N-COMPACT SPACES AS LIMITS OF INVERSE SYSTEMS OF DISCRETE SPACES

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Abstract. Let N denote the discrete space of all natural numbers. A space X is N-compact if it is homeomorphic with some closed subspace of a product of copies of N. In this paper, N-compact spaces are characterized as homeomorphs of inverse limit space of inverse systems of copies of subsets of N. Also, it is shown that a space X is N-compact if and only if the space $\mathscr{C}(X)$ of all non-empty compact subsets of X with the finite topology is N-compact.

1. Introduction

All spaces are assumed to be Hausdorff. Given two spaces X and E, following [4], X is said to be E-compact if it is homeomorphic to some closed subspace of a product of copies of E. In [5, p. 303], compact spaces are characterized as homeomorphs of inverse limit spaces of inverse systems of polyhedra and in [6, theorem 2] realcompact spaces are shown to be just the homeomorphs of inverse limit spaces of inverse systems of separable metric spaces. Let N denote the discrete space of all natural numbers, the main result of this paper is a similar characterization of N-compact spaces, namely, N-compact spaces are just homeomorphs of inverse limit spaces of inverse systems of copies of subsets of N. Some other characterizations of N-compact spaces can be found in [1] and [4].

Let $\mathscr{C}(X)$ denote the space of all non-empty compact subsets of a space X with the finite topology [3]. Then X is compact if and only if $\mathscr{C}(X)$ is compact [3, 4.9.12] and X is realcompact if and only if $\mathscr{C}(X)$ is realcompact [7]. An analogous result for N-compact spaces is obtained as a consequence of the main result, namely, X is N-compact if and only if $\mathscr{C}(X)$ is N-compact.

2. Inverse Limit Characterization

Let α be a collection of coverings of a space X, following [2], a filter F of subsets of X is α-Cauchy if for every C in α there exists a b in F and a c in C

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with $b \subset c$. The collection α will be called complete if $\cap \bar{F} \neq \varnothing$ for every α-Cauchy filter F, where $\bar{F}$ denotes the family of all $\bar{b}$ with b in F and $\bar{b}$ is the closure of the set b in X.

Given a 0-dimensional space X, (i.e. X has a base for the topology consisting of clopen subsets), let α denote the collection of all countable clopen coverings of X and β denote the collection of all countable disjoint clopen coverings of X. Then it is easy to see that a filter of subsets of X is α-Cauchy if and only if it is β-Cauchy, hence it follows from [1, theorem C] that

LEMMA 1. *A 0-dimensional space X is N-compact if and only if the collection β of all countable disjoint clopen coverings of X is complete.*

Since an inverse limit space of an inverse system $\{X_a, f_b^a\}$ over a directed set D is closed in the product $\Pi \{X_a \mid a \in D\}$, we have

LEMMA 2. *The inverse limit space of an inverse system is E-compact if each coordinate space of the inverse system is E-compact.*

THEOREM 3. *A space X is N-compact if and only if it is homeomorphic to an inverse limit space of an inverse system of copies of subsets of N.*

PROOF. Suppose that X is N-compact and β is the collection of all countable disjoint clopen coverings of X. Then β is directed by $<$, where $B < B'$ if B' refines B. For each B in β, let $|B|$ denote the discrete space whose points are (non-empty) elements of B. If b is in B then $|b|$ will denote the corresponding point in $|B|$. For B, C in β and $B < C$, define the map f_B^C from $|C|$ onto $|B|$ by $f_B^C(|c|) = |b|$ if $b \supset c$. Then

$$Y = \{|B|, f_B^C : B, C \in \beta, B < C\}$$

is an inverse system of copies of subsets of N over the directed set β. Let Y_∞ be the inverse limit space of Y. We will show that X is homeomorphic with Y_∞.

For each x in X, let

$$h(x) = \{|y_B|\}_{B \in \beta}$$

where y_B is the element of B that contains the point x. Then h is a map from X into $\Pi \{|B| : B \in \beta\}$. Since X is N-compact, it is 0-dimensional. Thus h separates the points and closed sets of X, so h is a homeomorphism. Next, we show that $h[X] = Y_\infty$. It is clear that $h[X] \subset Y_\infty$. Let $\{|z_B|\}_{B \in \beta}$ be a point in Y_∞. Then $\{z_B\}_{B \in \beta}$ is a collection of clopen subsets of X with the finite intersection property. Let F be the clopen ultrafilter on X containing $\{z_B\}_{B \in \beta}$. Clearly, F is β-Cauchy and since X is N-compact, $\cap F = \cap \bar{F} \neq \varnothing$ by lemma 1. Therefore, there is a point x_0 in X with

$$h(x_0) = \{|z_B|\}_{B \in \beta},$$

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otherwise, for any x in X, $h(x) \neq \{|z_B|\}_{B \in \beta}$ so x does not belong to $\bigcap_{B \in \beta} z_B$, hence $\cap F = \emptyset$. This is a contradiction. Hence h is a homeomorphism from X onto Y_∞.

The converse follows from lemma 2.

For any space X, let $\mathscr{C}(X)$ denote the space of all non-empty compact subsets of X with the finite topology [3]. Then X is homeomorphic to a closed subset of $\mathscr{C}(X)$ and X is discrete (respectively 0-dimensional) iff $\mathscr{C}(X)$ is discrete (respectively 0-dimensional) [3, 4.13]. Let $f: X \to Y$ be continuous then the map $\tilde{f}: \mathscr{C}(X) \to \mathscr{C}(Y)$ defined by

$$\tilde{f}(H) = \{f(x): x \in H\}$$

is continuous and it is shown in [7, lemma B] that if $\{X_a, f^a_b\}$ is an inverse system then $\mathscr{C}$(inv. lim. $\{X_a, f^a_b\}$) is homeomorphic to inv. lim.$\{\mathscr{C}(X_a), \tilde{f}^a_b\}$. Therefore, theorem 3 together with the above facts yields

THEOREM 4. *X is N-compact iff $\mathscr{C}(X)$ is N-compact.*

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