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ON THE CLOSURE OF THE LINEAR SPAN OF A WEIGHTED SEQUENCE IN $L^p(0, \infty)$

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1. Let $\{\lambda_n\}$ be an increasing sequence of positive numbers. The question of the closure in $L^p(0, \infty)$ $(1 \leqq p \leqq \infty)$ of the linear span of the sequence $\Lambda = \{e^{-x}x^{\lambda_n}\}$ has been considered by several authors, notably by Boas (**1**) and Fuchs [**3**; **4**]. (We shall find it a convenient abuse in language to talk of the closure of Λ in $L^\infty(0, \infty)$ in the sense of the closure in $\mathscr{C}_0(0, \infty)$.) Fuchs [**4**] has shown that if $\{\lambda_n\}$ is a sequence of positive numbers such that $\lambda_{n+1} - \lambda_n \geqq c > 0$, then Λ is total in $L^2(0, \infty)$ if and only if

$$\int_1^\infty \frac{\psi(r)}{r^2}\, dr = \infty, \tag{1}$$

where ψ is defined as follows:

$$\log \psi(r) = \begin{cases} 2\lambda_1^{-1}, & \text{if } r \leqslant \lambda_1, \\ 2 \sum_{\lambda_n < r} \lambda_n^{-1}, & \text{if } r > \lambda_1. \end{cases} \tag{2}$$

He has further proved that condition (1) is also sufficient for the sequence Λ to be total in $L^p(0, \infty)$ $(1 \leqq p \leqq \infty)$.

In this paper, we show first that if the integral in (1) converges, Λ is not total but is topologically linearly independent in $L^p(0, \infty)$ $(1 \leqq p \leqq \infty)$.

It is known (cf. Nachbin [**6**]) that in a locally convex space E a subset $\{e_\nu\}_{\nu \in I}$ is topologically linearly independent if and only if there exists in the dual space E^* a subset $\{f_\nu\}_{\nu \in I}$ such that $\{e_\nu, f_\nu\}$ is a biorthogonal system in the sense that $f_\mu(e_\nu) = \delta_{\mu\nu}$, and then $\{f_\nu\}_{\nu \in I}$ is called an orthonormal system associated with $\{e_\nu\}_{\nu \in I}$. Moreover, $\{e_\nu\}_{\nu \in I}$ remaining topologically linearly independent, such an orthonormal system $\{f_\nu\}_{\nu \in I}$ is unique if and only if $\{e_\nu\}_{\nu \in I}$ is total. If $\{e_\nu\}_{\nu \in I}$ is topologically linearly independent and x belongs to the closed linear span of $\{e_\nu\}_{\nu \in I}$, then $x = \lim_j \sum c_\nu{}^j e_\nu$ implies that for all $\nu \in I$

$$\lim_j c_\nu{}^j = f_\nu(x) = c_\nu,$$

where $\{f_\nu\}_{\nu \in I}$ is an orthonormal system associated with $\{e_\nu\}_{\nu \in I}$. The c_ν's are uniquely determined independently of the choice of approximating finite

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linear combinations $\sum c_\nu{}^j e_\nu$. The formal expansion

$$\sum_{\nu \in I} f_\nu(x) e_\nu \tag{3}$$

of x corresponding to the biorthogonal system $\{e_\nu, f_\nu\}_{\nu \in I}$ does not, in general, characterize x in the sense that if the formal expansions of two elements x and y in the closed linear span of $\{e_\nu\}_{\nu \in I}$ coincide, then $x = y$.

Next, we construct explicitly an orthonormal system $\{f_k\}$ associated with the topologically linearly independent sequence Λ when

$$\int_1^\infty \frac{\psi(r)}{r^2}\, dr < \infty \tag{4}$$

and show that each function in the closed linear span of Λ in $L^p(0, \infty)$ is characterized by its formal expansion with respect to the orthonormal system $\{f_k\}$.

The results which we obtain here improve those established earlier by the author in [**8**].

2. We begin by proving the following theorem.

THEOREM 1. *If $\{\lambda_n\}$ is a sequence of positive numbers such that $\lambda_{n+1} - \lambda_n \geqq c > 0$ and*

$$\int_1^\infty \frac{\psi(r)}{r^2}\, dr < \infty, \tag{4}$$

where ψ is defined as in (2), *then the sequence $\Lambda = \{e^{-x} x^{\lambda_n}\}$ is not total and is topologically linearly independent in $L^p(0, \infty)$ $(1 \leqq p \leqq \infty)$.*

In order to prove this theorem, we need the following lemmas due to Fuchs [**4**] (cf. Boas [**2**], Mandelbrojt [**5**]). The constants appearing here and in the subsequent sections may be different at each appearance.

LEMMA 1. *The function G defined by*

$$G(z) = \prod_{n=1}^\infty \frac{\lambda_n - z}{\lambda_n + z} \exp(2z/\lambda_n) \qquad (z = x + iy),$$

is holomorphic and satisfies

$$|G(z)| \leqq \{A\psi(r)\}^x,$$

and

$$|G(z)| \geqq \{B\psi(r)\}^x,$$

outside circles of radius $c/3$ with centres at the λ_n.

LEMMA 2. *If* (4) *holds, there exists a function g holomorphic and without zeros in $x = \operatorname{Re} z > 0$ such that*

$$|g(z)| \leqq \{x/\psi(r)\}^x.$$

This function is defined by setting $g = \exp(-u + iv)$, where

$$u(x, y) = \frac{2x}{\pi} \int_{-\infty}^{\infty} \frac{\psi(t)}{x^2 + (t - y)^2} dt,$$

with $\psi(-t) = \psi(t)$ and v is the harmonic conjugate of u.

Proof of Theorem 1. Let g be the function as described above. Following Fuchs [**4**], we define a function J by

$$J(z) = (2 + z)^{-k} g(z + 1) H(z) A^{-z-1} \qquad (z = x + iy), \tag{5}$$

where $k = 2 + 2c^{-1}$, H is the function derived from G on replacing every λ_ν by $\lambda_\nu^* = \lambda_\nu + 1$ and z by $z + 1$, and A is a positive constant as in Lemma 1. The function J possesses the following properties in $x \geqq a > -1$:

(i) J is holomorphic and $J \neq 0$;

(ii) $J(\lambda_\nu) = 0$ for $\nu = 1, 2, \ldots$ and J does not have any other real zeros besides these;

(iii) J is such that

$$|J(z)| \leqq (x + 1)^{x+1} \{(x + 2)^2 + y^2\}^{-k/2}, \tag{6}$$

and

$$|J'(z)| \leqq B(x + 1)^{x+1} \{(x + 2)^2 + y^2\}^{-k/2} \psi(r). \tag{7}$$

All the assertions in (i) and (ii), except (7), follow from Lemma 2 if we observe that, in view of Lemma 1, H is holomorphic in $x \geqq -1$ and satisfies the inequality

$$|H(z)| \leqq \{A\psi(r)\}^{x+1} \qquad (x \geqq -1).$$

Taking the derivative of the logarithm of J, we get

$$\frac{J'(z)}{J(z)} = -\frac{k}{(2 + z)} + \frac{g'(z + 1)}{g(z + 1)} + \frac{H'(z)}{H(z)} - \log A. \tag{8}$$

Since g is holomorphic for $x > 0$, so is the function $\log g$. Hence

$$\frac{g'(z)}{g(z)} = -\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}.$$

Using the inequality $\psi(\lambda u) < C\lambda^{2/c}\psi(u)$ $(\lambda > 1)$ and (4), we get

$$\left|\frac{\partial u}{\partial x}\right|, \left|\frac{\partial u}{\partial y}\right| \leqslant \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\psi(t)}{x^2 + (t - y)^2} dt \leqslant C\psi(r).$$

Thus we have for $x > -1$,

$$|g'(z + 1)/g(z + 1)| \leqq C\psi(r). \tag{9}$$

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Taking the derivative of the logarithm of H, we have

$$\frac{H'(z)}{H(z)} = -2(z+1)^2 \sum_{n=1}^{\infty} \frac{1}{(\lambda_n+1)(\lambda_n - z)(\lambda_n + z + 2)}$$

so that for $x \geqq -1$

$$|H'(z)| \leqslant 2|z+1|^2 \sum_{n=1}^{\infty} \frac{|H_n(z)|}{(\lambda_n+1)|\lambda_n+z+2|^2} \exp\{2(z+1)/(\lambda_n+1)\},$$

where

$$H_n(z) = \prod_{k\neq n} \frac{\lambda_k - z}{\lambda_k + z + 2} \exp\{2(z+1)/(\lambda_k+1)\}.$$

It is easily seen that

$$|H_n(z)| \leqq \{C\psi(r)\}^{x+1} \qquad (x \geqq -1),$$

so that

$$|H'(z)| \leqslant 2|z+1|^2\{C\psi(r)\}^{x+1} \sum_{n=1}^{\infty} \frac{1}{(\lambda_n+1)|\lambda_n+z+2|^2}.$$

But for $x \geqq -1$, the series on the right is majorized by the series $\sum_{n=1}^{\infty} \lambda_n^{-2}$ which converges since $\lambda_{n+1} - \lambda_n \geqq c > 0$. Hence for $x \geqq a > -1$

(10) $$|H'(z)| \leqq \{C\psi(r)\}^{x+1}.$$

It follows from (8), (9) and (5) that for $x \geqq a > -1$

$$\begin{aligned}|J'(z)| &\leqq |J(z)|\{B\psi(r) + |H'(z)/H(z)| + C\} \\ &\leqq B\psi(r)|J(z)| + |z+2|^{-k}|H'(z)||g(z+1)|A^{-x-1}.\end{aligned}$$

Using (6) and (10) and the fact that

$$|g(z+1)| \leqslant \left\{\frac{x+1}{\psi(r)}\right\}^{x+1},$$

we have

$$\begin{aligned}|J'(z)| &\leqq B|z+2|^{-k}(x+1)^{x+1}\psi(r) \\ &\quad + |z+2|^{-k}\{C\psi(r)\}^{x+1}\{(x+1)/\psi(r)\}^{x+1}A^{-x-1} \\ &\leqq B(x+1)^{x+1}|z+2|^{-k}\psi(r),\end{aligned}$$

where A is suitably chosen, which establishes (7).

Let

(11) $$h(t) = t^{-1}\int_{-\infty}^{\infty} J(x+iy)t^{-x-iy}dy \qquad (x \geqslant a > -1).$$

It follows from (6) that the integral on the right exists and is independent of x and hence defines the function h unambiguously for all $t \in (0, \infty)$. The same inequality shows that the function $J_x : J_x(y) = J(x+iy)$ belongs to $L^p(-\infty, \infty)$ for all $1 \leqq p \leqq \infty$.

We now prove that for all $x \geqq a > -1$

(12) $$\int_0^\infty t^{qx+q-1}|h(t)|^q dt \leqslant A_q(x+1)^{qx+q/2} \quad (1 \leqslant q < \infty)$$

$$|t^{x+1}h(t)| \leqslant \pi(x+1)^{x+1/2} \quad (q = \infty),$$

where $p^{-1} + q^{-1} = 1$. If we denote by $\hat{J}_x$ the Fourier transform of J_x, then (12) can be written as

(12′) $$||\hat{J}_x||_q \leqq A_q(x+1)^{x+1/2}, \quad (1 \leqq q \leqq \infty).$$

We first consider the case $1 \leqq p \leqq 2$ $(2 \leqq q \leqq \infty)$. Since $J_x \in L^p(-\infty, \infty)$ for $1 < p \leqq 2$, the function $\hat{J}_x \in L^q(-\infty, \infty)$ and by the Parseval-Riesz formula, we have

$$||\hat{J}_x||_q \leqq (2\pi)^{1/q}||J_x||_p \leqq A_q(x+1)^{x+1/2} \quad (2 \leqq q < \infty),$$

where A_q is some positive constant depending on q. Since $J_x \in L(-\infty, \infty)$,

(13′) $$||\hat{J}_x||_\infty \leqq \pi(x+1)^{x+1/2}.$$

We next consider the case $2 < p \leqq \infty$ $(1 \leqq q < 2)$. It follows from (7) that $J_x' \in L^2(-\infty, \infty)$ for all $x \geqq a > -1$, where $J_x'(y) = J'(x+iy)$ and that

(13″) $$||J_x'||_2 \leqslant B(x+1)^{x+1}\left[\int_{-\infty}^\infty \{(x+2)^2 + y^2\}^{-k}\psi^2(r)dy\right]^{1/2}$$
$$\leqslant C(x+1)^{x+1/2}.$$

Since $J_x \in L(-\infty, \infty)$ and (6) holds, on intergrating by parts, we get

$$\hat{J}_x(t) = \frac{1}{t}\int_{-\infty}^\infty e^{-ity}J_x'(y)dy = t^{-1}\hat{J}_x'(t)$$

and

$$||\hat{J}_x||_q \leqslant \left[\int_{|t|<1} |\hat{J}_x(t)|^q dt\right]^{1/q} + \left[\int_{|t|\geqq 1} |\hat{J}_x(t)|^q dt\right]^{1/q}.$$

Applying Hölder's inequality, Plancherel's theorem and (13″), we get

$$I_2 = \left[\int_{|t|\geqq 1} \left|\frac{\hat{J}_x'(t)}{t}\right|^q dt\right]^{1/q}$$
$$\leqslant \left[\int_{|t|\geqq 1} |t|^{2/(q-2)}dt\right]^{(2-q)/2q} ||\hat{J}_x'||_2$$
$$= A_q||J_x'||_2 \leqslant A_q(x+1)^{x+1/2},$$

proving (12′) since a similar inequality holds for I_1, in view of (13′).

If $1 \leqq q < \infty$, putting $qx + q - 1 = n$, it follows from (12) that

(14) $$\int_0^\infty t^n|h(t)|^q dt \leqslant A_q\left(\frac{n+1}{q}\right)^{n+1/2}.$$

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If $1 < q < \infty$, then

$$\int_0^\infty e^{t/p}|h(t)|^q dt = \sum_{n=0}^\infty \frac{1}{p^n n!}\int_0^\infty t^n|h(t)|^q dt$$
$$= O\left(\sum_{n=0}^\infty \frac{1}{(pq)^n}\frac{(n+1)^{n+1/2}}{n!}\right)$$
$$= O\left(\sum_{n=0}^\infty (e/pq)^n\right) = O(1),$$

using Stirling's formula.

Consequently we have for $1 < q < \infty$

$$\int_0^\infty e^{qt}|h(pqt)|^q dt < \infty. \tag{15}$$

If $q = 1, \infty$, using (14), we similarly get

$$\int_0^\infty e^t|h(\alpha t)|dt < \infty \tag{16}$$

and

$$e^t h(\alpha t) \in L^\infty(0, \infty) \tag{17}$$

respectively, where $\alpha > e$. Let

$$f(t) = \begin{cases} e^t h(pqt) & \text{when } 1 < q < \infty, \\ e^t h(\alpha t) & \text{when } q = 1 \text{ or } \infty. \end{cases} \tag{18}$$

Since (12) holds, by Mellin's inversion formula, we get

$$J(z) = \frac{1}{2\pi}\int_0^\infty h(t)t^z dt \qquad (x \geqslant a > -1).$$

$J(\lambda_n) = 0$ and consequently, by (18), we have

$$\int_0^\infty e^{-t}t^{\lambda_n}f(t)dt = 0, f \in L^q(0, \infty) \qquad (1 < q \leqslant \infty)$$
$$\int_0^\infty e^{-t}t^{\lambda_n}dF(t) = 0, F \in V(0, \infty) \qquad (q = 1) \tag{19}$$

for $n = 1, 2, \ldots,$ where

$$F(t) = \int_0^t f(u)du, \qquad f \in L(0, \infty). \tag{20}$$

Since $J \neq 0$, the functions f and F are also not identically zero. Thus Λ is not total in $L^p(0, \infty)$ $(1 \leqq p \leqq \infty)$.

J does not have any real zeros besides $\{_{N^\nu}\}$. Hence the equations (19) and (20) are not satisfied by any λ outside the given sequence. It follows that if $x > 0$, $x \neq \lambda_n$ for $n = 1, 2, \ldots, e^{-t}t^x$ does not belong to the closed linear span of Λ in $L^p(0, \infty)$ $(1 \leqq p \leqq \infty)$. In particular, none of the elements $e^{-t}t^{\lambda_n}$ belongs to the closed linear span of the rest. Thus Λ is topologically linearly independent.

We note that when the sequence Λ is total in $L^p(0, \infty)$ $(1 \leqq p \leqq \infty)$, it remains total if we suppress any one of its elements. Hence, in this case, each element depends on the others.

Theorem 1 taken in conjunction with the theorems of Fuchs stated in the beginning of § 1 enables us to assert the following theorem.

THEOREM 2. *If* (1) *holds, then the sequence* $\Lambda = \{e^{-x}x^{\lambda_n}\}$ *is total and is topologically linearly dependent in each* $L^p(0, \infty)$ $(1 \leqq p \leqq \infty)$. *If* (4) *holds, the sequence* Λ *is not total but is topologically linearly independent in each* $L^p(0, \infty)$ $(1 \leqq p \leqq \infty)$.

3. We now proceed to construct in $L^p(0, \infty)$ $(1 \leqq p \leqq \infty)$ an orthonormal system associated with the sequence $\Lambda = \{e^{-x}x^{\lambda_n}\}$, assuming that (4) holds.

Let

$$J_\mu(z) = \frac{J(z)}{J'(\lambda_\mu)(z - \lambda_\mu)} \qquad (z = x + iy),$$

where J is defined by (5). It follows from Lemmas 1 and 2 that J_μ possesses the following properties in $x \geqq a > -1$:

(i) J_μ is holomorphic and $J_\mu \not\equiv 0$;

(ii) $J_\mu(\lambda_\nu) = \delta_{\mu\nu}$ for $\mu, \nu = 1, 2, \ldots$ and J_μ does not possess any other real zeros besides $\{\mu_\nu\}_{\nu \neq \mu}$;

(iii) $|J_\mu(z)| \leqq |J'(\lambda_\mu)|^{-1}(x + 1)^{x+1}[(x + 2)^2 + y^2]^{-(k+1)/2}$.

For $x \geqq a > -1$, if we set

$$th_\mu(t) = \int_{-\infty}^{\infty} J_\mu(x + iy)t^{-x-iy}dy$$

and repeat the reasoning used in the proof of Theorem 1, we first obtain the inequalities:

$$(21) \qquad \begin{aligned} &\int_0^\infty t^{qx+q-1}|h_\mu(t)|^q dt \leqslant A_q|J(\lambda_\mu)|^{-q}(x + 1)^{qx+q/2} && (1 \leqslant q < \infty) \\ &|t^{x+1}h_\mu(t)| \leqslant \pi|J(\lambda_\mu)|^{-1}(x + 1)^{x+1/2} && (q = \infty) \end{aligned}$$

valid for $x \geqq a > -1$ and these, in turn, lead to the following inequalities:

$$(22) \qquad \begin{aligned} &\int_0^\infty e^{qt}|h_\mu(pqt)|^q dt \leqslant A_q|J'(\lambda_\mu)|^{-q} < \infty && (1 < q < \infty) \\ &\int_0^\infty e^t|h_\mu(\alpha t)|dt \leqslant A_1|J'(\lambda_\mu)|^{-1} < \infty && (q = 1) \\ &|e^t h_\mu(\alpha t)| \leqslant A_\infty|J'(\lambda_\mu)|^{-1} < \infty && (q = \infty), \end{aligned}$$

where $\alpha > e$.

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Set

$$f_\mu(t) = \begin{cases} \dfrac{(pq)^{\lambda_\mu+1}}{2\pi} e^t h_\mu(pqt) & \text{when } 1 < q < \infty \\ \dfrac{\alpha^{\lambda_\mu+1}}{2\pi} e^t h_\mu(\alpha t) & \text{when } q = 1 \text{ or } \infty. \end{cases} \tag{23}$$

It follows from (22) that $f_\mu \in L^q(0, \infty)$ for $1 \leqq q \leqq \infty$ and that

$$\|f_\mu\|_q \leqq A_q \kappa^{\lambda_\mu+1} |J'(\lambda_\mu)|^{-1} \qquad (1 \leqq q \leqq \infty), \tag{24}$$

where $\kappa = pq$ if $1 < q < \infty$ and $\kappa = \alpha$ if $q = 1, \infty$.

For $f_\mu \in L(0, \infty)$, define

$$F_\mu(t) = \int_0^t f_\mu(x)dx. \tag{25}$$

We assert that

$$\begin{aligned} &\int_0^\infty e^{-t} t^{\lambda_\nu} f_\mu(t)dt = \delta_{\mu\nu} \qquad (1 < q \leqslant \infty) \\ &\int_0^\infty e^{-t} t^{\lambda_\nu} dF_\mu(t) = \delta_{\mu\nu} \qquad (q = 1). \end{aligned} \tag{26}$$

In fact, since $J_x^\mu : J_x^\mu(y) = J_\mu(x + iy)$ belongs to $L^p(-\infty, \infty)$ for all $1 \leqq p \leqq \infty$ and (21) holds, by Mellin's inversion formula, we get

$$J_\mu(z) = \frac{1}{2\pi} \int_0^\infty h_\mu(t) t^z dt \qquad (x \geqslant a > -1).$$

Hence

$$J_\mu(\lambda_\nu) = \begin{cases} \dfrac{(pq)^{\lambda_\nu+1}}{2\pi} \displaystyle\int_0^\infty t^{\lambda_\nu} h_\mu(pqt)dt = \delta_{\mu\nu} & (1 < q < \infty) \\ \dfrac{\alpha^{\lambda_\nu+1}}{2\pi} \displaystyle\int_0^\infty t^{\lambda_\nu} h_\mu(\alpha t)dt = \delta_{\mu\nu} & (q = 1, \infty), \end{cases}$$

which proves (26) in view of (23) and (25).

4. Let $A^p(\Lambda)$ denote the closed linear span of $\Lambda = \{e^{-x}x^{\lambda_k}\}$ in $L^p(0, \infty)$ $(1 \leqq p \leqq \infty)$. If (4) holds, then Λ is topologically linearly independent and, therefore, every $G \in A^p(\Lambda)$ has a formal expansion $\sum f_k(G)e^{-x}x^{\lambda_k}$ corresponding to the associated orthonormal system $\{f_k\}$ as constructed in § 3. Using a technique developed by L. Schwartz in [**7**], we establish the following representation theorem which enables us to affirm the uniqueness of this expansion.

THEOREM 3. *Under the conditions of Theorem* 1 *each function G belonging to the closed linear span of the sequence* $\Lambda = \{e^{-x}x^{\lambda_n}\}$ *in* $L^p(0, \infty)$ $(1 \leqq p \leqq \infty)$ *possesses the following properties:*

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(1) *G is analytic in $(0, \infty)$ and G can be continued analytically to a function G whose principal branch is holomorphic in the entire z-plane $(z = x + iy)$ except perhaps for the negative real axis $(-\infty, 0]$.*

(2) *G can be expanded in a convergent series*

$$G(z) = e^{-z}\sum_{k=1}^{\infty} c_k z^{\lambda_k},$$

where the c_k's are determined by G and by the topologically linearly independent sequence Λ.

(3) *G satisfies the inequality*

$$|G(z)| \leqslant A_q e^{-x}\left(\sum_{k=1}^{\infty} \{B\psi(\lambda_k)\}^{-\lambda_k}|z|^{\lambda_k}\right) \cdot ||G||_p,$$

where $B > 0$ is an absolute constant depending on Λ.

In order to prove the theorem we need the following lemma.

LEMMA 3. *If* (4) *holds, the function J defined by* (5) *satisfies the inequality*

$$|J'(\lambda_\nu)| \geqq \{B\psi(\lambda_\nu)\}^{\lambda_\nu},$$

where B is a positive constant.

Proof. Since

$$\frac{J'(z)}{J(z)} = -\frac{k}{(2+z)} + \frac{g'(z+1)}{g(z+1)} + \frac{H'(z)}{H(z)} - \log A,$$

we have

$$|J'(\lambda_\nu)| = \left|\frac{J(\lambda_\nu)H'(\lambda_\nu)}{H(\lambda_\nu)}\right|$$
$$\geqslant \exp\left\{-\frac{2(\lambda_\nu+1)}{\pi}\int_{-\infty}^{\infty}\frac{\psi(t)}{(\lambda_\nu+1)^2+t^2}\,dt\right\}$$
$$\times (\lambda_\nu+2)^{-k-1}\cdot\prod_{n\neq\nu}\left|\frac{\lambda_n-\lambda_\nu}{\lambda_n+\lambda_\nu+2}\right| \exp\{2(\lambda_\nu+2)/(\lambda_n+1)\}.$$

In the above inequality, the first factor on the right is bounded below by $B^{-\lambda_\nu-1}$ and by Lemma 1, the second factor is bounded below by $\{C\psi(\lambda_\nu)\}^{\lambda_\nu}$. Hence the result follows.

Proof of Theorem 3. If $G \in A^p(\Lambda)$, there exists a sequence

$$\left\{\sum_{1}^{m_n} c_k^{(n)} e^{-x} x^{\lambda_k}\right\}$$

such that

$$G(x) = \lim_{n\to\infty}\sum_{k=1}^{m_n} c_k^{(n)} e^{-x} x^{\lambda_k} \tag{27}$$

in the norm of $L^p(0, \infty)$ $(1 \leqq p \leqq \infty)$. Since Λ is topologically linearly independent,

$$\lim_{n\to\infty} c_k^{(n)} = c_k$$

exists. If we construct the orthonormal system $\{f_k\}$ and $\{dF_k\}$ associated with Λ as described in § 3 above, we get

$$c_k = \int_0^\infty G(x) f_k(x) dx. \tag{28}$$

Hence, for $1 \leqq p \leqq \infty$,

$$|c_k| < ||G||_p ||f_k||_q \leqslant \frac{A_q \kappa^{\lambda_k+1}}{|J'(\lambda_k)|} ||G||_p, \tag{29}$$

where $p^{-1} + q^{-1} = 1$.

Consider the series $\sum_{k=1}^\infty c_k z^{\lambda_k}$. Using (29) and Lemma 3, we get

$$\sum_{k=1}^\infty |c_k||z|^{\lambda_k} \leqslant A_q \cdot ||G||_p \sum_{k=1}^\infty \frac{|z|^{\lambda_k}}{\{B\psi(\lambda_k)\}^{\lambda_k}}.$$

If $\sum \lambda_k^{-1} = \infty$, the series

$$\sum_{k=1}^\infty \frac{|z|^{\lambda_k}}{\{B\psi(\lambda_k)\}^{\lambda_k}}$$

converges for all z and it converges uniformly in each circle $\{z : |z| \leqq R\}$. In fact, since $\lambda_n \geqq cn$, given any z, there exists a positive integer N such that for all $k > N$

$$\sum_{N+1}^\infty \frac{|z|^{\lambda_k}}{\{B\psi(\lambda_k)\}^{\lambda_k}} \leqslant \sum_{N+1}^\infty (\tfrac{1}{2})^{\lambda_k} \leqslant \sum_{N+1}^\infty (\tfrac{1}{2})^{cn}$$

and from this the assertion follows.

If we put $G_1(z) = \sum_{k=1}^\infty c_k e^{-z} z^{\lambda_k}$, then G_1 is a function defined for all values of z and its principal branch is holomorphic in the entire z-plane except perhaps for the negative real axis $(-\infty, 0]$. Hence

$$|G_1(z)| \leqslant A_q e^{-x} \sum_{k=1}^\infty \{B\psi(\lambda_k)\}^{-\lambda_k} |z|^{\lambda_k} \cdot ||G||_p. \tag{30}$$

We now show that $G_1(x) = G(x)$ a.e. Since for $1 \leqq k \leqq m_n$

$$c_k - c_k^{(n)} = \int_0^\infty f_k(x)\{G(x) - c_k^{(n)} e^{-x} x^{\lambda_k}\} dx$$
$$= \int_0^\infty f_k(x)\left\{G(x) - \sum_{\nu=1}^{m_n} c_\nu^{(n)} e^{-x} x^{\lambda_\nu}\right\} dx,$$

and for $k > m_n$

$$c_k = \int_0^\infty f_k(x) G(x) dx = \int_0^\infty f_k(x)\left\{G(x) - \sum_{\nu=1}^{m_n} c_\nu^{(n)} e^{-x} x^{\lambda_\nu}\right\} dx,$$

we have for $x \geqq 0$

$$\left|G_1(x) - \sum_{k=1}^{m_n} c_k^{(n)} e^{-x} x^{\lambda_k}\right| \leqslant \sum_{k=1}^{m_n} |c_k - c_k^{(n)}| e^{-x} x^{\lambda_k} + \sum_{m_n+1}^{\infty} |c_k| e^{-x} x^{\lambda_k}$$
$$\leqslant A_q e^{-x} \left(\sum_{k=1}^{\infty} \{B\psi(\lambda_k)\}^{-\lambda_k} x^{\lambda_k} \right)$$
$$\times \left\{ \int_0^\infty \left| G(x) - \sum_{k=1}^{m_n} c_k^{(n)} e^{-x} x^{\lambda_k} \right|^p dx \right\}^{1/p}.$$

It follows that the sequence of polynomials

$$\left\{ \sum_{k=1}^{m_n} c_k^{(n)} e^{-x} x^{\lambda_k} \right\}$$

converges pointwise to G_1 and hence $G_1 = G$ a.e.

If $\sum \lambda_k^{-1} < \infty$, we can enlarge the sequence $\{\lambda_k\}$ into $\{\lambda_l'\}$ in such a way that the new sequence satisfies (4) and $\sum \lambda_l'^{-1} = \infty$. If $G \in A^p(\Lambda)$ is given by (27), then $G \in A^p(\Lambda')$, where $\Lambda' = \{e^{-x} x^{\lambda_l}\}$ and (28) is replaced by

$$c_k = \int_0^\infty G(x) f_{l_k}(x) dx,$$

$\{f_l\}$ being an orthonormal system associated with Λ' as described in § 3. A repetition of the preceding analysis from this point onwards enables us to establish the properties (1) to (3) of Theorem 3. We need only observe that the inequality

$$|c_k| \leqslant \frac{A_q}{\{B\psi(\lambda_k)\}^k} ||G||_p,$$

which holds when ψ is defined with respect to $\{\lambda_l'\}$, holds *a fortiori* when ψ is defined with respect to its subsequence $\{\lambda_k\}$.

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