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SOME INEQUALITIES RELATED TO PLANAR CONVEX SETS

BY

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ABSTRACT. B. Grünbaum and J. N. Lillington have considered inequalities defined by three lines meeting in a compact convex subset of the plane. We prove a conjecture of Lillington and propose some conjectures of our own.

1. **Introduction.** In [1], B. Grünbaum defines a *measure of symmetry* to be a real-valued function f defined on the family $\mathcal{K}^n$ of all convex bodies in E^n such that:

(1) $0 \le f(X) \le 1$ for all $X \in \mathcal{K}^n$;

(2) $f(X)=1$ if and only if $X \in \mathcal{K}^n$ has a center of symmetry;

(3) $f(X)=f(T(X))$ for every $X \in \mathcal{K}^n$ and every nonsingular affine transformation T of E^n onto itself;

(4) $f(X)$ is a continuous function of X.

(More precisely, Grünbaum calls such an f an *affine invariant* measure of symmetry.)

The above paper of Grünbaum is a survey of the many different known measures of symmetry, and of some functions defined on $\mathcal{K}^n$ which are likely candidates, but which have not yet been proved measures of symmetry. Grünbaum himself introduced one such candidate, as follows.

Suppose three concurrent lines, L_1, L_2, L_3, divide a compact convex subset X of the plane into six regions, as shown in Fig. 1.

Grünbaum ([1], p. 260) assumed that $|X_1|=|X_2|=|X_3|=a$ and $|Y_1|=|Y_2|=|Y_3|=b$ (here we use $|E|$ to denote the area of the set E). He conjectured that $f(X)=\inf(a/b)$, the infimum taken over all such partitions, satisfies $\frac{1}{2}\le f(X)\le 1$, with $f(X)=\frac{1}{2}$ if and only if X is a triangle, and $f(X)=1$ if and only if X is centrally symmetric. It would easily follow that f is a measure of symmetry.

The conjecture was the motivation of J. N. Lillington's paper [2]. Here, a function $k(X)$ is defined by:

$$k(X; L_1, L_2, L_3) = \max_{1\le i\le 3} \frac{|X_i|}{|Y_i|}$$

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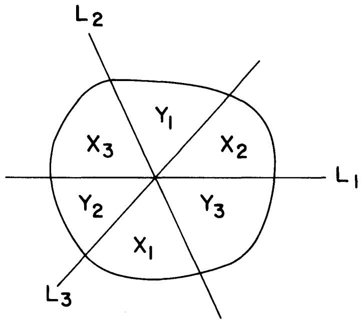

Figure 1.

and

$$k(X)=\inf_{L_1,L_2,L_3} k(X;L_1,L_2,L_3),$$

in the notation of Fig. 1, and without restriction on the areas involved.

Lillington proved that $k(X)\leq 1$, with equality if and only if X is centrally symmetric; it follows that if X is centrally symmetric then $f(X)=1$. By considering the lower bound of a further function, and using rather complicated geometric arguments, Lillington also showed that $f(X)\geq\frac{1}{2}$, with equality if and only if X is a triangle. (So, to prove Grünbaum's conjecture, it only remains to show that $f(X)=1$ only for centrally symmetric sets.) Though it easily follows from Lillington's work that $k(X)$ is a measure of symmetry, he could only conjecture a lower bound: $k(X)\geq\frac{1}{2}$, with equality if and only if X is a triangle.

In this paper we take up the challenge and study lower bounds of f, k and another function defined on compact convex subsets of the plane. We introduce the function $j(X)$, defined by:

$$j(X;L_1,L_2,L_3)=\max_{1\leq i\leq 3}\frac{|X_{i-1}|+|X_{i+1}|}{|Y_i|}$$

and

$$j(X)=\inf_{L_1,L_2,L_3} j(X;L_1,L_2,L_3).$$

(Here and throughout, values of i lying outside the set $\{1,2,3\}$ are defined by $i\equiv i+3$.)

In Section 2 we note, as did Lillington, that in considering lower bounds of all the above functions, X may be assumed to be a triangle. Here our approach differs from his; we obtain fairly simple algebraic expressions for all the areas involved and, in Section 3, use these to prove that $j(X)\geq 1$, with equality if and only if X is a triangle. As a corollary, we get a concise, algebraic proof of the lower bound for $f(X)$ conjectured by Grünbaum. A much more involved argument is needed to prove Lillington's conjecture; this is presented in Section 4.

In the final section we give some stronger conjectures of our own, the algebraic forms of which should be of independent interest.

2. **Algebraic expressions for the areas**. Suppose K is a compact convex set in the plane and concurrent lines L_1, L_2 and L_3 meet in a point o in K. Then a triangle X may be found such that functions such as f, j and k do not take larger values at X than at K. (See Fig. 2.) The details are given in Theorem 1 of [2].

Let $p_1, p_2, p_3, q_1, q_2, q_3$ be the intersections of L_1, L_2 and L_3 with the sides of X, as in Fig. 2.

We use areal coordinates, setting $A_1=(1,0,0)$, $A_2=(0,1,0)$, $A_3=(0,0,1)$, and $o=(x_1,x_2,x_3)$, where $x_1+x_2+x_3=|X|=1$. For $i=1,2,3$ we take L_i to be the line $x'_{i+1}-x_{i+1}=\lambda_i(x_{i-1}-x'_{i-1})$.

It follows that

$$p_i=(p_1^{(i)},p_2^{(i)},p_3^{(i)}) \quad \text{and} \quad q_i=(q_1^{(i)},q_2^{(i)},q_3^{(i)}),$$

where, using $x_1+x_2+x_3=1$, we have

$$p_{i-1}^{(i)}=x_{i-1}+x_{i+1}/\lambda_i, \qquad p_i^{(i)}=x_i+(1-1/\lambda_i)x_{i+1}, \qquad p_{i+1}^{(i)}=0 \quad \text{and}$$
$$q_{i-1}^{(i)}=0, \qquad q_i^{(i)}=x_i+(1-\lambda_i)x_{i-1}, \qquad q_{i+1}^{(i)}=x_{i+1}+\lambda_i x_{i-1}.$$

Applying the determinant formula for area, we find

$$\left.\begin{aligned} &|X_i|=x_i^2(\lambda_{i+1}+(1/\lambda_{i-1})-1) \\ \text{and} \\ &|Y_i|=(x_{i-1}+x_{i+1})^2-\lambda_{i-1}x_{i+1}^2-x_{i-1}^2/\lambda_{i+1}, \qquad i=1,2,3 \end{aligned}\right\} \tag{1}$$

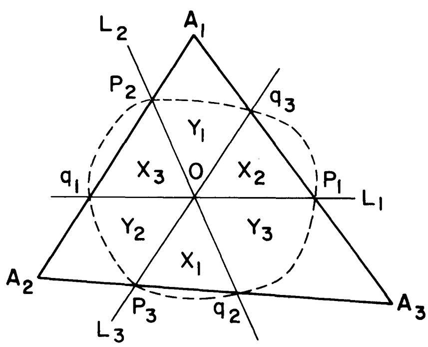

Figure 2.

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where we have used $|Y_i| = |\Delta(A_i p_{i+1} o)| + |\Delta(A_i o q_{i+2})|$, $i = 1, 2, 3$, and the equality $x_1 + x_2 + x_3 = 1$.

In (1), all $\lambda_i > 0$, $x_i \geq 0$, and, from the geometry,

$$0 \leq x_i + (1 - 1/\lambda_i)x_{i+1} \quad \text{and} \quad 0 \leq x_i + (1 - \lambda_i)x_{i-1}, \qquad i = 1, 2, 3. \tag{2}$$

3. **Grünbaum's inequality.** Before proving Theorem 3.1, which will yield Grünbaum's inequality as an easy corollary, we will comment briefly on the difficulty in establishing lower bounds for functions such as f, j and k. This seems, to us, to stem from the fact that

$$|X_1| + |X_2| + |X_3| \geq \tfrac{1}{2}(|Y_1| + |Y_2| + |Y_3|) \tag{3}$$

clearly does not hold in general. Equality holds here (and for all our other functions) when

$$(x_1, x_2, x_3, \lambda_1, \lambda_2, \lambda_3) = (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 1, 1, 1),$$

but (3) is not even true in a neighborhood of this point. To see this, suppose $\frac{1}{4} < d < \frac{1}{2}$ and put $x_1 = x_2 = d$, $x_3 = 1 - 2d$, $\lambda_1 = d/(1-2d)$, $\lambda_2 = (1-2d)/d$, $\lambda_3 = 1$. Then (3) becomes $-(1-3d)^2 \geq 0$, which is false if $d \neq \frac{1}{3}$. This means one cannot utilize the considerable algebraic simplification which results from adding the areas as in (3).

THEOREM 3.1. *$j(X) \geq 1$ with equality if and only if X is a triangle; further, equality holds only if the lines L_i pass through the centroid of X and are parallel to $A_{i+1}A_{i+2}$ $(i = 1, 2, 3)$.*

Proof. Suppose X is a triangle and $j(X) < 1$, i.e. the lines L_1, L_2, and L_3 are such that

$$|X_{i-1}| + |X_{i+1}| < |Y_i|, \qquad i = 1, 2, 3.$$

From (1), we have

$$2x_{i-1}^2((1/\lambda_{i+1}) - 1) + 2x_{i+1}^2(\lambda_{i-1} - 1) + (x_{i+1} - \lambda_i x_{i-1})^2/\lambda_i < 0$$

which implies, dropping the last term, that

$$x_{i-1}^2((1/\lambda_{i+1}) - 1) + x_{i+1}^2(\lambda_{i-1} - 1) < 0, \qquad i = 1, 2, 3. \tag{4}$$

For these inequalities to have a solution, either $\lambda_i < 1$ for $i = 1, 2, 3$ or $\lambda_i > 1$ for $i = 1, 2, 3$. Suppose the former.

The inequalities (4) give

$$x_1^2 < \frac{(1-\lambda_1)(1-\lambda_2)(1-\lambda_3)}{\left(\frac{1}{\lambda_1} - 1\right)\left(\frac{1}{\lambda_2} - 1\right)\left(\frac{1}{\lambda_3} - 1\right)} x_1^2 = \lambda_1\lambda_2\lambda_3 x_1^2,$$

which is impossible. The other case is dealt with similarly. In view of the

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remarks of Section 2, this suffices to prove that $j(X) \geq 1$, with equality if and only if X is a triangle.

Suppose that $j(X) = 1$. Then we obtain a contradiction as above unless

$$|X_{i-1}| + |X_{i+1}| = |Y_i|, \qquad i = 1, 2, 3, \quad \text{so that}$$

$$2x_{i-1}^2((1/\lambda_{i+1}) - 1) + 2x_{i+1}^2(\lambda_{i-1} - 1) + (x_{i+1} - \lambda_i x_{i-1})^2/\lambda_i = 0, \quad \text{for} \quad i = 1, 2, 3.$$

If, for some i, $(x_{i+1} - \lambda_i x_{i-1}) \neq 0$, then for this i,

$$x_{i-1}^2((1/\lambda_{i+1}) - 1) + x_{i+1}^2(\lambda_{i-1} - 1) < 0,$$

and once again we have a contradiction as before. So $\lambda_i = x_{i+1}/x_{i-1}$, $i = 1, 2, 3$, and

$$x_{i-1}^2((1/\lambda_{i+1}) - 1) + x_{i+1}^2(\lambda_{i-1} - 1) = 0, \qquad i = 1, 2, 3.$$

Substituting in the latter for λ_i, we obtain

$$x_i(x_{i-1} + x_{i+1}) = x_{i-1}^2 + x_{i+1}^2, \qquad i = 1, 2, 3.$$

Adding two of these equations and subtracting the third gives

$$x_i^2 = x_{i-1}x_{i+1}, \qquad i = 1, 2, 3,$$

from which it easily follows that $x_1 = x_2 = x_3$. So $\lambda_i = 1$, $i = 1, 2, 3$, and L_i passes through $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ parallel to $A_{i+1}A_{i+2}$, $i = 1, 2, 3$.

COROLLARY 3.2. *$f(X) \geq \frac{1}{2}$, with equality if and only if X is a triangle.*

Proof. From the first part of Theorem 3.1 it follows that $f(X) \geq \frac{1}{2}j(X) \geq \frac{1}{2}$ with equality only if X is a triangle. If X is a triangle, $x_1 = x_2 = x_3$, and $\lambda_i = 1$, $i = 1, 2, 3$, then $|X_1| = |X_2| = |X_3|$ and $|Y_1| = |Y_2| = |Y_3|$; furthermore $|X_i| = \frac{1}{2}|Y_i|$, so $f(X) = \frac{1}{2}$.

The method of Theorem 3.1 can be applied to Lillington's conjecture, but seems to give a proof only in the special case when $\lambda_i \leq 1$ for $i = 1, 2, 3$, or when $\lambda_i \geq 1$ for $i = 1, 2, 3$.

4. Lillington's Conjecture.

THEOREM 4.1. *$k(X) \geq \frac{1}{2}$, with equality if and only if X is a triangle.*

Proof. Suppose X is a triangle and $k(X) < \frac{1}{2}$, so that the lines L_1, L_2, and L_3 are such that

$$|X_i| < \tfrac{1}{2}|Y_i|, \qquad i = 1, 2, 3.$$

We may rewrite these inequalities, using (1), as

$$x_i^2(\lambda_{i+1} + (1/\lambda_{i-1}) - 1) < \tfrac{1}{2}[(x_{i-1}^2 + x_{i+1}^2) - \lambda_{i-1}x_{i+1}^2 - x_{i-1}^2/\lambda_{i+1}] \tag{5}$$

for $i = 1, 2, 3$. Note that both sides of (5) must be non-negative.

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To prove the theorem it suffices to show that these inequalities are inconsistent. We will need several lemmas in the course of the proof. We remind the reader that we have $x_1+x_2+x_3=1$, for we will often use this equality.

LEMMA 4.2. *Without loss of generality we may assume that* $x_i<\sqrt{2}-1$, *for* $i=1,2,3$.

Proof. Suppose that for some i, $x_i\geq\sqrt{2}-1$. Then $(x_{i-1}+x_{i+1})=(1-x_i)\leq 2-\sqrt{2}$. It follows that

$$x_i^2(\lambda_{i+1}+(1/\lambda_{i-1})-1)\geq(\lambda_{i+1}+(1/\lambda_{i-1})-1)(3-2\sqrt{2})$$
$$\geq\tfrac{1}{2}(\lambda_{i+1}+(1/\lambda_{i-1})-1)(x_{i-1}+x_{i+1})^2. \quad (6)$$

By expanding it can be verified that the last quantity is equal to

$$\tfrac{1}{2}[(x_{i-1}+x_{i+1})^2-\lambda_{i-1}x_{i+1}^2-x_{i-1}^2/\lambda_{i+1}]+\tfrac{1}{2}A,$$

where

$$A=\lambda_{i+1}(x_{i+1}+(1-(1/\lambda_{i+1}))x_{i-1})^2+(1/\lambda_{i-1})(x_{i-1}+(1-\lambda_{i-1})x_{i+1})^2, \quad \text{so} \quad A\geq 0.$$

This, together with (6), implies that $k(X)\geq\frac{1}{2}$.

Next we rearrange the inequalities (5) as

$$\tfrac{1}{2}x_{i+1}^2\lambda_{i-1}+x_i^2/\lambda_{i-1}+x_i^2\lambda_{i+1}+\tfrac{1}{2}x_{i-1}^2/\lambda_{i+1}<\tfrac{1}{2}(1-x_i)^2+x_i^2, \quad (7)$$

for $i=1,2,3$. Adding the inequalities for $i=2$ and 3 in (7), and using $x_3^2\lambda_1+x_2^2/\lambda_1\geq 2x_2x_3$, we obtain

$$(\tfrac{1}{2}x_1^2\lambda_2+x_3^2/\lambda_2)+(x_2^2\lambda_3+\tfrac{1}{2}x_1^2/\lambda_3)<\tfrac{3}{2}(1-x_1)^2+x_1-6x_2x_3 \quad (8)$$

and we also have from (7), with $i=1$,

$$(x_1^2\lambda_2+\tfrac{1}{2}x_3^2/\lambda_2)+(\tfrac{1}{2}x_2^2\lambda_3+x_1^2/\lambda_3)<\tfrac{1}{2}(1-x_1)^2+x_1^2. \quad (9)$$

Our aim is to replace (8) and (9) by inequalities involving only x_i, $i=1,2,3$.

LEMMA 4.3. *Suppose that for some positive* u, v:

$$(\tfrac{1}{2}u+p^2/u)+(v+\tfrac{1}{2}q^2/v)<a \quad \text{and} \quad (u+\tfrac{1}{2}p^2/u)+(\tfrac{1}{2}v+q^2/v)<b \quad (10)$$

Then $a>\sqrt{2}(p+q)$, $b>\sqrt{2}(p+q)$, *and one of the following conditions holds*:

$$a>5\sqrt{2}(p+q)/4 \quad (11)$$

$$b>5\sqrt{2}(p+q)/4 \quad (12)$$

$$a^2+b^2-5ab/2+9(p+q)^2/8<0 \quad (13)$$

Proof. First choose a and b so that we have equality in (10). Then

$$a^2+b^2-\frac{5ab}{2}+\frac{9}{8}(p+q)^2=\frac{9}{8}\left[2pq-pq\left(\frac{uv}{pq}+\frac{pq}{uv}\right)\right]\leq\frac{9}{8}pq\left[2-\min_{\gamma}\left(\gamma+\frac{1}{\gamma}\right)\right]=0.$$

The curve $a^2+b^2-5ab/2+9(p+q)^2/8=0$ in the (a, b)-plane lies to the right of the tangent $a=\sqrt{2}(p+q)$ at $(\sqrt{2}(p+q), 5\sqrt{2}(p+q)/4)$, and above the tangent $b=\sqrt{2}(p+q)$ at $(5\sqrt{2}(p+q)/4, \sqrt{2}(p+q))$. From this it is easily seen that if $a'>a$ and $b'>b$, then one of the conditions (11)–(13) (with a' replacing a, and b' replacing b) must hold.

We apply Lemma 4.3 to the inequalities (8) and (9), with $u=x_1^2\lambda_2$, $v=x_2^2\lambda_3$, $p=x_1x_3$ and $q=x_1x_2$. We see that one of the following conditions holds:

$$\tfrac{3}{2}(1-x_1)^2+x_1-6x_2x_3-5\sqrt{2}x_1/4+5\sqrt{2}x_1^2/4>0 \tag{14}$$

$$(\tfrac{1}{2}(1-x_1)^2+x_1^2-5\sqrt{2}x_1/4+5\sqrt{2}x_1^2/4>0) \quad \text{and}$$
$$(\tfrac{3}{2}(1-x_1)^2+x_1-6x_2x_3-\sqrt{2}x_1+\sqrt{2}x_1^2>0) \tag{15}$$

$$(\tfrac{3}{2}(1-x_1)^2+x_1-6x_2x_3)^2+(\tfrac{1}{2}(1-x_1)^2+x_1^2)^2-\tfrac{5}{2}(\tfrac{3}{2}(1-x_1)^2+x_1 -6x_2x_3)(\tfrac{1}{2}(1-x_1)^2+x_1^2)+\tfrac{9}{8}x_1^2(1-x_1^2)<0 \tag{16}$$

In (15), we have included the inequality $b>\sqrt{2}(p+q)$ from Lemma 4.3; the inequality $a>\sqrt{2}(p+q)$ is not useful.

The idea of the rest of the proof of Theorem 4.1 is to show that none of the conditions (14)–(16) hold in the set

$$S=\{x_1\le\sqrt{2}-1,\ x_2\le 1/3,\ x_3\le 1/3\}\cup\{1/3\le x_2\le\sqrt{2}-1,\ 1/3\le x_3\le\sqrt{2}-1\}.$$

By symmetry it will follow that (14)–(16) do not hold in the region $\{x_i\le\sqrt{2}-1, i=1,2,3\}$, and by Lemma 4.2 this is enough to prove the theorem.

The first step is to rewrite the conditions (14)–(16) in more recognizable forms:

$$\left(\frac{3}{2}+\frac{5\sqrt{2}}{4}\right)(x_2^2+x_3^2)+\left(\frac{5\sqrt{2}}{2}-3\right)x_2x_3-\left(1+\frac{5\sqrt{2}}{4}\right)(x_2+x_3)+1>0; \tag{17}$$

$$\left(\frac{3}{2}+\frac{5\sqrt{2}}{4}\right)(x_1-(2-\sqrt{2}))\left(x_1-\left(\frac{2\sqrt{2}}{7}-\frac{1}{7}\right)\right)>0 \quad \text{and} \tag{18a}$$

$$(\tfrac{3}{2}+\sqrt{2})(x_2^2+x_3^2)+(2\sqrt{2}-3)x_2x_3-(1+\sqrt{2})(x_2+x_3)+1>0; \tag{18b}$$

$$(\tfrac{9}{2}(x_2x_3-\tfrac{1}{12})(x_2+(5+2\sqrt{6})x_3-(2+2\sqrt{6}/3))(x_2+(5-2\sqrt{6})x_3-(2-2\sqrt{6}/3))<0; \tag{19}$$

With the conditions in this form, it is a matter of simple but tedious computation to show that none hold in the set S. Inequality (17) represents the exterior of an ellipse in the (x_2, x_3)-plane; by the symmetry in x_2 and x_3 and convexity it suffices to check that the points $\mathbf{a}=(\sqrt{2}-1, \frac{5}{3}-\sqrt{2}, \frac{1}{3})$, $\mathbf{b}=(\frac{5}{3}-\sqrt{2}, \frac{1}{3}, \sqrt{2}-1)$ and $\mathbf{c}=(3-2\sqrt{2}, \sqrt{2}-1, \sqrt{2}-1)$ (or rather the points represented by their last two coordinates) lie inside this ellipse. By direct computation one shows that (17) does not hold for $\mathbf{a}, \mathbf{b}$, or $\mathbf{c}$.

The points $\mathbf{b}$ and $\mathbf{c}$ do not satisfy (18b), which again represents the exterior of an ellipse. However the point **a**, and in fact the point $\mathbf{o}=(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ do satisfy (18b). Now (18a) represents the exterior of an infinite strip, and it can be verified that **a** and **o** do not satisfy (18a). Next, check that the point $\mathbf{d}=(2\frac{2\sqrt{2}}{7}-\frac{1}{7}, \frac{1}{3}, \frac{17}{21}-2\frac{2\sqrt{2}}{7})$, where the line $\{x_1=\frac{2\sqrt{2}}{7}-\frac{1}{7}\}$ meets the boundary of S, does not satisfy (18b). Then, by symmetry in x_2 and x_3, and convexity, it follows that (18a) and (18b) do not jointly hold at any point in S.

Finally, we must show that (19) is not satisfied in S. Note that the three factors of (19) represent a hyperbola (symmetric in x_2 and x_3) and a pair of straight lines. By checking the points **a** and **b**, one verifies that S lies inside the hyperbola (i.e. in the region $x_2x_3>\frac{1}{12}$). Each of the straight lines meets S only in the point **o**; then it is easily seen that for points in S the product of the two corresponding factors in (19) is non-negative. Consequently (19) does not hold in S, and the theorem is proved.

5. **Conjectures.** We propose the following conjectures.

CONJECTURE 5.1

$$\frac{|X_1|}{|Y_1|}+\frac{|X_2|}{|Y_2|}+\frac{|X_3|}{|Y_3|}\geq\frac{3}{2}.$$

CONJECTURE 5.2

$$\frac{|X_2|+|X_3|}{|Y_1|}+\frac{|X_3|+|X_1|}{|Y_2|}+\frac{|X_1|+|X_2|}{|Y_3|}\geq 3.$$

In Conjectures 5.1 and 5.2, we further conjecture that equality holds if and only if X is a triangle and the lines L_i pass through the centroid parallel to $A_{i+1}A_{i+2}$, $i=1, 2, 3$.

Note that Theorem 4.1 follows from Conjecture 5.1, while Conjecture 5.2 implies our Theorem 3.1.

We thank Mr. Neill Robertson for writing computer programs which yield strong evidence for the truth of these conjectures.

Substituting from (1), we see that Conjecture 5.1 is equivalent to

$$\sum_{i=1}^{3}\frac{x_i^2(\lambda_{i+1}+(1/\lambda_{i-1})-1)}{(x_{i-1}+x_{i+1})^2-\lambda_{i-1}x_{i+1}^2-x_{i-1}^2/\lambda_{i+1}}\geq\frac{3}{2} \tag{20}$$

where $x_i\geq 0$, $\lambda_i>0$, and

$$0\leq x_i+(1-1/\lambda_i)x_{i+1} \quad\text{and}\quad 0\leq x_i+(1-\lambda_i)x_{i-1}, \qquad i=1, 2, 3. \tag{21}$$

As (20) is homogeneous in the x_i's, all six variables may be taken as independent. Notice that (20) ensures that numerators and denominators in (20) are non-negative.

It is possible to show, by the usual calculations involving derivatives, that (20) holds in a neighborhood of (1, 1, 1, 1, 1, 1). In the special case $x_1=x_2=x_3$, and in the case $\lambda_1 = \lambda_2 = \lambda_3$, (20) may be verified by an appropriate use of the geometric-arithmetic mean inequality. The following partial result is more interesting:

THEOREM 5.3. *Conjecture* 5.1 *is true if* $\lambda_1\lambda_2\lambda_3 = 1$.

Proof. Replace the term $(2x_{i-1}x_{i+1})$ in the denominators of (20) by $(\lambda_{i-1}\lambda_{i+1}x_{i+1}^2 + x_{i-1}^2/\lambda_{i-1}\lambda_{i+1})$; the sum cannot increase. Then (20) reduces to

$$\sum_{i=1}^{3} \frac{x_i^2}{\lambda_{i-1}x_{i+1}^2 + x_{i-1}^2/\lambda_{i+1}} \geq \frac{3}{2},$$

or, using $\lambda_1\lambda_2\lambda_3 = 1$, to

$$\frac{\lambda_2 x_1^2}{(x_2^2/\lambda_1) + x_3^2} + \frac{x_2^2/\lambda_1}{x_3^2 + \lambda_2 x_1^2} + \frac{x_3^2}{\lambda_2 x_1^2 + (x_2^2/\lambda_1)} \geq \frac{3}{2}.$$

But this is equivalent to the known inequality (see below)

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2} \tag{22}$$

for $a, b, c \geq 0$.

The inequality (22) and its natural generalization to one with n terms have been the subject of many papers. For a proof of (22), due to L. J. Mordell, and an interesting historical account, see D. S. Mitrinović ([3], p. 144).

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