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A NON-NEGATIVE REPRESENTATION OF THE LINEARIZATION COEFFICIENTS OF THE PRODUCT OF JACOBI POLYNOMIALS

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1. Introduction. The problem of linearizing products of orthogonal polynomials, in general, and of ultraspherical and Jacobi polynomials, in particular, has been studied by several authors in recent years [**1**, **2**, **9**, **10**, **13–16**]. Standard defining relation [**7**, **18**] for the Jacobi polynomials is given in terms of an ordinary hypergeometric function:

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; (1-x)/2)$$

with $\operatorname{Re}\alpha > -1$, $\operatorname{Re}\beta > -1$, $-1 \leqq x \leqq 1$. However, for linearization problems the polynomials $R_n^{(\alpha,\beta)}(x)$, normalized to unity at $x = 1$, are more convenient to use:

$$(1.1)\qquad R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1) = {}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; (1-x)/2).$$

Roughly speaking, the linearization problem consists of finding the coefficients $g(k, m, n; \alpha, \beta)$ in the expansion

$$(1.2)\qquad R_n^{(\alpha,\beta)}(x)R_m^{(\alpha,\beta)}(x) = \sum_{k=|n-m|}^{n+m} g(k, m, n; \alpha, \beta)R_k^{(\alpha,\beta)}(x).$$

Using orthogonality these coefficients can be expressed in terms of the integral of a product of three Jacobi polynomials. Thus

$$(1.3)\qquad g(k, m, n) \equiv g(k, m, n; \alpha, \beta) = \frac{2^{-(\alpha+\beta+1)}\Gamma(\alpha+\beta+1)(\alpha+1)_k(\alpha+\beta+1)_k(2k+\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)k!(\beta+1)_k\,h(k, m, n)}$$

where

$$(1.4)\qquad h(k, m, n) = \int_{-1}^{1} dx(1-x)^\alpha(1+x)^\beta R_n^{(\alpha,\beta)}(x)R_m^{(\alpha,\beta)}(x)R_k^{(\alpha,\beta)}(x).$$

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Orthogonality ensures that if any one of the three integers k, m, n vanishes the other two must be equal for a non-zero value of the integral. So we need only consider the case $\min(k, m, n) \geqq 1$. For the sake of definiteness let us assume $n \geqq m$. Then (1.2) implies $n - m \leqq k \leqq n + m$. Orthogonality of Jacobi polynomials implies further that $h(k, m, n)$ vanishes if any one of k, m, n exceeds the sum of other two. Let us set $s = n - m$, $k = s + j \geqq 1$, $s, j \geqq 0$. Then we need to consider the integral $h_j \equiv h(s + j, n - s, n)$ for fixed n, s with $s + 1 \leqq n$ and for all j, $0 \leqq j \leqq 2n - 2s$.

The integration in (1.4) is quite elementary if one is interested in any finite expression of the coefficients. In fact, by using (1.1) or its many equivalent forms [**7**, p. 180], h_j can be obtained in a number of forms, each involving a double series. One of the possible representations is the following

$$(1.5)\quad h_j = 2^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2+2n)} \times \frac{(\beta+1)_n(\alpha+\beta+1+n-s)_{n-s}(\alpha+\beta+1+s+j)_s}{(\alpha+1)_s(\alpha+1)_{n-s}} \times \frac{(s+j)!\,n!}{s!\,j!}C_j$$

where

$$(1.6)\quad C_j = \sum_{r=0}^{j}\frac{(-j)_r(j+2s+\alpha+\beta+1)_r}{r!\,(\alpha+\beta+2+2n)_r} \times {}_4F_3\left[\begin{matrix}-r, & r+2s+1, & s-n, & s-n-\beta \\ & s+1, & \alpha+s+1, & 2s-2n-\alpha-\beta\end{matrix}\right].$$

The interesting property of this and the other representations that we have found is that the ${}_4F_3(1)$ series above is balanced in each case, that is, the sum of the denominator parameters exceeds that of the numerator parameters by 1.

However, the interesting problem is not just any representation of h_j but the one in which the nonnegativity of the coefficients $g(s + j, n - s, n; \alpha, \beta)$ is apparent for some range of values of the parameters α, β. The importance of non-negativity of these coefficients has been discussed in [**8**, **11**]. Non-negative representations have been found in the ultraspherical case $\alpha = \beta \geqq -\frac{1}{2}$ in [**6**], [**12**] and [**13**]. Hylleraas' method of recurrence relations also enabled him to compute the coefficients h_j in another special case $\alpha = \beta + 1$. His recurrence relations were used by Gasper to establish the non-negativity of h_j first for $\alpha \geqq \beta > -1$, $\alpha + \beta + 1 \geqq 0$ in [**8**] and then for a slightly larger region in [**9**]. Noting that Gasper's proof was quite computational Koornwinder [**14**] came up with a positivity proof that is valid for all orthogonal polynomials in one or more variables that satisfy an addition formula. This last condition as applied to Jacobi polynomials means, however, that his proof is valid in the more restrictive region $\alpha \geqq \beta \geqq -\frac{1}{2}$.

In the present paper we report a complete non-negative representation of the coefficients $g(s + j, n - s, n; \alpha, \beta)$ that has eluded researchers for many years. Our findings are embodied in the following theorem.

THEOREM. *Let j, s, n be non-negative integers satisfying the inequalities $s + 1 \leqq n, 0 \leqq j \leqq 2n - 2s$. Then the coefficients $g_j \equiv g(s + j, n - s, n; \alpha, \beta)$ defined in* (1.3) *through* (1.4) *are non-negative if $\alpha \geqq \beta > -1$ and $\alpha + \beta + 1 \geqq 0$. They have distinct explicit representations for even and odd values of j:*

$$(1.7) \quad g_j = \frac{\alpha + \beta + 1 + 2s + 2j}{\alpha + \beta + 1}$$

$$\times \frac{(\alpha + 1)_{s+j}(\beta + 1)_n(\alpha + \beta + 1)_{2s+j}(\alpha + \beta + 1)_j n!}{(\alpha + 1)_s(\alpha + 1)_{n-s}(\beta + 1)_{s+j}(\alpha + \beta + 2)_{2n+j} s!\, j!}$$

$$\times \frac{(s - n)_{j/2}(\alpha + \beta + n + 1)_{j/2}}{\left(s - n - \frac{\alpha + \beta}{2}\right)_{j/2}(\alpha + s + 1)_{j/2}}$$

$$\times \frac{(s - n - \alpha)_{j/2}(\beta + n + 1)_{j/2}(\frac{1}{2})_{j/2}}{\left(\frac{1}{2} + s - n - \frac{\alpha + \beta}{2}\right)_{j/2}(s + 1)_{j/2}(\alpha + 1)_{j/2}}$$

$$\times {}_9F_8\left[\begin{matrix} \alpha, 1 + \frac{\alpha}{2}, \alpha + \frac{1}{2}, \frac{\alpha - \beta}{2}, \frac{\alpha - \beta + 1}{2}, \alpha + \beta + n + 1 + j/2, \\ \frac{\alpha}{2}, \frac{1}{2}, \frac{\alpha + \beta}{2} + 1, \frac{\alpha + \beta + 1}{2}, -\beta - n - j/2, \end{matrix}\right.$$

$$\left.\begin{matrix} s - n + j/2, & -s - j/2, & -j/2 \\ \alpha + n + 1 - s - j/2, & \alpha + s + 1 + j/2, & \alpha + 1 + j/2 \end{matrix}\right],$$

when j is even; and when j is odd

$$(1.8) \quad g_j = \frac{\alpha + \beta + 1 + 2s + 2j}{\alpha + \beta + 1}$$

$$\times \frac{(\alpha + 1)_{s+j}(\beta + 1)_n(\alpha + \beta + 1)_{2s+j}(\alpha + \beta + 1)_j n!}{(\alpha + 1)_s(\alpha + 1)_{n-s}(\beta + 1)_{s+j}(\alpha + \beta + 2)_{2n+j} s!\, j!}$$

$$\times \frac{(s - n)_{(j+1)/2}(\alpha + \beta + n + 1)_{(j+1)/2}}{\left(s - n - \frac{\alpha + \beta}{2}\right)_{(j+1)/2}(\alpha + s + 1)_{(j+1)/2}}$$

$$\times \frac{(s - n - \alpha)_{(j-1)/2}(\beta + n + 1)_{(j-1)/2}(3/2)_{(j-1)/2}}{\left(\frac{1}{2} + s - n - \frac{\alpha + \beta}{2}\right)_{(j-1)\ 2}(s + 1)_{(j-1)/2}(\alpha + 2)_{(j-1)/2}}$$

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$$\times \frac{\alpha-\beta}{\alpha+\beta+1}\, {}_9F_8\left[\begin{matrix}\alpha+1, \frac{\alpha+3}{2}, \alpha+\frac{1}{2}, \frac{\alpha-\beta}{2}+1, \frac{\alpha-\beta+1}{2}, \\ \frac{\alpha+1}{2}, \quad 3/2, \frac{\alpha+\beta}{2}+1, \frac{\alpha+\beta+3}{2},\end{matrix}\right.$$

$$\alpha+\beta+n+3/2+j/2, s-n+\frac{1}{2}+j/2,$$

$$\frac{1-j}{2}-\beta-n, \qquad \alpha+n+3/2-s-j/2,$$

$$\left.\begin{matrix}\frac{1}{2}-s-j/2, \quad \frac{1-j}{2} \\ \alpha+s+3/2+j/2, \alpha+3/2+j/2\end{matrix}\right].$$

The non-negativity of the g_j's is quite clear from these representations since only four of the 17 parameters in each of the above ${}_9F_8(1)$ series are negative over the whole range of the respective series while their coefficients are strictly non-negative under the stated conditions. The special property of these hypergeometric series is that they are both very well-poised and 2-balanced, meaning that the sum of the denominator parameters exceeds that of the numerator parameters by 2. This property is particularly pleasing not only because it is the only case where a ${}_9F_8(1)$ series has a known transformation formula [**4**, p. 27] but also because it reduces to a very well-poised 2-balanced ${}_7F_6(1)$ if any one of the numerator parameters equals any in the denominator in which event the series is exactly summable by Dougall's theorem [**4**, p. 26]. Thus, when $\alpha = \beta + 1$ both series in (1.7) and (1.8) become ${}_7F_6$'s and the result produces Hylleraas' formula. In the ultraspherical case, of course, the odd series in (1.8) vanishes because of the factor $(\alpha - \beta)$ in front, and the even series in (1.7) reduces to a ratio of gamma functions since one of numerator parameters in ${}_9F_8$ vanishes when $\alpha = \beta$.

We achieve these representations by exploiting a few summation and transformation theorems of generalized hypergeometric series of argument 1. Since many of these theorems are not probably too well-known we present a list of them in § 2.

It may be mentioned that the conditions $\alpha \geqq \beta > -1, \alpha + \beta + 1 \geqq 0$ are only sufficient for the non-negativity of the g_j's, but the second inequality is not necessary. In fact, Gasper devotes his second paper [**9**] to the case $\alpha \geqq \beta > -1$ and $-1 < \alpha + \beta + 1 < 0$ and obtains a necessary and sufficient criterion for the non-negativity. Our representations (1.7) and (1.8) do not demonstrate this property explicitly in this region but we believe that our formulas can be used to work out a proof of Gasper's results. But since the main thrust of the paper is an explicit nonnegative representation rather than an indirect proof of positivity which has been already provided by Gasper we refrain from addressing ourselves to this latter problem.

In § 3 we shall see that there is a more compact formula for the linearization coefficients whose nonnegativity is self-evident in the case $-\frac{1}{2} \leqq \beta \leqq \alpha$:

$$(1.9) \quad g_j = \frac{\alpha+\beta+1+2s+2j}{\alpha+\beta+1} \cdot \frac{n!}{s!\,j!}$$

$$\times \frac{(\beta+1)_n(\alpha+\beta+1)_{2n-2s}}{(\alpha+1)_{n-s}(\beta+1)_s(\alpha+\beta+1)_{n-s}}$$

$$\times \frac{(\alpha+\beta+1)_{2s+j}(2s-2n)_j(2\alpha+2\beta+2n+2)_j}{(\alpha+\beta+2)_{2n+j}(2s-2n-\alpha-\beta)}$$

$$\times \frac{(\alpha-\beta)_j}{(2\beta+2s+2)_j}$$

$$\times {}_9F_8\left[\begin{matrix}\beta+s+\frac{1}{2}, 1+\frac{\beta+s+\frac{1}{2}}{2}, \beta+\frac{1}{2}, \beta+n+1, s-n-\alpha, \frac{\alpha+\beta+1}{2}+s+j/2, \\ \frac{\beta+s+\frac{1}{2}}{2}, s+1, s-n+\frac{1}{2}, \alpha+\beta+n+3/2, \frac{\beta-\alpha}{2}+\frac{2-j}{2}, \end{matrix}\right.$$

$$\left.\begin{matrix}\frac{\alpha+\beta+2}{2}+s+j/2, \frac{1-j}{2}, -j/2 \\ \frac{\beta-\alpha}{2}+\frac{1-j}{2}, \beta+s+1+j/2, \beta+s+3/2+j/2\end{matrix}\right],$$

$$j = 0, 1, 2, \ldots, 2n-2s.$$

2. Some useful formulas. In this section we shall first list a few summation and transformation formulas for generalized hypergeometric series that are used heavily in this paper. The two main sources for these formulas are [**4**] and [**17**]. First, the summation theorem of Pfaff-Saalschutz (commonly known as Saalschutz's theorem) [**4**, p. 9]:

$$(2.1) \quad {}_3F_2\left[\begin{matrix}a, & b, & -n \\ c, & 1+a+b+c-n\end{matrix}\right] = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}, \; n = 0, 1, \ldots$$

Next is Dougall's summation theorem [**4**, p. 26] for a very well-poised ${}_7F_6(1)$ series:

(2.2) $$ {}_7F_6\left[\begin{matrix} a, 1+\frac{a}{2}, & b, & c, & d, & e, & -n \\ \frac{a}{2}, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a+n \end{matrix}\right] = \frac{(1+a)_n(1+a-b-c)_n(1+a-b-d)_n(1+a-c-d)_n}{(1+a-b)_n(1+a-c)_n(1+a-d)_n(1+a-b-c-d)_n}, $$

provided $n = 0, 1, \ldots$ and

(2.3) $$1 + 2a = b + c + d + e - n.$$

Condition (2.3) implies that the ${}_7F_6(1)$ series is 2-balanced. Special cases of (2.2) include the sum for a very well-poised ${}_5F_4(1)$ and that for a well-poised ${}_3F_2(1)$.

Among the transformation formulas that we shall make use of is the following for a terminating ${}_3F_2(1)$ series [**4**, p. 19–22; **10**]:

(2.3) $$ {}_3F_2\left[\begin{matrix} -n, a, b \\ c, d \end{matrix}\right] = \frac{(d-b)_n}{(d)_n}\,{}_3F_2\left[\begin{matrix} -n, c-a, b \\ c, 1+b-d-n \end{matrix}\right]. $$

This is a special case of Whipple's [**4**, p. 56] transformation formula for a balanced and terminating ${}_4F_3(1)$ series:

(2.4) $$ {}_4F_3\left[\begin{matrix} x, y, z, -n \\ u, v, w \end{matrix}\right] = \frac{(v-z)_n(w-z)_n}{(v)_n(w)_n}\,{}_4F_3\left[\begin{matrix} u-x, u-y, & z, & -n \\ u, & 1+z-v-n, & 1+z-w-n \end{matrix}\right], $$

provided

(2.5) $$u + v + w = x + y + z + 1 - n.$$

Whipple [**4**, p. 55] also provides a connection between such a ${}_4F_3(1)$

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and a very well-poised ${}_7F_6(1)$:

$$(2.6)\quad {}_4F_3\left[\begin{matrix} x, y, z, -n \\ u, v, w \end{matrix}\right] = \frac{(u-y)_n(u-z)_n}{(u)_n(u-y-z)_n}\,{}_7F_6\left[\begin{matrix} a, 1+\frac{a}{2}, w-x, v-x, & y, & z, & -n \\ \frac{a}{2}, \quad v, \quad w, & 1+z-u-n, & 1+y-u-n, & 1+y+z-u \end{matrix}\right],$$

where

$$(2.7)\quad a = y+z-u-n = w+v-x-1.$$

A transformation formula due to Bailey [**4**, p. 27] gives us a handle to transform one very well-poised terminating 2-balanced ${}_9F_8(1)$ series into another:

$$(2.8)\quad {}_9F_8\left[\begin{matrix} a, 1+\frac{a}{2}, & b, & c, & d, & e, & f, & g, & -n \\ \frac{a}{2}, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f, & 1+a-g, & 1+a+n \end{matrix}\right]$$
$$= \frac{(1+a)_n(1+k-e)_n(1+k-f)_n(1+k-g)_n}{(1+k)_n(1+a-e)_n(1+a-f)_n(1+a-g)_n}$$
$$\times\, {}_9F_8\left[\begin{matrix} k, 1+\frac{k}{2}, & k+b-a, & k+c-a, & k+d-a, & e, & f, & g, & -n \\ \frac{k}{2}, & 1+a-b, & 1+a-c, & 1+a-d, & 1+k-e, & 1+k-f, & 1+k-g, & 1+k+n \end{matrix}\right],$$

where

$$(2.9)\quad k = 1+2a-b-c-d$$

and

$$(2.10)\quad b+c+d+e+f+g-n = 2+3a.$$

Finally, the transformation formula that enables us to transform the ${}_9F_8(1)$ series in (1.9) to one in (1.7) or (1.8) is also given by Bailey

[**4**, p. 63]:

$$(2.11)\quad {}_9F_8\left[\begin{matrix} a, 1+\frac{a}{2}, & b, & c, & d, & e, & f, & g, & -n \\ \frac{a}{2}, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f, & 1+a-g, & 1+a-n \end{matrix}\right]$$

$$= \frac{(1+a)_n(1+a-b-c)_n(1+a-b-d)_n \times (1+a-b-e)_n(1+a-b-f)_n(g)_n}{(1+a-b)_n(1+a-c)_n(1+a-d)_n(1+a-e)_n \times (1+a-f)_n(g-b)_n}$$

$$\times {}_9F_8\left[\begin{matrix} b-g-n, 1+\frac{1}{2}(b-g-n), & b, & 1+a-c-g, 1+a-d-g, 1+a-e-g, & 1+a-f-g, b-a-n, & -n \\ \frac{1}{2}(b-g-n), & 1-g-n, & b+c-a-n, b+d-a-n, b+e-a-n, & b+f-a-n, 1+a-g, & 1+b-g \end{matrix}\right]$$

where the parameters are related by (2.10).

3. Proof of the theorem. For the time being let us assume that $n \leqq 2s$. This assumption is not crucial to our proof and we shall be able to drop it later by invoking analytic continuation. For any integer i between 0 and $n-s$ formula (2.1) enables us to write

$$\sum_{q=0}^{n-s-i} \frac{(s-n+i)_q(s-n-\alpha)_q(n+\alpha+\beta+1)_q}{q!(\beta+1)_q(2s-n+1+i)_q}$$

$$= \frac{(n+\alpha+\beta+1-s)_{n-s}(\alpha+s+1)_{n-s}}{(\beta+1)_{n-s}(2s+1-n)_{n-s}} \times \frac{(s-n-\beta)_i(2s+1-n)_i}{(\alpha+s+1)_i(2s-2n-\alpha-\beta)_i}.$$

Thus

$$(3.1)\quad \frac{(s-n)_i(s-n-\beta)_i}{(\alpha+s+1)_i(2s-2n-\alpha-\beta)_i}$$

$$= \frac{(\beta+1)_{n-s}(2s-n+1)_{n-s}}{(\alpha+s+1)_{n-s}(\alpha+\beta+1+n-s)_{n-s}} \times \sum_{q=0}^{n-s} \frac{(s-n)_q(s-n-\alpha)_q(n+\alpha+\beta+1)_q}{q!(\beta+1)_q(2s-n+1)_q} \times \frac{(s-n+q)_i}{(2s-n+1+q)_i}.$$

Using this in (1.6) we get

$$C_j = \frac{(\beta+1)_{n-s}(2s-n+1)_{n-s}}{(\alpha+s+1)_{n-s}(\alpha+\beta+1+n-s)_{n-s}}$$
$$\times \sum_{q=0}^{n-s} \frac{(s-n)_q(s-n-\alpha)_q(n+\alpha+\beta+1)_q}{q!(\beta+1)_q(2s-n+1)_q}$$
$$\times \sum_{r=0}^{j} \frac{(-j)_r(j+2s+\alpha+\beta+1)_r}{r!(\alpha+\beta+2+2n)_r}$$
$$\times {}_3F_2\left[\begin{matrix}-r, r+2s+1, & s-n+q\\ s+1, 2s-n+1+q\end{matrix}\right].$$

However, by (2.1)

$${}_3F_2\left[\begin{matrix}-r, r+2s+1, & s-n+q\\ s+1, 2s-n+1+q\end{matrix}\right]$$
$$= \frac{(-s-r)_r(n+1-q)_r}{(s+1)_r(n-2s-q-r)_r} = \frac{(n+1-q)_r}{(2s-n+1+q)_r}$$

and so the sum over r in (3.2) becomes

$${}_3F_2\left[\begin{matrix}-j, j+2s+\alpha+\beta+1, & n+1-q\\ \alpha+\beta+2+2n, 2s-n+1+q\end{matrix}\right]$$

which, by virtue of (2.3), transforms to

$$[(-1)^j(2s-2n)_j/(\alpha+\beta+2+2n)_j]$$
$$\times {}_3F_2\left[\begin{matrix}-j, j+2s+\alpha+\beta+1, & 2s-2n+2q\\ 2s-2n, 2s-n+1+q\end{matrix}\right].$$

Now,

$$\text{(3.3)} \quad \sum_{q=0}^{n-s} \frac{(s-n)_q(s-n-\alpha)_q(n+\alpha+\beta+1)_q}{q!(\beta+1)_q(2s-n+1)_q}$$
$$\times {}_3F_2\left[\begin{matrix}-j, j+2s+\alpha+\beta+1, & 2s-2n+2q\\ 2s-2n, 2s-n+1+q\end{matrix}\right]$$
$$= \sum_{k=0}^{j} \frac{(-j)_k(j+2s+\alpha+\beta+1)_k}{k!(2s-n+1)_k}$$
$$\times \sum_q \frac{(s-n)_q(s-n-\alpha)_q(n+\alpha+\beta+1)_q(2s-2n+k)_{2q}}{q!(\beta+1)_q(2s-n+1+k)_q(2s-2n)_{2q}}$$
$$= \sum_{k=0}^{j} \frac{(-j)_k(j+2s+\alpha+\beta+1)_k}{k!(2s-n+1)_k}$$
$$\times {}_4F_3\left[\begin{matrix}s-n+k/2, \ s-n+\frac{k+1}{2}, s-n-\alpha, n+\alpha+\beta+1\\ \beta+1, 2s-n+1+k, s-n+\frac{1}{2}\end{matrix}\right],$$

by using the duplication formula for gamma functions.

Since $0 \leqq j \leqq 2n - 2s$, the ${}_4F_3(1)$ series above is terminating. It is also balanced. Assuming, as an intermediate step, that $\alpha + n - s$ is a non-negative integer we obtain, on using (2.5),

$$(3.4)\quad {}_4F_3[\ \ ] = \frac{\Gamma(\alpha+\beta+1+2n-2s-k)\Gamma(2s-n+1+k) \times \Gamma(\beta+1)\Gamma(\alpha+n+1)}{\Gamma(\beta+1+n-s-k)\Gamma(\alpha+s+1+k) \times \Gamma(\alpha+\beta+1+n-s)\Gamma(s+1)}$$
$$\times {}_4F_3\left[\begin{matrix} -k/2, & \frac{1-k}{2}, \alpha+\beta+1+n, & s-n-\alpha \\ & s-n+\frac{1}{2}, & s+1, \beta+1+n-s-k \end{matrix}\right].$$

Simplifying the coefficients in (3.2) and (3.4) we then get

$$(3.5)\quad C_j = \frac{(-1)^j(2s-2n)_j}{(\alpha+\beta+2+2n)_j} \times \sum_{k=0}^{j} \frac{(-j)_k(j+2s+\alpha+\beta+1)_k(s-n-\beta)_k}{k!(\alpha+s+1)_k(2s-2n-\alpha-\beta)_k}$$
$$\times {}_4F_3\left[\begin{matrix} -k/2, & \frac{1-k}{2}, \alpha+\beta+1+n, & s-n-\alpha \\ & s-n+\frac{1}{2}, & s+1, \beta+1+n-s-k \end{matrix}\right].$$

Since the right hand side is a rational function of the parameters and the factors with $\Gamma(2s - n + 1)$ have cancelled out in the previous steps we may now drop the assumption $n \leqq 2s$ made earlier. Because of the rationality we may also drop the second assumption made prior to (3.4).

Using (2.6) and simplifying, one can show that

$$(3.6)\quad C_j = \frac{(-1)^j(2s-2n)_j}{(\alpha+\beta+2+2n)_j} \times \sum_{k=0}^{j} \frac{(-j)_k(j+2s+\alpha+\beta+1)_k(2s-2n-2\beta)_k}{k!(2s-2n-\alpha-\beta)_k(2\alpha+2s+2)_k}$$
$$\times {}_7F_6\left[\begin{matrix} \alpha+s+\frac{1}{2}, 1+\frac{\alpha+s+\frac{1}{2}}{2}, \alpha+\frac{1}{2}, \alpha+n+1, \\ \alpha+\beta+n+1, \frac{1-k}{2}, -\frac{k}{2} \\ \frac{\alpha+s+\frac{1}{2}}{2}, s+1, s-n-\frac{1}{2}, \\ s-n-\beta+\frac{1}{2}, \alpha+s+1+k/2, \alpha+s+3/2+k/2 \end{matrix}\right].$$

Since

$$(-k/2)_r(\tfrac{1}{2}-k/2)_r/(\alpha+s+1+k/2)_r(\alpha+s+3/2+k/2)_r = (-k)_{2r}/(2\alpha+2s+2+k)_{2r},$$

a transformation of the summation variable k to $k + 2r$ yields

$$(3.7)\quad C_j = \frac{(-1)^j(2s-2n)_n}{(\alpha+\beta+2+2n)_j} \times \sum_{r=0}^{[j/2]} \frac{(-j)_{2r}(j+2s+\alpha+\beta+1)_{2r}(2s-2n-2\beta)_{2r}}{r!(2s-2n-\alpha-\beta)_{2r}(2\alpha+2s+2)_{4r}}$$

$$\times \frac{(\alpha+\frac{1}{2})_r(\alpha+s+\frac{1}{2})_r\left(1+\frac{\alpha+s+\frac{1}{2}}{2}\right)_r(\alpha+n+1)_r \times (\alpha+\beta+n+1)_r}{\left(\frac{\alpha+s+\frac{1}{2}}{2}\right)_r(s+1)_r(s-n-\frac{1}{2})_r(s-n-\beta+\frac{1}{2})_r}$$

$$\times\, {}_3F_2\left[\begin{matrix}2r-j, j+2s+\alpha+\beta+1+2r, 2s-2n-2\beta+2r\\ 2s-2n-\alpha-\beta+2r,\quad 2\alpha+2s+2+4r\end{matrix}\right].$$

The ${}_3F_2(1)$ series is balanced and hence by (2.1)

$${}_3F_2[\ ] = \frac{(2\alpha+2\beta+2n+2+2r)_{j-2r}(\alpha-\beta+1+2r-j)_{j-2r}}{(2\alpha+2s+2+4r)_{j-2r}(\alpha+\beta+2n+1-2s-j)_{j-2r}}.$$

When we use this in (3.7) and simplify by means of the standard formulas for the shifted factorials [**17**, p. 239] and the duplication formula, we obtain

$$(3.8)\quad C_j = \frac{(-1)^j(2s-2n)_j(2\alpha+2\beta+2n+2)_j(\beta-\alpha)_j}{(\alpha+\beta+2+2n)_j(2\alpha+2s+2)_j(2s-2n-\alpha-\beta)_j}$$

$$\times\, {}_9F_8\left[\begin{matrix}\alpha+s+\frac{1}{2}, 1+\frac{\alpha+s+\frac{1}{2}}{2}, \alpha+\frac{1}{2}, \alpha+n+1, s-n-\beta, \frac{\alpha+\beta+1}{2}+s+j/2, \frac{\alpha+\beta+2}{2}+s+j/2, \frac{1-j}{2}, -j/2 \\ \frac{\alpha+s-\frac{1}{2}}{2}, s+1, s-n+\frac{1}{2}, \alpha+\beta+n+3/2, \frac{\alpha-\beta}{2}+\frac{1-j}{2}, \frac{\alpha-\beta}{2}+\frac{2-j}{2}, \alpha+s+1+j/2, \alpha+s+3/2+j/2\end{matrix}\right].$$

This series is terminating, very well-poised and 2-balanced. However it does not have the obvious non-negativity properties, so a transformation seems in order. First, we construct a k by using the first, third, fourth and fifth parameters on the top according to the prescription (2.9). Thus

$$k = 1 + 2(\alpha+s+\tfrac{1}{2}) - (\alpha+\tfrac{1}{2}) - (\alpha+n+1) - (s-n-\beta) = \beta+s+\tfrac{1}{2}.$$

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Next, we use (2.8) and simplify. This leads to

$$(3.9)\quad C_j = \frac{(2s-2n)_j(2\alpha+2\beta+2n+2)_j(\beta+s+1)_j(\alpha-\beta)_j}{(2s-2n-\alpha-\beta)_j(\alpha+\beta+2+2n)_j(\alpha+s+1)} \times (2\beta+2s+2)_j$$

$$\times {}_9F_8\left[\begin{matrix}\beta+s+\tfrac{1}{2},\, 1+\dfrac{\beta+s+\frac{1}{2}}{2},\, \beta+\tfrac{1}{2},\, \beta+n+1, \\ \dfrac{\beta+s+\frac{1}{2}}{2},\, s+1,\, s-n+\tfrac{1}{2},\end{matrix}\right.$$

$$s-n-\alpha,\quad \frac{\alpha+\beta+1}{2}+s+j/2,\, \frac{\alpha+\beta+2}{2}+s+j/2,$$

$$a+\beta+n+3/2,\quad \frac{\beta-\alpha}{2}+\frac{2-j}{2},\quad \frac{\beta-\alpha}{2}+\frac{1-j}{2},$$

$$\left.\begin{matrix}\dfrac{1-j}{2}, \quad -j/2 \\ \beta+s+1+j/2,\, \beta+s+3/2+j/2\end{matrix}\right],$$

$$j = 0, 1, 2, \ldots, 2n-2s.$$

This immediately leads to (1.9) when we make use of (1.3) and (1.5). Non-negativity of the C_j's is quite obvious when $-\frac{1}{2} \leqq \beta \leqq \alpha$. However, there is scope for further transformation when we note that (2.11) is just the thing we need. Assuming general values of α, β within the restriction $-1 < \beta \leqq \alpha, 0 \leqq \alpha+\beta+1$ we must now make a distinction between odd and even j's in order to apply (2.11). Thus, for even j, we take $\alpha+\frac{1}{2}$ as b, $(1-j)/2$ as g and $j/2$ as n in (2.11). So the ${}_9F_8(1)$ series in (3.8) transforms to

$$\frac{(\alpha+s+3/2)_{j/2}(s-n-\alpha)_{j/2}(\beta+n+1)_{j/2} \times \left(\frac{1}{2}-\frac{\alpha+\beta}{2}-j/2\right)_{j/2}\left(-\frac{\alpha+\beta}{2}-j/2\right)_{j/2}\left(\frac{1-j}{2}\right)_{j/2}}{(s+1)_{j/2}(s-n+\frac{1}{2})_{j/2}(\alpha+\beta+n+3/2)_{j/2} \times \left(\frac{\alpha-\beta}{2}+\frac{2-j}{2}\right)_{j/2}\left(\frac{\alpha-\beta}{2}+\frac{1-j}{2}\right)_{j/2}(-\alpha-j/2)_{j/2}}$$

$$\times {}_9F_8\left[\begin{matrix}\alpha,\, 1+\alpha/2,\, \alpha+1/2,\, \dfrac{\alpha-\beta}{2},\, \dfrac{\alpha-\beta+1}{2}, \\ \alpha/2,\, \dfrac{1}{2},\, \dfrac{\alpha+\beta}{2}+1,\, \dfrac{\alpha+\beta+1}{2},\end{matrix}\right.$$

$$\begin{matrix}s-n+j/2, & \alpha+\beta+n+1+j/2, \\ \alpha+n+1-s-j/2, & -\beta-n-j/2,\end{matrix}$$

$$\left.\begin{matrix}-s-j/2, \quad -j/2 \\ \alpha+s+1+j/2,\, \alpha+1+j/2\end{matrix}\right].$$

Using this in (3.8) and simplifying the coefficients we obtain (1.7) for even j.

When j is odd, $(j - 1)/2$ is a non-negative integer and so we take $\alpha + \frac{1}{2}$ as b, $-j/2$ as g and $(j - 1)/2$ as n in (2.11). The ${}_9F_8(1)$ series in (3.8) now transforms to

$$\frac{(\alpha + s + 3/2)_{(j-1)/2}(s - n - \alpha)_{(j-1)/2}(\beta + n + 1)_{(j-1)/2} \times \left(-\frac{\alpha+\beta}{2} - j/2\right)_{(j-1)/2}\left(\frac{1}{2} - \frac{\alpha+\beta}{2} - j/2\right)_{(j-1)/2}(-j/2)_{(j-1)/2}}{(s+1)_{(j-1)/2}(s - n + \frac{1}{2})_{(j-1)/2}(\alpha + \beta + n + 3/2)_{(j-1)/2} \left(\frac{\alpha-\beta}{2} + \frac{2-j}{2}\right)_{(j-1)/2}\left(\frac{\alpha-\beta}{2} + \frac{1-j}{2}\right)_{(j-1)/2} \times (-\alpha - \frac{1}{2} - j/2)_{(j-1)2/}}$$

$$\times {}_9F_8\left[\begin{matrix}\alpha + 1, \alpha/2 + 3/2, \alpha + \frac{1}{2}, s - n + \frac{1}{2} + j/2, \\ \frac{\alpha+1}{2}, 3/2, \alpha + n + 3/2 - s - j/2,\end{matrix}\right.$$

$$\begin{matrix}\alpha + \beta + n + 3/2 + j/2, \frac{\alpha-\beta}{2} + 1, \frac{\alpha-\beta}{2} + \frac{1}{2}, \\ \frac{1}{2} - \beta - n - j/2, \frac{\alpha+\beta}{2} + 1, \frac{\alpha+\beta}{2} + \frac{3}{2},\end{matrix}$$

$$\left.\begin{matrix}\frac{1}{2} - s - j/2, & \frac{1-j}{2} \\ \alpha + s + \frac{3}{2} + j/2, & \alpha + \frac{3}{2} + j/2\end{matrix}\right].$$

Substituting this in (3.8) and carrying out one final set of simplifications on the coefficients leads to formula (1.8) for odd j.

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