ON FOURIER TRANSFORM MULTIPLIERS IN L^p

G. O. OKIKIOLU

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1. Introduction

We denote by R the set of real numbers, and by R^n, $n \geqq 2$, the Euclidean space of dimension n. Given any subset E of R^n, $n \geqq 1$, we denote the characteristic function of E by χ_E, so that $\chi_E(x) = 1$ if $x \in E$; and $\chi_E(x) = 0$ if $x \in R^n \setminus E$. The space $L^p(R^n) \equiv L^p$ consists of those measurable functions f on R^n such that

$$\left(\int_{R^n} |f(t)|^p \, dt \right)^{1/p} \equiv ||f||_p$$

is finite. Also, L^∞ represents the space of essentially bounded measurable functions with $||f||_\infty = \inf \{\alpha > 0;\ m(\{x : |f(x)| > \alpha\}) = 0\}$, where m represents the Lebesgue measure on R^n. The numbers p and p' will be connected by $1/p + 1/p' = 1$.

The Fourier transform of a function f in L^1 is defined by

$$\mathscr{F}(f)(x) = (2\pi)^{-\frac{1}{2}n} \int_{R^n} f(t) e^{ix \cdot t} dt,$$

where for $x = (x_1, x_2, \cdots, x_n)$, $t = (t_1, t_2, \cdots, t_n)$, we set $x \cdot t = x_1 t_1 + x_2 t_2 + \cdots x_n t_n$.

The symbol $\mathscr{F}$ will also represent the extension of the Fourier operator as a bounded linear operator in L^p, $1 \leqq p \leqq 2$.

Given a bounded measurable function g on R^n we denote by T_g the operator defined on $L^2(R^n)$ by

$$\mathscr{F} T_g(f) = g \mathscr{F}(f); \ (f \in L^2). \tag{1.1}$$

On using the fact that $||\mathscr{F}(f)||_2 = ||f||_2$ for f in L^2, we see that

$$||T_g(f)||_2 \leqq ||g||_\infty ||f||_2. \tag{1.1.1}$$

As usual we say that g is a multiplier in L^p, $p \neq 2$, if T_g can be extended to bounded linear operator in L^p. For properties of multipliers in L^p see for example, Hörmander [2] and Brainerd and Edwards [1]. The main purpose of this paper is to describe some classes of bounded measurable functions on R^n which are

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multipliers in L^p, $1 < p < \infty$. A particular case of the main theorem shows that some non-negative bounded symmetrically non-increasing function on R are multipliers in $L^p(R)$, $1 < p < \infty$. The conclusions obtained here supplement existing results on multipliers in L^p spaces; see S. G. Mihlin [6], Hörmander [2, Theorem 2.5], Stein [7], de Leeuw [3], Littman [4], and also [5]. See also [10].

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2. The main results

The main results of the paper are given in Theorem 2.4 and Corollary 2.5 of this Section. In the next Section we consider the special results involving the one-dimensional cases. First we state two results which will be employed in the proof of the main theorem.

2.1. DEFINITIONS

(2.1.1) Given any measurable function h on R^n we define the set H^a, $a > 0$, by

$$H^a = \{x \in R^n : |h(x)| > a\}.$$

(2.1.2) For any real-valued function ϕ on R and a function h on R^n, we denote by $\phi \circ h$ the function whose value at $x \in R^n$ is $\phi(h(x))$.

(2.1.3) We denote by $\mathscr{J}^n (n \geqq 1)$ the class of rectangles I in R^n of the form $I = I_1 \times I_2 \times \cdots \times I_n$, where each I_j is an interval of R (open, closed, or half open).

2.2. LEMMA. *Let ϕ be absolutely continuous on $[0, \infty)$, with $\phi(0) = 0$, and let h be measurable and non-negative on R^n. Then, with $\phi'(x) = (d/dx)\phi(x)$,*

$$\phi \circ h(x) = \int_0^\infty \phi'(a)\chi_{H^a}(x)\, da, \qquad (x \in R^n).$$

PROOF. This follows immediately from the fact that

$$\phi \circ h(x) = \int_0^{h(x)} \phi'(a)\, da.$$

2.3. LEMMA. *If $1 < p < \infty$, $n \geqq 1$, then there is a constant $k_{p,n}$ such that, for $J \in \mathscr{J}^n$,*

$$||T\chi_J(f)||_p \leqq k_{p,n}||f||_p.$$

PROOF. This is a well-known result which follows easily from the one-dimensional case. Note that if $J = [a, b) \subset R$, then

$$T\chi_I(f)(x) = \pi^{-1} \int_R (e^{-ia(t-x)} - e^{-ib(t-x)})(t-x)^{-1} f(t)\, dt,$$

and the boundedness of $T\chi_I$ follows by applying known results involving the Hilbert transform [8, Theorem 101].

(2.3.1) NOTE. If $M = \bigcup_{k=1}^{j} I_k$, where $\{I_1, I_2, \cdots, I_j\}$ is a finite disjoint subclass of $\mathscr{J}^n$, then we have

$$T\chi_M = \sum_{k=1}^{j} T_{\chi_{I_k}},$$

so that by 2.3,

$$\|T\chi_M(f)\|_p \leqq jk_{p,n}\|f\|_p, \qquad (f \in L^p(R^n)).$$

2.4. THEOREM. *Let the functions ϕ on $[0, \infty)$ and h on R^n satisfy the following conditions*:

(i) *ϕ is absolutely continuous on $[0, \infty)$, $\phi(0) = 0$, and $\phi' \in L^1(0, \infty)$,*

(ii) *h is non-negative on R^n, and there is an integer $j \geqq 1$ such that, for $a > 0$, we have*

$$H^a = \bigcup_{k=1}^{j(a)} H(a, k),$$

where $j(a) \leqq j$ or $H^a = \emptyset$, each set $H(a, k) \in \mathscr{J}^n$, and $H(a, k) \cap H(a, l) = \emptyset$ if $k \neq l$. Then for $1 < p < \infty$ and with $k_{p,n}$ defined as in Lemma 2.3; we have

$$\|T_{\phi\circ h}(f)\|_p \leqq jk_{p,n}\|\phi'\|_1\|f\|_p.$$

PROOF. Let $f \in L^1 \cap L^\infty$. Then by Lemma 2.2, we have

$$\mathscr{F}T_{\phi\circ h}(f)(x) = \int_0^\infty \phi'(a)\chi_{H^a}(x)\mathscr{F}(f)(x)\,da = \int_0^\infty \phi'(a)\mathscr{F}T_{\chi_{H^a}}(f)(x)\,da,$$

and it follows by applying Fubini's theorem and the Parseval relation for Fourier transforms that, for $g \in L^2$,

$$\int_{R^n} g(x)\mathscr{F}T_{\phi\circ h}(f)(x)\,dx = \int_0^\infty \phi'(a)\int_{R^n}\mathscr{F}(g)(x)T_{\chi_{H^a}}(f)(x)\,dx\,da$$
$$= \int_{R^n}\mathscr{F}(g)(x)\int_{R^n}\phi'(a)T_{\chi_{H^a}}(f)(x)\,da\,dx.$$

Further, on applying the Parseval relation once more and noting that the function g is arbitrary, we see that

$$T_{\phi\circ h}(f)(x) = \int_0^\infty \phi'(a)T_{\chi_{H^a}}(f)(x)\,da.$$

Hence it follows from the hypotheses and Note 2.3.1, and by applying Minkowski's integral inequality that

$$\|T_{\phi\circ h}(f)\|_p \leqq \int_0^\infty |\phi'(a)|\,\|T_{\chi_{H^a}}(f)\|_p\,da \leqq jk_{p,n}\|\phi'\|_1\|f\|_p.$$

The inequality can be extended to all of L^p in the usual way by constructing sequences of functions in $L^2 \cap L^p$ which converge to given functions in L^p-norm.

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2.5. COROLLARY. *Let the function h on R^n, $n \geqq 1$, be bounded and satisfy condition* (ii) *of Theorem* 2.4. *Then, for* $1 < p < \infty$ *and with* $k_{p,n}$ *defined as in Lemma* 2.3, *we have*

$$\|T_h(f)\|_p \leqq jk_{p,n}\|h\|_\infty\|f\|_p.$$

PROOF. Let ϕ_0 be any monotone increasing absolutely continuous function on $[0, \infty)$ with $\|\phi_0'\|_1 = 1$ and $\phi_0(0) = 0$; for example we may choose

$$\phi_0(t) = 1 - e^{-t}.$$

Now define the function $\phi(x)$ by $\phi(x) = (1+\varepsilon)\|h\|_\infty \phi_0(x)$, where $\varepsilon > 0$. We denote by ϕ^{-1} the inverse of ϕ, so that for $y \in [0, \|h\|_\infty]$,

$$\phi \circ \phi^{-1}(y) = y.$$

Since $\{x \in R^n : \phi^{-1} \circ h(x) > a\} = \{x \in R^n : h(x) > \phi(a)\}$ it follows that if $h(x)$ satisfies the condition (ii) of Theorem 2.4, then so also does $\phi^{-1} \circ h(x)$. Hence by Theorem 2.4, we have, for $1 < p < \infty$,

$$\|T_h(f)\|_p = \|T_{\phi\circ\phi^{-1}\circ h}(f)\|_p \leqq jk_{p,n}\|\phi'\|_1\|f\|_p = (1+\varepsilon)jk_{p,n}\|h\|_\infty\|f\|_p,$$

and the required conclusion follows since ε is arbitrary.

(2.5.1) REMARK. It is possible to prove Corollary 2.5 directly by using the same argument as that given in the proof of Theorem 2.4. We simply note that, as in Lemma 2.2,

$$h(x) = \int_0^{h(x)} da = \int_0^b \chi_{H^a}(x)\, da,$$

where $b = (1+\varepsilon)\|h\|_\infty$, $\varepsilon > 0$.

3. The one-dimensional case of the main result

The main feature of the one-dimensional case of Theorem 2.4 is that condition (ii) of that theorem can be put in a more suitable form involving in some cases the monotone-ness of the function h.

3.1. THEOREM. *Let the function ϕ on $[0, \infty)$ and h on R satisfy the following conditions*

(i) *ϕ is absolutely continuous on $[0, \infty)$, $\phi(0) = 0$, and $\phi' \in L^1(0, \infty)$,*

(ii) *h is non-negative on R, and there is a finite class $\{A_1, A_2, \cdots, A_j\}$ of disjoint intervals such that*

$$R = \bigcup_{k=1}^{j} A_k,$$

where h is monotone on each set A_k. Then for $1 < p < \infty$, there is a constant k_p such that

$$\|T_{\phi\circ h}(f)\|_p \leqq jk_p\|\phi'\|_1\|f\|_p.$$

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PROOF. This result is a special case of Theorem 2.4. We note that, for $a > 0$, we have

$$H^a = \{x \in R : h(x) > a\} = \bigcup_{k=1}^{j} H(a, k),$$

where $H(a, k) = \{x \in A_k : h(x) > a\}$. Since h is monotone on the interval A_k, the set $H(a, k) \in \mathscr{J}^1$, and so condition (ii) of Theorem 2.4 is satisfied.

3.2. COROLLARY. *Let the function h on R be bounded and satisfy the condition* (ii) *of Theorem* 3.1. *Then for* $1 < p < \infty$ *there is a constant* k_p *such that*

$$\|T_h(f)\|_p \leqq jk_p \|h\|_\infty \|f\|_p.$$

PROOF. In view of the remarks in the proof of Theorem 3.1, the result is easily seen to be a special case of Corollary 2.5.

3.3. REMARKS. (i) The conclusions of Corollary 3.2 provide examples of multipliers in $L^p(R)$, $1 < p < \infty$, which have an infinite set of discontinuities on some subinterval of R. The main condition required of such functions is that they should be non-increasing or non-decreasing on the interval. (ii) Since linear combinations of multipliers are also multipliers the condition in Corollary 3.2 that h be non-negative can be replaced by one requiring that h be a linear combination of functions satisfying the conditions given there. For this reason the conclusions of the Corollary apply to functions of the form

$$h(t) = \int_0^t g(\xi)\, d\xi,$$

where the integral is bounded on R.

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University of East Anglia
Norwich, England