V|+|E|}}{|\text{Aut}(\Gamma)|} \left(\frac{3}{2} \right)^{n+2|E|-|V|} \prod_{v \in V} \frac{(6g_v - 5 + 2n_v)!!}{g_v! 24^{g_v}} \leq \frac{\sqrt{g}}{|\text{Aut}(\Gamma)|} C_1^{|E|} \frac{\prod_{v \in V} \chi_v!}{\chi!} \rho^{3\chi} \prod_{i=1}^n \frac{L_i}{\sinh(L_i \rho)}$$

for some $C_1 = C_1(n) > 0$ and $g \gg 0$. Here, we also used the inequality $|V| \leq |E| + 1$ to write the overall constant as a power of the number of edges. However, for all $A \in \mathbb{R}_{\geq 1}$,

$$\begin{aligned} &[z^{3\chi}] \prod_{v \in V_B} A^{3\chi_v} \prod_{i=1}^n \frac{\sinh(L_i z)}{L_i} \prod_{e \in E} \left(\frac{\sinh(\ell_e z)}{\ell_e} \right)^2 \\ &\leq [z^{3\chi}] \left(\prod_{v \in V_B} \prod_{\lambda \in \Lambda(v)} \frac{\sinh(L_v A z)}{L_v} \right) \left(\prod_{v \notin V_B} \prod_{\lambda \in \Lambda(v)} \frac{\sinh(L_v z)}{L_v} \right) \prod_{e \in E} \left(\frac{\sinh(\ell_e A z)}{\ell_e} \right)^2. \end{aligned}$$

Here, $\Lambda(v)$ denotes the set of leaves attached to a vertex v . For any given convergent power series $\sum_{m \geq 0} a_m z^m$ with $a_m \geq 0$, we have $a_k \leq \rho^{-k} \sum_{m \geq 0} a_m \rho^m$ for all choices of $\rho > 0$ (within the disc of convergence). Thus, we find that for all $\ell \in \Delta_{\leq B}^E$,

$$\begin{aligned} &[z^{3\chi}] \prod_{v \in V_B} A^{3\chi_v} \prod_{i=1}^n \frac{\sinh(L_i z)}{L_i} \prod_{e \in E} \left(\frac{\sinh(\ell_e z)}{\ell_e} \right)^2 \\ &\leq \rho^{-3\chi} \left(\prod_{v \in V_B} \prod_{\lambda \in \Lambda(v)} \frac{\sinh(L_v A \rho)}{L_v} \right) \left(\prod_{v \notin V_B} \prod_{\lambda \in \Lambda(v)} \frac{\sinh(L_v \rho)}{L_v} \right) \prod_{e \in E} \left(\frac{\sinh(\ell_e A \rho)}{\ell_e} \right)^2 \\ &\leq C_2^{|E|} \rho^{-3\chi} \prod_{i=1}^n \frac{\sinh(L_i \rho)}{L_i} \end{aligned}$$

for some $C_2 = C_2(A, B, n) > 0$ and $g \gg 0$. In the first inequality, we chose $\rho = (6g - 6 + 3n)/|L|$ as above, while in the second inequality, we used the fact that for all $v \in V_B$ and $\ell \in \Delta_{\leq B}^E$,

$$\prod_{v \in V_B} \prod_{\lambda \in \Lambda(v)} \frac{\sinh(L_v A \rho)}{L_v} \leq c_1 \prod_{v \in V_B} \prod_{\lambda \in \Lambda(v)} \frac{\sinh(L_v \rho)}{L_v}, \quad \prod_{e \in E} \left(\frac{\sinh(\ell_e A \rho)}{\ell_e} \right)^2 \leq c_2^{|E|},$$

for some $c_i = c_i(A, B, n) > 0$ and $g \gg 0$. To conclude, we apply the identity $\int_{\Delta_{\leq B}^E} \prod_{e \in E} \ell_e d\ell_e = \frac{B^{2|E|}}{(2|E|)!}$. □

4.2. Summing over all topologies

We are now ready to estimate the sum of $\mathbb{E}[M_{A,B;\Gamma}]$ when Γ runs over all separating stable graphs. To prepare for it, let us start with a basic estimate on a sum of reciprocals of multinomial coefficients.

Lemma 4.3. *Let $2 \leq k \leq n$ be integers. The following bound holds true:*

$$\sum_{\substack{(n_1, \dots, n_k) \in \mathbb{Z}_{\geq 1}^k \\ n_1 + \dots + n_k = n}} \frac{n_1! \cdots n_k!}{n!} \leq \frac{4}{n}. \tag{51}$$

Proof. Denote by $F(n, k)$ the left-hand side of Equation (51). Let us first prove the claimed bound holds for $k = 2$. We start by rewriting $F(n, 2)$ as

$$F(n, 2) = \sum_{\substack{p, q \geq 1 \\ p+q=n}} \frac{p!q!}{n!} = \frac{2}{n} + \sum_{\substack{p, q \geq 2 \\ p+q=n}} \frac{p!q!}{n!}.$$

In the last sum, there are $n - 3$ summands, each bounded by $\frac{2}{n(n-1)}$. Thus, the last sum is bounded by $\frac{2(n-3)}{n(n-1)} \leq \frac{2}{n}$, proving the claimed bound. Let us now prove that $F(n, k) \leq F(n, 2)$. We have

$$F(n, k) = \frac{1}{n!} \sum_{r=2}^{n-(k-2)} \sum_{\substack{(n_1, \dots, n_{k-2}) \in \mathbb{Z}_{\geq 1}^{k-2} \\ n_1 + \dots + n_{k-2} = n-r}} \frac{n_1! \cdots n_{k-2}!}{n!} \sum_{\substack{p, q \geq 1 \\ p+q=r}} p!q!.$$

The first part of the proof implies that the innermost sum is bounded by $4(r-1)!$. Thus, after a relabelling of the index r as $n_{k-1} + 1$, we find

$$F(n, k) \leq \frac{4}{n!} \sum_{\substack{(n_1, \dots, n_{k-1}) \in \mathbb{Z}_{\geq 1}^{k-1} \\ n_1 + \dots + n_{k-1} = n-1}} n_1! \cdots n_{k-1}! = \frac{4}{n} F(n-1, k-1) \leq F(n-1, k-1),$$

where the last inequality holds for $n \geq 4$. By repeatedly applying the above inequality, we find $F(n, k) \leq F(n-k+2, 2) \leq \frac{4}{n}$, and hence the thesis. The cases with $n < 4$ can be checked independently. □

We are now ready to prove Theorem 3.5. For convenience, the proof makes use of labelled stable graphs – that is, a stable graph Γ together with bijections $V(\Gamma) \rightarrow \{1, \dots, |V(\Gamma)|\}$ and $H(\Gamma) \rightarrow \{1, \dots, |H(\Gamma)|\}$.

Proof of Theorem 3.5. Let Γ be a stable graph of type (g, n) with vertex set V , half-edges set H , and edge set E . Denote by $S(V)$ and $S(H)$ the symmetric group over V and H , respectively. Write $S(\Gamma) = S(V) \times S(H)$, and write π for the projection that maps a labelled stable graph to its underlying stable graph. The group $S(\Gamma)$ acts on $\pi^{-1}(\Gamma)$ by permuting the labels, and the stabiliser of any element in $\pi^{-1}(\Gamma)$ is isomorphic to $\text{Aut}(\Gamma)$. Thus, we have $|\pi^{-1}(\Gamma)| = |S(\Gamma)|/|\text{Aut}(\Gamma)|$, so for any function f defined on the set of labelled stable graphs that is constant along the fibres of π , we have

$$\sum_{\Gamma \in \mathbb{G}_{g,n}^{\text{sep}}} \frac{f(\Gamma)}{|\text{Aut}(\Gamma)|} = \sum_{\Gamma} \frac{f(\Gamma)}{|V|! |H|!}.$$

The sum on the right-hand side runs over the set of labelled separating stable graphs of type (g, n) . In light of Proposition 4.2, we consider the following choice of f :

$$f(\Gamma) := \sqrt{g} \frac{C^{|\mathbf{E}|}}{(2|\mathbf{E}|)!} \frac{\prod_{v \in V} (2g_v - 2 + n_v)!}{(2g - 2 + n)!}.$$

Note that $f(\Gamma)$ depends only on $\mathbf{e} = |\mathbf{E}|$, the genus decoration $(g_v)_{v \in V}$, and the valency decoration $(n_v)_{v \in V}$, so we can write $f(\mathbf{e}, (g_v), (n_v))$ for $f(\Gamma)$.

Given $m, k \in \mathbb{Z}_{\geq 1}$, we write

$$\begin{aligned} C_0(m, k) &:= \{ (m_1, \dots, m_k) \in \mathbb{Z}_{\geq 0}^k \mid m_1 + \dots + m_k = m \}, \\ C_1(m, k) &:= \{ (m_1, \dots, m_k) \in \mathbb{Z}_{\geq 1}^k \mid m_1 + \dots + m_k = m \}. \end{aligned}$$

We first claim that

$$\begin{aligned} \sum_{\Gamma} \frac{f(\Gamma)}{|\mathbf{V}|! |\mathbf{H}|!} &\leq \sum_{\substack{v \geq 2, \mathbf{e} \geq 1 \\ v \leq \mathbf{e} + 1}} \frac{1}{v! (2\mathbf{e})!} \sum_{\substack{s \geq 0, t \geq 1 \\ s+t=\mathbf{e}}} \sum_{\substack{g \in C_0(g-1-\mathbf{e}+v, v) \\ n \in C_0(n, v) \\ s \in C_0(s, v), t \in C_1(2t, v)}} \\ &\times \binom{n}{n_1, \dots, n_v} \binom{2\mathbf{e}}{2s_1+t_1, \dots, 2s_v+t_v} \left(\prod_{i=1}^v \binom{2s_i+t_i}{2s_i} (2s_i-1)!! \right) (2t-1)!! f(\mathbf{e}, g, n+2s+t). \end{aligned}$$

Indeed, any labelled stable graph of type (g, n) having $v \geq 2$ vertices and $\mathbf{e} \geq 1$ edges, out of which $s \geq 0$ are self-loops and $t \geq 1$ are not, can be constructed as follows. Fix v vertices and decorate them with their genus $g \in C_0(g-1-\mathbf{e}+v, v)$. Fix a splitting of the leaves – that is, $n \in C_0(n, v)$ – and distribute the labels. This can be achieved in $\binom{n}{n_1, \dots, n_v}$ different ways. Second, fix a splitting of the half-edges that are not leaves as self-loops and non-self-loops – that is, $s \in C_0(s, v)$ and $t \in C_1(2t, v)$. Attach $2s_i + t_i$ half-edges to the i -th vertex; there are $\binom{2\mathbf{e}}{2s_1+t_1, \dots, 2s_v+t_v}$ ways to label them. Third, among the $2s_i + t_i$ half-edges attached to the i -th vertex, choose $2s_i$ of them and pair them up to form s_i self-loops; there are $\binom{2s_i+t_i}{2s_i} (2s_i-1)!!$ ways to do so. Finally, we tie the remaining $2t$ half-edges two-by-two to form t edges connecting distinct vertices; there are at most $(2t-1)!!$ ways to do so. This is an overestimate, since we may create self-loops by connecting the last half-edges, or create a disconnected graph, or create a graph where the stability condition does not hold. Nonetheless, this proves the above claim.

Denote $\chi = 2g - 2 + n$. With our choice of f , we find

$$\begin{aligned} \sum_{\Gamma} \frac{f(\Gamma)}{|\mathbf{V}|! |\mathbf{H}|!} &\leq \sqrt{g} \sum_{\substack{v \geq 2, \mathbf{e} \geq 1 \\ v \leq \mathbf{e} + 1}} \frac{C^{\mathbf{e}}}{v! (2\mathbf{e})!} \sum_{\substack{s \geq 0, t \geq 1 \\ s+t=\mathbf{e}}} \frac{n! (2t-1)!!}{2^s \chi!} \sum_{\substack{g \in C_0(g-1-\mathbf{e}+v, v) \\ n \in C_0(n, v) \\ s \in C_0(s, v), t \in C_1(2t, v)}} \prod_{i=1}^v \frac{(2g_i - 2 + n_i + 2s_i + t_i)!}{n_i! s_i! t_i!} \\ &\leq \sqrt{g} \sum_{\substack{v \geq 2, \mathbf{e} \geq 1 \\ v \leq \mathbf{e} + 1}} \frac{C^{\mathbf{e}}}{v! (2\mathbf{e})!} \sum_{\substack{s \geq 0, t \geq 1 \\ s+t=\mathbf{e}}} \frac{n! (2t-1)!!}{2^s \chi!} \frac{v^{n+s+t}}{n! s! t!} \sum_{x \in C_1(\chi, v)} \prod_{i=1}^v x_i \\ &\leq 3\sqrt{g} \sum_{\substack{v \geq 2, \mathbf{e} \geq 1 \\ v \leq \mathbf{e} + 1}} \frac{v^n (2vC)^{\mathbf{e}}}{v! (2\mathbf{e})!} \frac{1}{\chi!} \sum_{x \in C_1(\chi, v)} \prod_{i=1}^v x_i. \end{aligned}$$

The second last inequality follows from the multinomial theorem, while the last inequality follows from $2^{-s} \frac{(2t-1)!!}{t!} \leq 2^{t-s} \leq 2^{\mathbf{e}}$ and $\sum_{s+t=\mathbf{e}} \frac{1}{s!} \leq \exp(1) \leq 3$. By Lemma 4.3, the above quantity is bounded by

$$\frac{12}{\chi} \sqrt{g} \sum_{\substack{v \geq 2, \mathbf{e} \geq 1 \\ v \leq \mathbf{e} + 1}} \frac{v^n (2vC)^{\mathbf{e}}}{v! (2\mathbf{e})!} \leq \frac{12}{\chi} \sqrt{g} \sum_{v=2}^{\infty} \frac{v^n}{v!} \exp(2vC) \leq \frac{12C'}{\chi} \sqrt{g} = O(g^{-1/2})$$

for some $C' = C'(A, B, n) > 0$. Now the result follows from Proposition 4.2. □

5. What is wrong with closed curves

Contrary to the hyperbolic case, in the metric ribbon graph setting, we consider the length spectrum of closed curves satisfying a specific condition: the cycle condition. The goal of this section is to explain why this is the case. In a nutshell, the reason is the absence of a collar lemma for metric ribbon graphs.

In [9, 10], Basmajian has shown that the length of any closed geodesic with self-intersection number k on any hyperbolic surface is bounded below by some universal constant M_k , and $M_k \rightarrow \infty$ as $k \rightarrow \infty$. In particular, if a closed geodesic intersects itself many times, then it cannot be very short. Basmajian’s proof relies on the (generalised) collar lemma, which says roughly that a short closed geodesic on a hyperbolic surface has a large tubular neighbourhood which is a topological cylinder.

In the context of metric ribbon graphs, the collar lemma fails dramatically. The distinction lies in the fact that metric ribbon graphs, unlike hyperbolic surfaces characterised by a constant nonzero sectional curvature, permit scaling. In particular, short curves of high topological complexity ($\omega(1)$ self-intersections or $\omega(1)$ intersections between two curves) can exist within a ribbon graph of low topological complexity, as we go deep into the thin part of the moduli space. This idea was used to establish the L^p -integrability of the combinatorial unit ball of measured foliations in [13], which exhibit different behaviour compared to its hyperbolic analogue [7]. In this section, we use the same idea to study the L^p -integrability of the functions $\bar{N}_{g,L,[a,b]}^\circ$ and $\bar{N}_{g,L,[a,b]}$.

Let us start with an estimate for the case of a one-holed torus.

Lemma 5.1. Fix $0 \leq a < b$.

- There exist $\epsilon^\circ, \delta^\circ, C^\circ > 0$ depending only on a and b such that, for any $L \leq \epsilon^\circ$ and $G \in \mathcal{M}_{1,1}^{comb}(L)$ trivalent with edge lengths in $[(1/3 - \delta^\circ)L, (1/3 + \delta^\circ)L]$, we have

$$\bar{N}_{1,L,[a,b]}^\circ(G) \geq \frac{C^\circ}{L^2}. \tag{52}$$

- There exist $\epsilon, \delta, C, c > 0$ depending only on a and b such that, for any $L \leq \epsilon$ and $G \in \mathcal{M}_{1,1}^{comb}(L)$ trivalent with edge lengths in $[(1/3 - \delta)L, (1/3 + \delta)L]$, we have

$$\bar{N}_{1,L,[a,b]}(G) \geq CL e^{c/L}. \tag{53}$$

Proof. Let us proceed with the first claim. Let $\epsilon^\circ > 0, 0 < \delta^\circ < 1/3$, and $G \in \mathcal{M}_{1,1}^{comb}(\epsilon^\circ)$ trivalent with edge lengths in $[(1/3 - \delta^\circ)L, (1/3 + \delta^\circ)L]$. The fundamental group of a torus with a single boundary component is a free group of rank 2. We can choose a generating set $\{\alpha, \beta\}$ such that α and β traverse exactly two edges. A cyclically reductive word in $\{\alpha, \beta\}$ of m letters has length (with respect to G) between $2(1/3 - \delta^\circ)\epsilon^\circ m$ and $2(1/3 + \delta^\circ)\epsilon^\circ m$. If ϵ° and m satisfy

$$\frac{at}{2(1/3 - \delta^\circ)} \leq \epsilon^\circ m < \frac{bt}{2(1/3 + \delta^\circ)} \quad \text{for some } t > 0,$$

which is possible whenever $\delta^\circ < (b - a)/(3a + 3b)$. Then every cyclically reductive word of m letters has length (with respect to G) in $[ta, tb]$, and we find

$$\frac{a}{1/3 - \delta^\circ} \frac{t}{\epsilon^\circ} \leq m < \frac{b}{1/3 + \delta^\circ} \frac{t}{\epsilon^\circ}.$$

It follows from [44, Theorem 2.7] that for ϵ° small enough,

$$\bar{N}_{1,\epsilon^\circ,[ta,tb]}^\circ(G) \geq \frac{1}{4\zeta(2)} \left(\frac{b}{1/3 + \delta^\circ} - \frac{a}{1/3 - \delta^\circ} \right)^2 \left(\frac{t}{\epsilon^\circ} \right)^2.$$

Therefore, for any $L \leq \epsilon^\circ$, we have

$$\bar{N}_{1,L,[a,b]}^\circ(G) = \bar{N}_{1,\epsilon^\circ, [\frac{\epsilon^\circ}{L}a, \frac{\epsilon^\circ}{L}b]}^\circ\left(\frac{\epsilon^\circ}{L} \cdot G\right) \geq \frac{1}{4\zeta(2)} \left(\frac{b}{1/3 + \delta^\circ} - \frac{a}{1/3 - \delta^\circ}\right)^2 \frac{1}{L^2},$$

as claimed.

As for the second inequality, the argument is similar. It can be shown that the number of conjugacy classes in a torus with a single boundary component representing primitive closed curves of m letters is asymptotically equivalent to $3^m/m$. Thus, for any small enough ϵ , we have

$$\bar{N}_{1,\epsilon,[ta,tb]}(G) \geq C(\epsilon/t) \exp(ct/\epsilon)$$

for some $C, c > 0$. □

Proof of Theorem D. The volume of subset of $\mathcal{M}_{1,1}^{\text{comb}}(\ell)$ consisting of those metric ribbon graphs where every edge has length in $[(1/3 - \delta^\circ)\ell, (1/3 + \delta^\circ)\ell]$ is $(2\delta^\circ)^3 V_{1,1}(\ell)$. Thus, it follows from the first claim of Lemma 5.1 that

$$\mathbb{E}\left[(\bar{N}_{g,L,[a,b]}^\circ)^k\right] \geq \frac{(C^\circ)^k (2\delta^\circ)^3}{V_{g,n}(L)} \int_0^{\epsilon^\circ} \ell^{-2k} \cdot V_{1,1}(\ell) \cdot V_{g-1,n+1}(L_1, \dots, L_n, \ell) \ell \, d\ell.$$

The right-hand side blows up if $k > 3/2$. Similarly, from the second claim of Lemma 5.1,

$$\mathbb{E}[\bar{N}_{g,L,[a,b]}] \geq \frac{C(2\delta)^3}{V_{g,n}(L)} \int_0^\epsilon \ell e^{C/\ell} \cdot V_{1,1}(\ell) \cdot V_{g-1,n+1}(L_1, \dots, L_n, \ell) \ell \, d\ell = \infty.$$

This completes the proof. □

6. Numerical evidence

Let us briefly outline the simulation for the unicellular case (i.e., when $n = 1$). We fix the unique boundary length as $L = 12g$. A random ribbon graph is almost surely trivalent, and in the unicellular case, each trivalent ribbon graph has an equal chance of being selected. Hence, $\mathbf{G}_{g,L}$ can be sampled in two steps: firstly, we sample a trivalent one-faced ribbon graph, and secondly, we endow it with a uniformly random metric. Once a random metric graph is generated, we can count cycles to get the bottom part of its length spectrum.

Generating random unicellular metric maps

The first step can be achieved as follows. Based on the work of Chapuy, Féray and Fusy [20], the generation of a random combinatorial unicellular trivalent map can be reduced to generating a random plane trivalent tree and a random permutation that satisfies certain conditions. The generation of random trees can be accomplished recursively using Rémy’s algorithm [55]. Starting with the tree having one vertex and two leaves, at each step, an edge is uniformly chosen, a vertex is added at its midpoint, and a leaf is attached to the new vertex. Once the tree accumulates $6g - 3$ edges, we randomly merge its leaves three-by-three, resulting in a random trivalent unicellular map of genus g . To conclude, each edge of the resulting graph is endowed with a length sampled according to a Dirichlet distribution of parameters (1^{6g-3}) . See Figure 9 for examples of random unicellular maps of genus 64.

Counting cycles

What remains now is to find all cycles of length falling within a fixed interval $[a, b]$. We performed this search using the FindCycle function in Mathematica and recording the lengths of the resulting cycles.

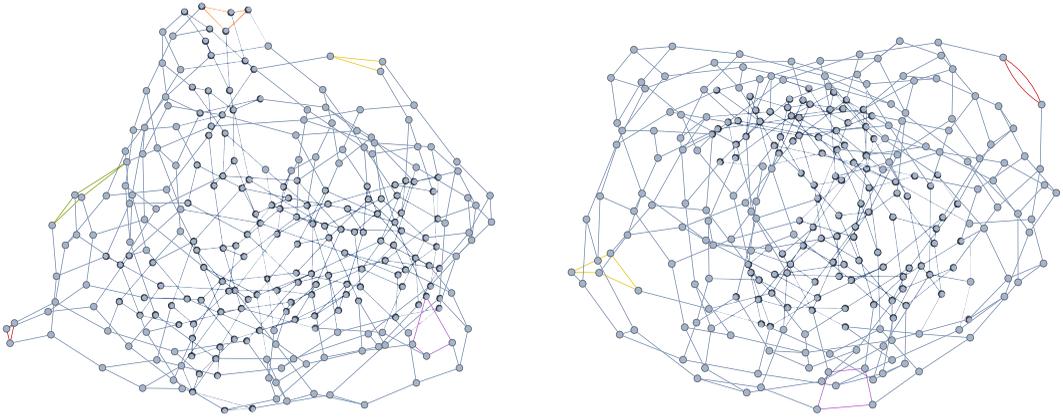


Figure 9. The graphs underlying two random unicellular maps of genus 64. The highlighted cycles include all cycles with at most 4 edges.

The simulation

For the simulations presented in Figure 2, we chose $[a, b] = [0, 4)$. For efficiency reasons, our search was limited to cycles with at most 12 edges. Since edge-lengths are, on average, equal to 1, only few cycles are missed in the search. Nonetheless, this accounts for why the histogram is slightly below the theoretical prediction.

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References

- [1] A. Aggarwal, ‘Large genus asymptotics for intersection numbers and principal strata volumes of quadratic differentials’, *Invent. Math.* **226**(3) (2021), 897–1010. doi: [10.1007/s00222-021-01059-9](https://doi.org/10.1007/s00222-021-01059-9). [arXiv:2004.05042\[math.AG\]](https://arxiv.org/abs/2004.05042).
- [2] N. Anantharaman, M. Ingremeau, M. Sabri and B. Winn, ‘Empirical spectral measures of quantum graphs in the Benjamini–Schramm limit’, *J. Funct. Anal.* **280**(12) (2021). Id/No 108988, 52. doi: [10.1016/j.jfa.2021.108988](https://doi.org/10.1016/j.jfa.2021.108988). [arXiv:2008.05709\[math.SP\]](https://arxiv.org/abs/2008.05709).
- [3] N. Anantharaman and L. Monk, ‘Friedman–Ramanujan functions in random hyperbolic geometry and application to spectral gaps’, Preprint, 2023, [arXiv:2304.02678\[math.SP\]](https://arxiv.org/abs/2304.02678).
- [4] N. Anantharaman and M. Sabri, ‘Quantum ergodicity on graphs: from spectral to spatial delocalization’, *Ann. Math.* **189**(3) (2019), 753–835. doi: [10.4007/annals.2019.189.3.3](https://doi.org/10.4007/annals.2019.189.3.3). [arXiv:1704.02766\[math.SP\]](https://arxiv.org/abs/1704.02766).
- [5] J. E. Andersen, G. Borot, S. Charbonnier, A. Giacchetto, D. Lewański and C. Wheeler, ‘On the Kontsevich geometry of the combinatorial Teichmüller space’, Preprint, 2020, [arXiv:2010.11806\[math.DG\]](https://arxiv.org/abs/2010.11806).
- [6] O. Angel, G. Chapuy, N. Curien and G. Ray, ‘The local limit of unicellular maps in high genus’, *Electron. Commun. Probab.* **18** (2013), 8. doi: [10.1214/ECP.v18-3037](https://doi.org/10.1214/ECP.v18-3037). [arXiv:1309.6254\[math.PR\]](https://arxiv.org/abs/1309.6254).
- [7] F. Arana-Herrera and J. S. Athreya, ‘Square-integrability of the Mirzakhani function and statistics of simple closed geodesics on hyperbolic surfaces’, *Forum Math. Sigma* **8** (2020), e9. doi: [10.1017/fms.2019.49](https://doi.org/10.1017/fms.2019.49). [arXiv:1907.06287\[math.DS\]](https://arxiv.org/abs/1907.06287).

- [8] F. Arana-Herrera and A. Calderon, ‘The shapes of complementary subsurfaces to simple closed hyperbolic multi-geodesics’, Preprint, 2022, [arXiv:2208.04339\[math.GT\]](https://arxiv.org/abs/2208.04339).
- [9] A. Basmajian, ‘The stable neighborhood theorem and lengths of closed geodesics’, *Proc. Am. Math. Soc.* **119**(1) (1993), 217–224. doi: [10.2307/2159845](https://doi.org/10.2307/2159845).
- [10] A. Basmajian, ‘Universal length bounds for non-simple closed geodesics on hyperbolic surfaces’, *J. Topol.* **6**(2) (2013), 513–524. doi: [10.1112/jtopol/jtt005](https://doi.org/10.1112/jtopol/jtt005).
- [11] I. Benjamini and O. Schramm, ‘Recurrence of distributional limits of finite planar graphs’, *Electron. J. Probab.* **6** (2001), Id/No 23, 13. doi: [10.1214/EJP.v6-96](https://doi.org/10.1214/EJP.v6-96). [arXiv:math/0011019](https://arxiv.org/abs/math/0011019).
- [12] B. Bollobás, *Random Graphs* (Cambridge, Cambridge University Press, 2001). doi: [10.1017/CBO9780511814068](https://doi.org/10.1017/CBO9780511814068).
- [13] G. Borot, S. Charbonnier, V. Delecroix, A. Giacchetto and C. Wheeler, ‘Around the combinatorial unit ball of measured foliations on bordered surfaces’, *Int. Math. Res. Not.* **2023**(17) (2022), 14464–14514. doi: [10.1093/imrn/rnac231](https://doi.org/10.1093/imrn/rnac231). [arXiv:2110.12538\[math.GT\]](https://arxiv.org/abs/2110.12538).
- [14] B. H. Bowditch and D. B. A. Epstein, ‘Natural triangulations associated to a surface’, *Topol.* **27**(1) (1988), 91–117. doi: [10.1016/0040-9383\(88\)90008-0](https://doi.org/10.1016/0040-9383(88)90008-0).
- [15] T. Budzinski, G. Chapuy and B. Louf, ‘Distances and isoperimetric inequalities in random triangulations of high genus’, *Annals of probability*. <https://imstat.org/journals-and-publications/annals-of-probability/annals-of-probability-future-papers/>, 2023, [arXiv:2311.04005\[math.PR\]](https://arxiv.org/abs/2311.04005).
- [16] T. Budzinski, N. Curien and B. Petri, ‘On Cheeger constants of hyperbolic surfaces’.
- [17] T. Budzinski, N. Curien and B. Petri, ‘On the minimal diameter of closed hyperbolic surfaces’, *Duke Math. J.* **170**(2) (2021), 365–377. doi: [10.1215/00127094-2020-0083](https://doi.org/10.1215/00127094-2020-0083). [arXiv:1909.12283\[math.GT\]](https://arxiv.org/abs/1909.12283).
- [18] T. Budzinski and B. Louf, ‘Local limits of uniform triangulations in high genus’, *Invent. Math.* **223**(1) (2021), 1–47. doi: [10.1007/s00222-020-00986-3](https://doi.org/10.1007/s00222-020-00986-3). [arXiv:1902.00492\[math.PR\]](https://arxiv.org/abs/1902.00492).
- [19] T. Budzinski and B. Louf, ‘Local limits of bipartite maps with prescribed face degrees in high genus’, *Ann. Probab.* **50**(3) (2022), 1059–1126. doi: [10.1214/21-AOP1554](https://doi.org/10.1214/21-AOP1554). [arXiv:2012.05813\[math.PR\]](https://arxiv.org/abs/2012.05813).
- [20] G. Chapuy, V. Féray and E. Fusy, ‘A simple model of trees for unicellular maps’, *J. Comb. Theory Ser. A.* **120**(8) (2013), 2064–2092. doi: [10.1016/j.jcta.2013.08.003](https://doi.org/10.1016/j.jcta.2013.08.003). [arXiv:1202.3252\[math.CO\]](https://arxiv.org/abs/1202.3252).
- [21] N. Curien, I. Kortchemski and C. Marzouk, ‘The mesoscopic geometry of sparse random maps’, *J. É. c. Polytech., Math.* **9** (2022), 1305–1345. doi: [10.5802/jep.207](https://doi.org/10.5802/jep.207). [arXiv:2112.10719\[math.PR\]](https://arxiv.org/abs/2112.10719).
- [22] V. Delecroix, E. Goujard, P. Zograf and A. Zorich, ‘Masur–Veech volumes, frequencies of simple closed geodesics and intersection numbers of moduli spaces of curves’, *Duke Math. J.* **170**(12) (2021), 2633–2718. doi: [10.1215/00127094-2021-0054](https://doi.org/10.1215/00127094-2021-0054). [arXiv:1908.08611\[math.GT\]](https://arxiv.org/abs/1908.08611).
- [23] V. Delecroix, E. Goujard, P. Zograf and A. Zorich, ‘Large genus asymptotic geometry of random square-tiled surfaces and of random multicurves’, *Invent. Math.* **230**(1) (2022), 123–224. doi: [10.1007/s00222-022-01123-y](https://doi.org/10.1007/s00222-022-01123-y). [arXiv:2007.04740\[math.GT\]](https://arxiv.org/abs/2007.04740).
- [24] V. Delecroix and M. Liu, ‘Length partition of random multicurves on large genus hyperbolic surfaces’, *J. Eur. Math. Soc.* (2024) <https://ems.press/journals/jems/articles/14297869>.
- [25] N. Do, ‘The asymptotic Weil–Petersson form and intersection theory on $\overline{\mathcal{M}}_g, n$ ’, [arXiv: https://arxiv.org/abs/1010.4126](https://arxiv.org/abs/1010.4126).
- [26] B. Eynard, E. Garcia-Falde, A. Giacchetto, P. Gregori and D. Lewański, ‘Resurgent large genus asymptotics of intersection numbers’, [arXiv: https://arxiv.org/abs/2309.03143](https://arxiv.org/abs/2309.03143).
- [27] P. Flajolet and R. Sedgewick, *Analytic Combinatoric* (Cambridge, Cambridge University Press, 2009). doi: [10.1017/CBO9780511801655](https://doi.org/10.1017/CBO9780511801655).
- [28] C. Gilmore, E. Le Masson, T. Sahlsten and J. Thomas, ‘Short geodesic loops and L_p norms of eigenfunctions on large genus random surfaces’, *Geom. Funct. Anal.* **31**(1) (2021), 62–110. doi: [10.1007/s00039-021-00556-6](https://doi.org/10.1007/s00039-021-00556-6). [arXiv:1912.09961\[math.SP\]](https://arxiv.org/abs/1912.09961).
- [29] J. Guo and D. Yang, ‘On the large genus asymptotics of psi-class intersection numbers’, *Math. Ann.* **388**(2022), 1–37. doi: [10.1007/s00208-022-02505-6](https://doi.org/10.1007/s00208-022-02505-6). [arXiv:2110.06774\[math.AG\]](https://arxiv.org/abs/2110.06774).
- [30] L. Guth, H. Parlier and R. Young, ‘Pants decompositions of random surfaces’, *Geom. Funct. Anal.* **21**(5) (2011), 1069–1090. doi: [10.1007/s00039-011-0131-x](https://doi.org/10.1007/s00039-011-0131-x). [arXiv:1011.0612\[math.GT\]](https://arxiv.org/abs/1011.0612).
- [31] J. L. Harer, ‘The virtual cohomological dimension of the mapping class group of an orientable surface’, *Invent. Math.* **84**(1) (1986), 157–176. doi: [10.1007/BF01388737](https://doi.org/10.1007/BF01388737).
- [32] J. L. Harer and D. Zagier, ‘The Euler characteristic of the moduli space of curves’, *Invent. Math.* **85**(3) (1986), 457–485. doi: [10.1007/BF01390325](https://doi.org/10.1007/BF01390325).
- [33] Y. He, S. Yang, Y. Wu and Y. Xue, ‘Non-simple systoles on random hyperbolic surfaces for large genus’, [arXiv: https://arxiv.org/abs/2308.16447](https://arxiv.org/abs/2308.16447).
- [34] W. Hide and M. Magee, ‘Near optimal spectral gaps for hyperbolic surfaces’, *Ann. Math.* **198**(2) (2023), 791–824. doi: [10.4007/annals.2023.198.2.6](https://doi.org/10.4007/annals.2023.198.2.6). [arXiv:2107.05292\[math.SP\]](https://arxiv.org/abs/2107.05292).
- [35] S. Janson and B. Louf, ‘Short cycles in high genus unicellular maps’, *Ann. Inst. Henri Poincaré, Probab. Stat.* **58**(3) (2022), 1547–1564. doi: [10.1214/21-AIHP1218](https://doi.org/10.1214/21-AIHP1218). [arXiv:2103.02549\[math.PR\]](https://arxiv.org/abs/2103.02549).
- [36] S. Janson and B. Louf, ‘Unicellular maps vs. hyperbolic surfaces in large genus: simple closed curves’, *Ann. Probab.* **51**(3) (2023), 899–929. doi: [10.1214/22-AOP1601](https://doi.org/10.1214/22-AOP1601). [arXiv:2111.11903\[math.PR\]](https://arxiv.org/abs/2111.11903).

- [37] M. Kontsevich, 'Intersection theory on the moduli space of curves and the matrix Airy function', *Comm. Math. Phys.* **147**(1) (1992), 1–23. doi: [10.1007/BF02099526](https://doi.org/10.1007/BF02099526).
- [38] P. Kuchment, 'Quantum graphs. I: Some basic structures', *Waves Random Media* **14**(1) (2004), s107–s128. issn: 0959-7174. doi: [10.1088/0959-7174/14/1/014](https://doi.org/10.1088/0959-7174/14/1/014).
- [39] E. Le Masson and T. Sahlsten, 'Quantum ergodicity for Eisenstein series on hyperbolic surfaces of large genus', *Math. Ann.* **389**(2023), 1–54. doi: [10.1007/s00208-023-02671-1](https://doi.org/10.1007/s00208-023-02671-1).
- [40] M. Lipnowski and A. Wright, 'Towards optimal spectral gaps in large genus', *Ann. Probab.* **52**(2) (2024), 545–575. doi: [10.1214/23-AOP1657](https://doi.org/10.1214/23-AOP1657). arXiv:2103.07496[math.GT].
- [41] B. Louf, 'Large expanders in high genus unicellular maps', *Comb. Theory* **2**(3) (2022). Id/No 7, 19. doi: [10.5070/C62359155](https://doi.org/10.5070/C62359155). arXiv:2102.11680[math.CO].
- [42] B. Louf, *Combinatorial Maps and Hyperbolic Surfaces in High Genus*. 2023. <https://library.cirm-math.fr/Record.htm?idlist=1&record=19391802124911190849>.
- [43] B. D. McKay, N. C. Wormald and B. Wysocka, 'Short cycles in random regular graphs', *Electron. J. Comb.* **11**(1) (2004). doi: [10.37236/1819](https://doi.org/10.37236/1819).
- [44] G. McShane and I. Rivin, 'A norm on homology of surfaces and counting simple geodesics', *Int. Math. Res. Not.* 1995 (1995), 61. doi: [10.1155/S1073792895000055](https://doi.org/10.1155/S1073792895000055). arXiv:math/0005222.
- [45] M. Mirzakhani, 'Simple geodesics and Weil–Petersson volumes of moduli spaces of bordered Riemann surfaces', *Invent. Math.* **167**(1) (2007). doi: [10.1007/s00222-006-0013-2](https://doi.org/10.1007/s00222-006-0013-2).
- [46] M. Mirzakhani, 'Growth of Weil–Petersson volumes and random hyperbolic surface of large genus', *J. Differ. Geom.* **94**(2) (2013), 267–300. doi: [10.4310/jdg/1367438650](https://doi.org/10.4310/jdg/1367438650).
- [47] M. Mirzakhani and B. Petri, 'Lengths of closed geodesics on random surfaces of large genus', *Comment. Math. Helv.* **94**(4) (2019), 869–889. doi: [10.4171/CMH/477](https://doi.org/10.4171/CMH/477). arXiv:1710.09727[math.GT].
- [48] M. Mirzakhani and P. Zograf, 'Towards large genus asymptotics of intersection numbers on moduli spaces of curves', *Geom. Funct. Anal.* **25**(4) (2015), 1258–1289. doi: [10.1007/s00039-015-0336-5](https://doi.org/10.1007/s00039-015-0336-5). arXiv:1112.1151[math.AG].
- [49] L. Monk, 'Benjamini–Schramm convergence and spectra of random hyperbolic surfaces of high genus', *Anal. PDE* **15**(3) (2022), 727–752. doi: [10.2140/apde.2022.15.727](https://doi.org/10.2140/apde.2022.15.727). arXiv:2002.00869[math.PR].
- [50] F. Naud, 'Determinants of Laplacians on random hyperbolic surfaces', *J. Anal. Math.* **151**(1) (2023), 265–291.
- [51] A. Okounkov and R. Pandharipande, 'Gromov–Witten theory, Hurwitz theory, and completed cycles', *Ann. Math.* **163** (2006), 517–560. doi: [10.4007/annals.2006.163.517](https://doi.org/10.4007/annals.2006.163.517). arXiv:math/0204305[math.AG].
- [52] R. C. Penner, 'The decorated Teichmüller space of punctured surfaces', *Commun. Math. Phys.* **113** (1987), 299–339. doi: [10.1007/BF01223515](https://doi.org/10.1007/BF01223515).
- [53] B. Petri, 'Random regular graphs and the systole of a random surface', *J. Topol.* **10**(1) (2017), 211–267. doi: [10.1112/topo.12005](https://doi.org/10.1112/topo.12005). arXiv:1311.5140[math.DG].
- [54] G. Ray, 'Large unicellular maps in high genus', *Ann. Inst. Henri Poincaré, Probab. Stat.* **51**(4) (2015), 1432–1456. doi: [10.1214/14-AIHP618](https://doi.org/10.1214/14-AIHP618).
- [55] J.-L. Rémy, 'Un procédé itératif de dénombrement d'arbres binaires et son application à leur génération aléatoire', *RAIRO Inform. Th é or. Appl.* **19**(2) (1985), 179–195.
- [56] A. Roig-Sanchis, 'The length spectrum of random hyperbolic 3-manifolds', *Adv. Math.*, **466**, 110158.
- [57] Z. Rudnick, 'GOE statistics on the moduli space of surfaces of large genus', *Geom. Funct. Anal.* **33** (2023), 1581–1607. doi: [10.1007/s00039-023-00655-6](https://doi.org/10.1007/s00039-023-00655-6). arXiv:2202.06379[math.ST].
- [58] W. T. Tutte, 'A census of planar maps', *Can. J. Math.* **15** (1963), 249–271. doi: [10.4153/CJM-1963-029-x](https://doi.org/10.4153/CJM-1963-029-x).
- [59] E. Witten, 'Two-dimensional gravity and intersection theory on moduli space', *Surveys in Differential Geometry*. Conference on geometry and topology (Harvard University, Cambridge, MA, USA, Apr. 27, 1990–Apr. 29, 1990) vol. 1. (1991), 243–310. doi: [10.4310/SDG.1990.v1.n1.a5](https://doi.org/10.4310/SDG.1990.v1.n1.a5).
- [60] S. Wolpert, 'On the Weil–Petersson geometry of the moduli space of curves', *Am. J. Math.* **107**(4) (1985), 969–997. doi: [10.2307/2374363](https://doi.org/10.2307/2374363).
- [61] N. C. Wormald, 'Models of random regular graphs', *Surveys in Combinatorics*. 17th British combinatorial conference (University of Kent at Canterbury, UK, July 12, 1999–July 16, 1999) (Cambridge, Cambridge University Press, 1999), 239–298. doi: [10.1017/CBO9780511721335.010](https://doi.org/10.1017/CBO9780511721335.010).
- [62] Y. Wu and Y. Xue, 'Prime geodesic theorem and closed geodesics for large genus'. To appear in *J. Eur. Math. Soc.* arXiv:2209.10415[math.GT].
- [63] Y. Wu and Y. Xue, 'Random hyperbolic surfaces of large genus have first eigenvalues greater than $\frac{3}{16} - \epsilon$ ', *Geom. Funct. Anal.* **32**(2) (2022), 340–410. doi: [10.1007/s00039-022-00595-7](https://doi.org/10.1007/s00039-022-00595-7). arXiv:2102.05581[math.DG].