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ON PERIODIC SOLUTIONS TO AUTONOMOUS RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

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Under the assumption that $C_a = C\big([-r,0], S^{n-1}(a)\big)$ is positively invariant for $a > 0$, two necessary and sufficient conditions are obtained for an autonomous retarded functional differential equation to have a non-trivial periodic solution in C_a. Moreover, a feasible sufficient condition is given, which is better for $n = 2$ than that given by Dos Reis and Baroni.

1. INTRODUCTION

Let $\mathbf{R}$ be the set of real numbers, $\mathbf{R}^+$ the set of non-negative numbers and $\mathbf{R}^n$ the real Euclidean space. For $r \geqslant 0$ let $C = C([-r,0], \mathbf{R}^n)$ be the space of continuous functions from $[-r,0]$ to $\mathbf{R}^n$ with the topology of uniform convergence given by the norm $\|\phi\| = \sup\limits_{-r \leqslant \theta \leqslant 0} |\phi(\theta)|$.

Consider the autonomous retarded functional differential equation

$$X'(t) = f(X_t) \tag{1}$$

where $f: C \to \mathbf{R}^n$ is a continuous map that takes bounded sets into bounded sets and X_t is defined as $X_t(\theta) = X(t+\theta)$ for $-r \leqslant \theta \leqslant 0$. Suppose that unicity and continuity with respect to initial values hold and that the solutions are defined on $[-r,\infty)$. Then equation (1) defines a semi-dynamical system $\pi: C \times \mathbf{R}^+ \to C$ given by $\pi(\phi,t) = X_t(\phi)$.

Let $X(t,t_0,\phi)$ be a solution to (1) on $[-r,A)$ with $X_{t_0}(t_0,\phi) = \phi$. For the sake of convenience, we denote the solution $X(t,0,\phi)$ by $X(t,\phi)$ and $X_t(0,\phi)$ by $X_t(\phi)$. A solution $X(t,\phi)$ defined on $[-r,\infty)$ is called *periodic* if there is a $T > 0$ such that $\phi = \pi(\phi,T)$, or equivalent, $\pi(\phi,t) = \pi(\phi,t+T)$ for all $t \geqslant 0$. A set $M \subseteq C$ is called *positively invariant* if $\pi(\phi,t) \in M$ holds for all $\phi \in M$ and $t \geqslant 0$.

For $a > 0$, let $S^{n-1}(a) = \{x \in \mathbf{R}^n \colon |x| = a\}$ and $C_a = C\big([-r,0], S^{n-1}(a)\big)$. Suppose that C_a is positively invariant. The problems which concern us are these: Does (1) necessarily have periodic solutions in C_a? What is the essential condition

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ensuring that (1) has a non-trivial periodic solution in C_a? Many examples (see [3], p.326) have shown that there may not be any periodic solutions to (1) although C_a is positively invariant. Thus the first problem has already been solved. In the case of $n = 2$, [1] presented not only a good result but also a better method for the solution of the second problem. The aim of this paper is to generalise the result in [1] to all cases of $n \geqslant 2$. Indeed, we shall give necessary and sufficient conditions for (1) to have a non-trivial periodic solution in C_a.

2. Main results

Theorem 1. *Suppose that $f\colon C \to \mathrm{R}^n$ is a continuous map that takes bounded sets into bounded sets. Let π be the semi-dynamical system defined by (1). If C_a is positively invariant, then (1) has a non-trivial periodic solution in C_a if and only if there is a closed continuous curve L_a on $S^{n-1}(a)$, a positively invariant closed set $C_L \subseteq C([-r,0], L_a) \subseteq C_a$, a point $X_0 \in L_a$ and a $T_1 > 0$ satisfying:*

(i) *$J = \{\phi \in C_L\colon \phi(0) = X_0\}$ is non-empty;*

(ii) *$f(\phi) \neq 0$ holds for all $\phi \in J$;*

(iii) *for any $\phi \in J$, there is a $t' \in (0, T_1]$ such that $X(t', \phi) = X_0$.*

To prove Theorem 1, the following lemmas are needed.

Lemma 1. *Let r be the delay in (1). If the operator $\pi^t\colon C \to C$ defined by $\pi^t(\phi) = \pi(\phi, t)$ takes bounded sets into bounded sets, then for any bounded set $B \subset C$ and $t \geqslant r$, $\mathrm{cl}(\pi^t B)$ (closure of $\pi^t B$) is compact. Moreover, $\mathrm{cl}\left(\bigcup_{r\leqslant t\leqslant \bar{t}} \pi^t B\right)$ is also compact for any $\bar{t} \geqslant r$.*

Proof: By the Ascoli-Arzela theorem, the compactness of $\mathrm{cl}(\pi^t B)$ can easily be proved for $t \geqslant r$. This, together with continuity, implies that for any $t_0 \in [r, \bar{t}]$, $\lim_{t\to t_0} \pi^t\phi = \pi^{t_0}\phi$ holds uniformly for $\phi \in B$. Suppose $\{\psi_n\} \subseteq \bigcup_{r\leqslant t\leqslant \bar{t}} \pi^t B$. We will show that there is a convergent subsequence $\{\psi_{n_k}\} \subseteq \{\psi_n\}$. Clearly there are $\{\phi_n\} \subseteq B$ and $\{t_n\} \subseteq [r, \bar{t}]$ such that $\pi^{t_n}\phi_n = \psi_n$ for $n = 1, 2, \ldots$. By the compactness of $[r, \bar{t}]$, we may suppose without loss of generality that $\lim_{n\to\infty} t_n = t_0$. Then $\pi^{t_0}\phi_n \varepsilon \pi^{t_0} B$ corresponds to $\pi^{t_n}\phi_n \varepsilon \pi^{t_n} B$. Because $\lim_{t\to t_0} \pi^t\phi_n = \pi^{t_0}\phi_n$ holds uniformly for n, for any $\varepsilon > 0$, there is a natural number N such that

$$\left\|\pi^{t_n}\phi_n - \pi^{t_0}\phi_n\right\| < \frac{\varepsilon}{2}$$

holds for any $n > N$. Then the compactness of $\mathrm{cl}(\pi^{t_0} B)$ implies that there exists a subsequence $\{\pi^{t_0}\phi_{n_k}\} \subseteq \{\pi^{t_0}\phi_n\}$ and a $\psi_0 \in \mathrm{cl}(\pi^{t_0} B)$ such that $\lim_{k\to\infty} \pi^{t_0}\phi_{n_k} = \psi_0$.

We assume that $\|\pi^{t_0}\phi_{n_k} - \psi_0\| < \varepsilon/2$ for $k > k_1$, where $n_{k_1} > N$ holds. Then,

$$\|\psi_{n_k} - \psi_0\| \leqslant \|\pi^{t_{n_k}}\phi_{n_k} - \pi^{t_0}\phi_{n_k}\| + \|\pi^{t_0}\phi_{n_k} - \psi_0\| < \varepsilon$$

holds for $k > k_1$. Thus, $\lim_{k\to\infty} \psi_{n_k} = \psi_0$. Namely, $\mathrm{cl}\left(\bigcup_{r\leqslant t\leqslant \bar{t}} \pi^t B\right)$ is compact. □

LEMMA 2. *Suppose E is a Banach space, K either a cone or a truncated cone in E, $G \subseteq E$ an open bounded set with $0 \in G$ and ∂G the boundary of G. If $A\colon \partial G \cap K \to K$ is completely continuous and $\inf\{\|A\phi\| : \phi \in \partial G \cap K\} > 0$, then A has an eigenvector in $\partial G \cap K$.*

LEMMA 3. *Let $F \subseteq E$ be a bounded, closed and convex set with $0 \notin F$. Then the set $K(F) = \{x \in E\colon (\exists z \in F)(\exists t \geqslant 0)(x = tz)\}$ is a cone in E.*

The proofs of Lemma 2 and Lemma 3 can be found in [**1, 2, 4**].

Proof of Theorem 1:

NECESSITY. Let $X(t, \phi_0)$ be a non-trivial periodic solution in C_a. Then there is a $T > 0$ such that $X(t+T, \phi_0) = X(t, \phi_0)$ holds for all $t \geqslant 0$. Let $L_a = \{X(t, \phi_0)\colon 0 \leqslant t \leqslant T\}$; then L_a is a closed continuous curve on $S^{n-1}(a)$. It is obvious that $C_L = \{X_t(\phi_0)\colon 0 \leqslant t \leqslant T\} \subseteq C([-r, 0], L_a)$ is positively invariant. Because $X(t, \phi_0)$ is non-trivial, there is a $t_0 \in [0, T]$ such that $d/dt\, X(t, \phi_0)\mid_{t_0} \neq 0$, that is $f(X_{t_0}(\phi_0)) \neq 0$. We denote $X(t_0, \phi_0)$ by X_0; then we have $J = \{\phi \in C_L\colon \phi(0) = X_0\} = \{X_{t_0}(\phi_0)\}$. Thus $f(\phi) \neq 0$ holds for $\phi \in J$. Let $T_1 = T$ and $t' = T_1$. Then $X(t', \phi) = X_0$ for any $\phi \in J$. Hence the necessity holds.

SUFFICIENCY. For $r = 0$ the result is trivial, so we assume $r > 0$.

Let $F = \{\phi \in C\colon \phi(0) = X_0 \text{ and } \|\phi\| \leqslant 2a\}$. It is clear that F is closed, bounded, convex and $0 \notin F$. Then Lemma 3 implies that $K(F)$ is a cone and $K = K(F) \cap \{\phi \in C\colon \|\phi\| \leqslant a\}$ a truncated cone in C. Let

$$G = \{\phi \in C\colon \|\phi\| < 2a\} - \{\phi \in C\colon (\exists \phi_1 \in C_L)(\exists \alpha \in [1, 3])(\phi = \alpha\phi_1)\}.$$

Then G is an open bounded set and $K \cap \partial G = \{\phi \in C_L\colon \phi(0) = X_0\} = J$.

We define the map $\tau\colon J \to [r, \infty)$ by $\tau(\phi) = \inf\{t \geqslant r\colon \pi(\phi, t) \in J\}$. Then (iii) implies $r \leqslant \tau(\phi) \leqslant (k_0+1)T_1$ for all $\phi \in J$ and some natural number k_0 with $k_0 T_1 \geqslant r$. We assert that τ is continuous in J. In fact, for any ϕ_n, $\overline{\phi} \in J$ and $t_n = \tau(\phi_n)$, $\bar{t} = \tau(\overline{\phi})$ with $\phi_n \to \overline{\phi}$ as $n \to \infty$, we only need to show $t_n \to \bar{t}$ as $n \to \infty$. By (ii) we know that $d/dt\, X(t, \overline{\phi})\mid_{\bar{t}} = f(X_{\bar{t}}(\overline{\phi})) \neq 0$, which, together with the definition of τ, implies the existence of $s_1 \in [0, r)$ and $s_2 > \bar{t}$ such that $X(t, \overline{\phi}) \neq X_0$ holds on $[s_1, s_2]$ except at $t = \bar{t}$. By continuity, $X(t, \phi_n) \to X(t, \overline{\phi})$

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holds uniformly on $[s_1, s_2]$ as $n \to \infty$. Because $\{X(t, \overline{\phi}) : s_1 \leqslant t \leqslant s_2\}$, as well as $\{X(t, \phi_n) : s_1 \leqslant t \leqslant s_2\}$, is a section of the curve L_a, we must have $t_n \in [s_1, s_2]$ when n is large enough. If $t_n \not\to \bar{t}$ as $n \to \infty$, then there is a subsequence $\{t_{n_k}\}$ such that $t_{n_k} \to \bar{\bar{t}} \in [s_1, s_2]$ as $k \to \infty$ but $\bar{\bar{t}} \neq \bar{t}$. Thus $X_0 = X(t_{n_k}, \phi_{n_k}) \to X\left(\bar{\bar{t}}, \overline{\phi}\right) \neq X_0$ as $k \to \infty$, which is a contradiction. Therefore we have $t_n \to \bar{t}$ as $n \to \infty$.

Let $A : J \to C$ be defined by $A\phi = \pi(\phi, \tau(\phi))$. Then $AJ \subseteq J$. Since $\pi(\phi, t)$ is continuous in (ϕ, t) and $\tau(\phi)$ is continuous in ϕ, A is continuous. Clearly, we have $AJ \subseteq \bigcup_{r \leqslant t \leqslant (t_0+1)T_1} \pi^t J$ which, by Lemma 1, implies that A is completely continuous. Moreover, $\inf\{\|A\phi\| : \phi \in J\} = a > 0$ holds. By Lemma 2, there exist a $\phi_0 \in J$ and a $\mu \in R$ such that $A\phi_0 = \mu\phi_0$. As $A\phi_0 \in J$, we have $\|\phi_0\| = a = \|A\phi_0\| = \|\mu\phi_0\| = |\mu| \cdot \|\phi_0\|$ thus, $|\mu| = 1$.

By the definition of a cone [1], ϕ_0 and $-\phi_0$ cannot both belong to K. Thus $\mu = 1$, that is $A\phi_0 = \phi_0$. Since $f(\phi) \neq 0$ holds on J, it is obvious that $X_t(\phi_0)$ is a non-trivial periodic solution. ∎

Theorem 2. *Under the same general assumption as above, (1) has a non-trivial periodic solution in C_a if and only if there is a closed continuous curve L_a on $S^{n-1}(a)$, a positively invariant compact set $C_L \subseteq C([-r, 0], L_a) \subseteq C_a$ and a point $X_0 \in L_a$ satisfying:*

(i) $J = \{\phi \in C_L : \phi(0) = X_0\}$ *is not empty;*
(ii) $f(\phi) \neq 0$ *holds for any* $\phi \in J$;
(iii) *for any* $\phi \in J$, *there is a* $t' > 0$ *such that* $X_{t'}(\phi) \in J$.

Proof: From (i) — (iii) we know that for any $\phi \in J$, there is a $t'(\phi) > 0$ such that $X_{t'}(\phi) \in J$ but that $X_t(\phi) \notin J$ for $0 < t < t'$. Suppose $\{t'(\phi) : \phi \in J\}$ is unbounded. Then by the compactness of C_L, we can select a sequence $\{\phi_n\} \subseteq J$ and a corresponding $\{t_n\} = \{t_n(\phi_n)\}$ such that both $\phi_n \to \phi_0 \in J$ and $t_n \to \infty$ hold as $n \to \infty$. By the continuity with respect to initial values, we conclude that $X_t(\phi_0) \notin J$ for all $t \geqslant 0$, which contradicts (iii). Hence $\{t'(\phi) : \phi \in J\}$ is a bounded set. By Theorem 1, (1) has a non-trivial periodic solution. Thus the sufficiency is proved. The necessity is obvious from Theorem 1. ∎

Theorem 3. *Under the same general assumption as above, if there is a closed continuous curve L_a on $S^{n-1}(a)$ such that $C([-r, 0], L_a) \subseteq C_a$ is positively invariant and $f(\phi) \neq 0$ holds for $\phi \in C([-r, 0], L_a) - B$, then (1) has a non-trivial periodic solution in C_a. Here*

$B = \{\phi \in C([-r, 0], L_a) : f(\phi) = 0,\ \phi(s)$ *is not constant for* $s \in [-r, 0]\}$ *belongs to one of the following cases:*

(i) B *is empty;*

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(ii) *B is a finite set;*

(iii) *all $\phi \in B$, except for a finite number of elements of B, satisfy $\phi(s) \to \phi(0)(s \to 0)$ along L_a in the same direction. Moreover, $\{\phi(0)\colon \phi \in B\} \neq L_a$.*

PROOF: Suppose that B satisfies (iii). If there is a $\phi_0 \in B$ and a $\phi \in C([-r,0], L_a)$ such that $X_{t'}(\phi) = \phi_0 = X_{t''}(\phi)$ holds for some $t'' > t' \geqslant 0$, then $X_t(\phi)$ is a non-trivial periodic solution as $\phi_0(s)$ is not constant. If no such solution exists, we first show that for any $X_0 \in L_a$ and any $\phi \in C([-r,0], L_a)$, there is a $t' > 0$ such that $X(t', \phi) = X_0$.

In fact, $X(t,\phi)$ obviously exists on $[0,\infty)$ and moves along L_a. For convenience, we denote one direction of L_a by $(+)$ and the other by $(-)$. Suppose that all $\phi \in B$, except for a finite number of elements of B, are in $(+)$. If $X(t,\phi)$ changes direction at some $\bar{t} > 0$, then $X'(\bar{t},\phi) = 0$ holds, that is $f(X_{\bar{t}}(\phi)) = 0$, which implies $X_{\bar{t}}(\phi) \in B$ since $f(\phi) \neq 0$ on $C([-r,0], L_a) - B$. If $X(t,\phi)$ changes direction infinitely often, then there is a sequence $\{t_n\}$ such that $t_{n+1} > t_n > 0$, $X_{t_n}(\phi) \in B$, $X_{t_{2n+1}}(\phi)$ in $(+)$ and $X_{t_{2n}}(\phi)$ in $(-)$ hold for all positive integers n. Since $X_t(\phi)$ cannot coincide with any element of B more than once, we have $X_{t_i}(\phi) \neq X_{t_j}(\phi)$ for $i \neq j$. Therefore B has an infinite number of elements that are in $(-)$. This contradicts the assumption. Thus $X(t,\phi)$ changes direction at most a finite number of times. Hence there must be a $T_1 > 0$ such that $X(t,\phi)$ moves along L_a in a definite direction for $t \geqslant T_1$. If $X(t,\phi) \to X_1 \in L_a$ as $t \to \infty$, then $f(X_t(\phi)) \to 0$ as $t \to \infty$. Because f is continuous and $X_t(\phi) \to \phi_1$ as $t \to \infty$, where $\phi_1(s) = X_1$ for $s \in [-r,0]$, we have $f(\phi_1) = 0$ which implies $\phi_1 \in B$. This contradicts the assumption too. Thus $\lim_{t\to\infty} X(t,\phi)$ does not exist. Therefore there must be a $t' > T_1$, such that $X(t',\phi) = X_0$.

Suppose $\phi_0 \in C([-r,0], L_a)$ and $\nu(\phi_0) = \{X_t(\phi_0)\colon t \geqslant 0\}$. Let $\omega(\phi_0)$ be the limit set of $\nu(\phi_0)$. Then $\omega(\phi_0) \subseteq C([-r,0], L_a)$ is non-empty, compact and positively invariant [**3**]. By the above conclusion we know that $J = \{\phi \in \omega(\phi_0) : \phi(0) = X_0\}$ is non-empty for any $X_0 \in L_a$. By $\{\phi(0) : \phi \in B\} \neq L_a$ we can choose $X_0 \in L_a - \{\phi(0) : \phi \in B\}$ so that $J \cap B$ is empty. Thus, $f(\phi) \neq 0$ holds on J. Furthermore, for any $\phi \in J$, there is a $t' > 0$ such that $X_{t'}(\phi) \in J$. By Theorem 2, (1) has a non-trivial periodic solution in C_a.

If B satisfies either (i) or (ii), the proof is similar to the above. ∎

3. Remarks and examples

REMARK. Our results generalise the theorem in [**1**] in the following aspects. Firstly, [**1**] dealt only with the case of $n = 2$, while our results can be used for all cases of $n \geqslant 2$. Secondly, [**1**] presented only a sufficient condition, whereas our results include necessary and sufficient conditions, which are normally regarded as the best solution to any problem. Even our sufficient condition (Theorem 3) is much better than that of [1] because the theorem in [1] is only the special case of Theorem 3, when $n = 2$ and B satisfies (i). Thus, our results can be used for a more general class of equations than [1] can.

EXAMPLE 1. Consider the system

(3.1)

$$\begin{cases} X_1'(t) = X_3(t) \int_{-1}^{0} [X_1(t+\theta) - \sin\theta]^2 d\theta [1 + X_2^2\left(t - \frac{1}{2}\right) + X_3^2(t-1)] \\ X_2'(t) = 2X_1(t)X_3(t) \int_{-1}^{0} [X_1(t+\theta) - \sin\theta]^2 d\theta [1 + X_2^2\left(t - \frac{1}{2}\right) + X_3^2(t-1)] \\ X_3'(t) = -X_1(t)[1 + 2X_2(t)] \int_{-1}^{0} [X_1(t+\theta) - \sin\theta]^2 d\theta [1 + X_2^2\left(t - \frac{1}{2}\right) + X_3^2(t-1)] \end{cases}$$

Let $X = (X_1, X_2, X_3)^T$, $f = (f_1, f_2, f_3)^T$,

$$f_1(\phi) = \phi_3(0) \int_{-1}^{0} [\phi_1(\theta) - \sin\theta]^2 d\theta [1 + \phi_2^2\left(-\frac{1}{2}\right) + \phi_3^2(-1)],$$

$$f_2(\phi) = 2\phi_1(0)\phi_3(0) \int_{-1}^{0} [\phi_1(\theta) - \sin\theta]^2 d\theta [1 + \phi_2^2\left(-\frac{1}{2}\right) + \phi_3^2(-1)] \text{ and}$$

$$f_3(\phi) = -\phi_1(0)[1 + 2\phi_2(0)] \int_{-1}^{0} [\phi_1(\theta) - \sin\theta]^2 d\theta [1 + \phi_2^2\left(-\frac{1}{2}\right) + \phi_3^2(-1)].$$

Then (3.1) can be written as $X'(t) = f(X_t)$. It is easy to verify that $d/dt[X_1^2(t) + X_2^2(t) + X_3^2(t)] = 0$, $d/dt[X_2(t) - X_1^2(t)] = 0$ and the general assumptions hold for (3.1). Let $L_a \subseteq S^2(a)$ be defined by $X_1^2 + X_2^2 + X_3^2 = a^2$ with $X_2 = X_1^2$. Then C_a, as well as $C([-1,0], L_a) \subset C_a$, is positively invariant. If $a \geqslant \sqrt{(\sin^2 1 + 1/2)^2 - 1/4}$ holds, we denote

$$\phi_0(s) = \left(\sin s, \sin^2 s, \sqrt{a^2 - \sin^2 s - \sin^4 s}\right),$$

$$\overline{\phi}_0(s) = \left(\sin s, \sin^2 s, -\sqrt{a^2 - \sin^2 s - \sin^4 s}\right),$$

and $B = \{\phi_0, \overline{\phi}_0\}$. Then $f(\phi) \neq 0$ holds for $\phi \in C([-1,0], L_a) - B$.

If $0 < a < \sqrt{(\sin^2 1 + 1/2)^2 - 1/4}$, then $f(\phi) \neq 0$ holds for all $\phi \in C([-1,0], L_a)$. By Theorem 3, (3.1) has a non-trivial periodic solution in C_a for any $a > 0$.

EXAMPLE 2. Consider the system

$$(3.2)\qquad \begin{cases} X_1'(t) = X_3(t)\displaystyle\int_{-1}^{0}\{[X_1(t+\theta)-\sin\theta]^2 \\ \qquad +[X_3(t+\theta)-\sqrt{4-\sin^2\theta-\sin^4\theta}]^2\}d\theta[1+X_3^2(t-2)] \\ X_2'(t) = 2X_1(t)X_3(t)\displaystyle\int_{-1}^{0}\{[X_1(t+\theta)-\sin\theta]^2 \\ \qquad +[X_3(t+\theta)-\sqrt{4-\sin^2\theta-\sin^4\theta}]^2\}d\theta[1+X_3^2(t-2)] \\ X_3'(t) = X_1(t)[1+2X_2(t)]\displaystyle\int_{-1}^{0}\{[X_1(t+\theta)-\sin\theta]^2 \\ \qquad +[X_3(t+\theta)-\sqrt{4-\sin^2\theta-\sin^4\theta}]^2\}d\theta[1+X_3^2(t-2)] \end{cases}$$

Let $L_a \subseteq S^2(a)$ be defined by $X_1^2+X_2^2+X_3^2=a^2$ with $X_2=X_1^2$. Then both C_a and $C([-2,0],\,L_a)$ are positively invariant. If $a\neq 2$ $(a>0)$, then $f(\phi)\neq 0$ holds for all $\phi\in C([-2,0],\,L_a)$. If $a=2$, we put

$$B=\{\phi\in C([-2,0],\,L_2):\phi(\theta)=\left(\sin\theta,\,\sin^2\theta,\,\sqrt{4-\sin^2\theta-\sin^4\theta}\right)\text{ for }\theta\in[-1,0]\}.$$

Then $f(\phi)\neq 0$ holds for $\phi\in C([-2,0],\,L_2)-B$. Furthermore, for all $\phi\in B\phi(s)\to\phi(0)(s\to 0)$ are in the same direction along L_2. By Theorem 3, (3.2) has a non-trivial periodic solution in C_a for any $a>0$.

REMARK. Although the two examples above can be reduced to the case of $n=2$, they cannot be treated by the theorem of [1] because $f(\phi)$ may have zeros in C_a. Because Theorem 3 is only a sufficient condition, there exist systems that cannot be treated by this. In this case Theorems 1 and 2 may be helpful. The next example will show this.

EXAMPLE 3. Consider the system

$$(3.3)\qquad \begin{cases} X_1'(t)=X_2(t)\left(2\pi\displaystyle\max_{-1\leqslant s\leqslant 0}|X_1(t+s)|\right)/[|X_1(t-1)|+|X_2(t-1)|] \\ X_2'(t)=-X_1(t)\left(2\pi\displaystyle\max_{-1\leqslant s\leqslant 0}|X_1(t+s)|\right)/[|X_1(t-1)|+|X_2(t-1)|] \end{cases}$$

for which $f=(f_1,\,f_2)^T$, $f_1=\phi_2(0)\left(2\pi\max_{-1\leqslant s\leqslant 0}|\phi_1(s)|\right)/(|\phi_1(-1)|+|\phi_2(-1)|)$, $f_2=-\phi_1(0)\left(2\pi\max_{-1\leqslant s\leqslant 0}|\phi_1(s)|\right)/(|\phi_1(-1)|+|\phi_2(-1)|)$ and $\phi=(\phi_1,\phi_2)^T\in C([-1,0],\,R^2)$. Clearly, the general assumptions hold for (3.3). Moreover, C_a is positively invariant for any $a>0$ as $d/dt[X_1^2(t)+X_2^2(t)]=0$. Let $L_a=S^1(a)=\{X\in\mathrm{R}^2:X_1^2+X_2^2=a^2\}$,

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then $C([-1,0], L_a) = C_a$. Let $\phi_0(s) = (a,0)^T$ for $-1 \leqslant s \leqslant 0$, $X_0 = (a,0)^T \in L_a$, $\nu^+(\phi_0) = \{X_t(\phi_0) : t \geqslant 0\}$, $C_L = \mathrm{cl}\,(\nu^+(\phi_0))$ and $J = \{\phi \in C_L : \phi(0) = X_0\}$. It is easy to verify that $C_L = \nu^+(\phi_0) \cup \omega(\phi_0)^{[3]}$ [$\omega(\phi_0)$ is the positive limit set of $\nu^+(\phi_0)$ with $\pi^t\omega(\phi_0) = \omega(\phi_0)$ for any $t \geqslant 0$]. By Lemma 1, C_L is a positively invariant compact set. It is obvious that $f(\phi) \neq 0$ holds on J. Furthermore, for any $\phi \in J$ and $t \geqslant 0$, the solution $X(t,\phi)$ satisfies

$$\begin{cases} X_1(t) = a\cos 2\pi \displaystyle\int_0^t \{\max_{-1\leqslant s\leqslant 0} |X_1(\ell+s)| \,/[|X_1(\ell-1)| + |X_2(\ell-1)|]\}d\ell \\ X_2(t) = -a\sin 2\pi \displaystyle\int_0^t \{\max_{-1\leqslant s\leqslant 0} |X_1(\ell+s)| \,/[|X_1(\ell-1)| + |X_2(\ell-1)|]\}d\ell. \end{cases}$$

Thus $\max_{-1\leqslant s\leqslant 0} |X_1(\ell+s)| = a$ holds for $0 \leqslant \ell \leqslant 1$. Since

$$a = |X(\ell-1)| \leqslant |X_1(\ell-1)| + |X_2(\ell-1)| \leqslant \sqrt{2}\,|X(\ell-1)| = \sqrt{2}a$$

holds for any $\ell \geqslant 0$, we have

$$\sqrt{2}\pi t \leqslant 2\pi \int_0^t \{\max_{-1\leqslant s\leqslant 0} |X_1(\ell+s)| \,/[|X_1(\ell-1)| + |X_2(\ell-1)|]\}d\ell \leqslant 2\pi t$$

for $t \in [0,1]$. Therefore there exists a $t_0 \in [1/2,\, 1/\sqrt{2}]$ such that $X_1(t_0) = -a$. Similarly, there is a $t' \in [t_0 + 1/2,\, t_0 + 1/\sqrt{2}]$ such that $X_{t'}(\phi) \in J$. By Theorem 2 (3.3) has a non-trivial periodic solution in C_a.

References

[1] J.G. Dos Reis and R.L.S. Baroni, 'On the existence of periodic solutions for autonomous retarded functional differential equations on R^2', *Proc. Royal Soc. Edinburgh Section A* **102** (1986), 259–262.

[2] J.K. Hale, *Functional Differential Equations*, (Appl. Math. Sci. 3) (Springer-Verlag, Berlin, Heidelberg, New York, 1971).

[3] J.K. Hale, *Theory of Functional Differential Equations*, (2nd edition), (Appl. Math. Sci. 3) (Springer-Verlag, Berlin Heidelberg, New York, 1977).

[4] M.A. Krasnosel'skii, *Positive Solutions of Operator Equations* (Noordhoff, Grongingen, 1964).

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