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SETS OF IDEMPOTENTS THAT GENERATE THE SEMIGROUP OF SINGULAR ENDOMORPHISMS OF A FINITE-DIMENSIONAL VECTOR SPACE

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If $\mathbf{M}$ is a mathematical system and $\operatorname{End}\mathbf{M}$ is the set of singular endomorphisms of $\mathbf{M}$, then $\operatorname{End}\mathbf{M}$ forms a semigroup under composition of mappings. A number of papers have been written to determine the subsemigroup $S_{\mathbf{M}}$ of $\operatorname{End}\mathbf{M}$ generated by the idempotents $E_{\mathbf{M}}$ of $\operatorname{End}\mathbf{M}$ for different systems $\mathbf{M}$. The first of these was by J. M. Howie [4]; here the case of $\mathbf{M}$ being an unstructured set $\mathbf{X}$ was considered. Howie showed that if $\mathbf{X}$ is finite, then $\operatorname{End}\mathbf{X}=S_{\mathbf{X}}$.

Soon afterwards, J. A. Erdös [3] considered the case of $\mathbf{M}$ being a finite-dimensional vector space $\mathbf{V}$ over an arbitrary field F. Erdös showed that in this case also $\operatorname{End}\mathbf{V}=S_{\mathbf{V}}$. I have given two alternative proofs in [2]. J. B. Kim [6] has also given a proof of this result if the field F is algebraically closed. The proofs given by Erdös and myself show that $\operatorname{End}\mathbf{V}$ is, in fact, generated by the subset E of $E_{\mathbf{V}}$ consisting solely of the elements of $E_{\mathbf{V}}$ with one-dimensional null-space. It is easy to show that, except in the trivial case of $\mathbf{V}$ being a one-dimensional vector space, $\operatorname{End}\mathbf{V}$ may be generated by a proper subset of E.

In this paper I determine conditions that are necessarily satisfied by a subset E' of E if E' generates $\operatorname{End}\mathbf{V}$. I then show, if the field F is finite, that these conditions are also sufficient. From this, again if F is finite, the minimum order of a generating set of idempotents is determined.

1. Notation and preliminary results

Definition 1.1. The semigroup of singular endomorphisms of an n-dimensional vector space $\mathbf{V}$ over a field F will be denoted by Sing_n. Let $\alpha\in\operatorname{Sing}_n$. The range of α will be denoted by $\mathbf{R}_\alpha$ and the null-space of α by $\mathbf{N}_\alpha$. Elements of Sing_n will be written on the right of elements of the vector space $\mathbf{V}$.

Using this "right mapping" convention the following lemma is immediate:

Lemma 1.2. *Let* $\alpha,\beta\in\operatorname{Sing}_n$. *Then:*

(a) $\mathbf{N}_\alpha\subseteq\mathbf{N}_{\alpha\beta}$,
(b) $\mathbf{R}_{\alpha\beta}\subseteq\mathbf{R}_\beta$,
(c) α, β *and* $\alpha\beta$ *all have the same rank if and only if* $\mathbf{N}_\alpha=\mathbf{N}_{\alpha\beta}$ *and* $\mathbf{R}_{\alpha\beta}=\mathbf{R}_\beta$.

The following simple lemma will also be used.

Lemma 1.3. ([1, Exercise 2.2.6.]) *Let* $\alpha, \beta \in \mathrm{Sing}_n$. *Then:*

(a) $\alpha \mathscr{L} \beta$ *if and only if* $\mathbf{R}_\alpha = \mathbf{R}_\beta$,
(b) $\alpha \mathscr{R} \beta$ *if and only if* $\mathbf{N}_\alpha = \mathbf{N}_\beta$,
(c) $\alpha \mathscr{D} \beta$ *if and only if* α *and* β *have the same rank,*
(d) $\alpha \mathscr{J} \beta$ *if and only if* $\alpha \mathscr{D} \beta$.

Definition 1.4. The principal factor of Sing_n containing those elements of rank $n-1$ will be denoted by PF_{n-1}^0. The set of elements of Sing_n of rank $n-1$ will be denoted by PF_{n-1}. Thus PF_{n-1} consists of the non-zero elements of PF_{n-1}^0.

The remainder of this section is devoted to introducing (and using) a new notation for the $\mathscr{H}$-classes of PF_{n-1}^0. This can quickly be adapted to serve as a new notation for elements of E (the non-zero idempotents of PF_{n-1}^0).

Definition 1.5. Let ξ, χ be automorphisms of the field F such that $(\chi\xi^{-1})^2$ is the identity mapping. Let $\mathbf{a} = (a_1, a_2, a_3, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, b_3, \ldots, b_n)$ be elements of $\mathbf{V}$. The (ξ, χ)-*stroke product* (or simply *stroke product*) of $\mathbf{a}$ with $\mathbf{b}$ is denoted by $\langle \mathbf{a} \mid \mathbf{b} \rangle_{(\xi,\chi)}$ (or simply $\langle \mathbf{a} \mid \mathbf{b} \rangle$) and defined by

$$\langle \mathbf{a} \mid \mathbf{b} \rangle = \sum_{i=1}^{n} (a_i\xi)(b_i\chi).$$

I shall regard ξ and χ as fixed in advance and shall not make explicit reference to them in definitions and statements.

Definition 1.6. If $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ are elements of $\mathbf{V}$, we shall say that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular if $\langle \mathbf{a} \mid \mathbf{b} \rangle = 0$. It is simple to check that perpendicularity is a symmetric relation.

If $\mathbf{A}$ is a subset of $\mathbf{V}$, we shall define the perpendicular of $\mathbf{A}$ to be

$$\mathbf{A}^\perp = \{\mathbf{x} \in \mathbf{V} : \langle \mathbf{x} \mid \mathbf{a} \rangle = \mathbf{0} \quad (\forall \mathbf{a} \in \mathbf{A})\}.$$

It should be noted that in general $\mathbf{A}$ and $\mathbf{A}^\perp$ are not disjoint. It should also be noted that $\mathbf{A}^\perp$ is a subspace of $\mathbf{V}$.

Using this definition of perpendicularity, the following lemma is simple to prove.

Lemma 1.7. *Let* $\mathbf{U}$ *and* $\mathbf{W}$ *be subspaces of an n-dimensional vector space* $\mathbf{V}$. *Then:*

(a) $\dim \mathbf{U}^\perp = n - \dim \mathbf{U}$,
(b) $(\mathbf{U}^\perp)^\perp = \mathbf{U}^\perp$,
(c) *if* $\mathbf{U} \subset \mathbf{W}$, *then* $\mathbf{W}^\perp \subset \mathbf{U}$.

Notation 1.8. Since every element in any particular $\mathscr{R}$-class of PF_{n-1}^0 has the same one-dimensional null-space we can label the $\mathscr{R}$-classes of PF_{n-1}^0 in the obvious way with an element of $\mathbf{V}$ that generates this one-dimensional subspace of $\mathbf{V}$. Similarly, the $\mathscr{L}$-classes of PF_{n-1}^0 could be labelled in the obvious way with $n-1$ elements of $\mathbf{V}$ that generate the common range. But, since, if $\dim \mathbf{U} = n-1$, we have (by Lemma 1.7) that $\dim \mathbf{U}^\perp = 1$, it follows that we can label the $\mathscr{L}$-classes of PF_{n-1}^0 in an obvious way with an element of $\mathbf{V}$ that generates the one-dimensional subspace of $\mathbf{V}$ perpendicular to the common range of the elements in that $\mathcal{L}$-class. Thus if α is a non-zero element of PF_{n-1}^0 such that $\mathbf{N}_\alpha = \langle \mathbf{n} \rangle$ and $\mathbf{R}_\alpha = \langle \mathbf{r} \rangle$ then we can label the $\mathcal{L}$-class containing α by $L_\mathbf{r}$, the $\mathcal{R}$-class containing α by $R_\mathbf{n}$ and the $\mathcal{H}$-class containing α by $H_{\mathbf{n},\mathbf{r}}$. As $H_{\mathbf{n},\mathbf{r}}$ is rather unwieldy this will in future be denoted by $[\mathbf{n}:\mathbf{r}]$. It is clear that $[\mathbf{n}:\mathbf{r}]$ denotes exactly one $\mathcal{H}$-class for any choice of $\mathbf{n}$ and $\mathbf{r}$ in $\mathbf{V}$ (the fact that $[\mathbf{n}:\mathbf{r}]$ represents at least one $\mathcal{H}$-class of PF_{n-1}^0 is a result of Lemma 1.3). It is also clear that for any non-zero scalars λ and μ we have $[\mathbf{n}:\mathbf{r}] = [\lambda\mathbf{n}:\mu\mathbf{r}]$.

Having adopted this notation, it is then reasonable to introduce the following: If $[\mathbf{n}:\mathbf{r}]$ is a group $\mathcal{H}$-class of PF_{n-1}^0 we shall denote the idempotent in $[\mathbf{n}:\mathbf{r}]$ by $(\mathbf{n}:\mathbf{r})$. $(\mathbf{n}:\mathbf{r})$ is clearly unique since no $\mathcal{H}$-class contains more than one idempotent. With this notation there is a very simple way to tell if a particular $\mathcal{H}$-class of PF_{n-1}^0 contains an idempotent.

Lemma 1.9. *$[\mathbf{n}:\mathbf{r}]$ is a group $\mathcal{H}$-class if and only if $\langle \mathbf{n} | \mathbf{r} \rangle \neq 0$.*

Proof. Suppose that $[\mathbf{n}:\mathbf{r}]$ is a group $\mathcal{H}$-class. Then $[\mathbf{n}:\mathbf{r}]$ contains the idempotent $\varepsilon = (\mathbf{n}:\mathbf{r})$. Now $\mathbf{N}_\varepsilon \cap \mathbf{R}_\varepsilon = \{\mathbf{0}\}$ (for if $\mathbf{x} \in \mathbf{N}_\varepsilon \cap \mathbf{R}_\varepsilon$ then $\mathbf{x} = \mathbf{x}\varepsilon = \mathbf{0}$) and since $\mathbf{n} \in \mathbf{N}_\varepsilon$ and $\mathbf{n} \neq \mathbf{0}$ we have $\mathbf{n} \notin \mathbf{R}_\varepsilon = (\mathbf{R}_\varepsilon^\perp)^\perp$. But, since $\mathbf{r} \in \mathbf{R}_\varepsilon$ and $\mathbf{R}_\varepsilon^\perp$ is one-dimensional, we have $\langle \mathbf{n} | \mathbf{r} \rangle \neq 0$.

Conversely, suppose $\langle \mathbf{n} | \mathbf{r} \rangle \neq 0$. Now, there exists an element $\alpha \in PF_{n-1}^0$ such that $\mathbf{N}_\alpha = \langle \mathbf{n} \rangle$ and $\mathbf{R}_\alpha^\perp = \langle \mathbf{r} \rangle$. Since $\langle \mathbf{n} | \mathbf{r} \rangle \neq 0$, we have $\lambda\mathbf{n} \notin (\mathbf{R}_\alpha^\perp)^\perp = \mathbf{R}_\alpha$ for any non-zero scalar λ in F, i.e. $\mathbf{R}_\alpha \cap \mathbf{N}_\alpha = \{\mathbf{0}\}$. Let $\mathbf{x} \in \mathbf{N}_{\alpha^2}$. Then $\mathbf{x}\alpha \in \mathbf{R}_\alpha \cap \mathbf{N}_\alpha$. Thus $\mathbf{x}\alpha = \mathbf{0}$ and so $\mathbf{x} \in \mathbf{N}_\alpha$. Consequently $\mathbf{N}_{\alpha^2} \subseteq \mathbf{N}_\alpha$. But $\mathbf{N}_\alpha \subseteq \mathbf{N}_{\alpha^2}$ and so $\mathbf{N}_\alpha = \mathbf{N}_{\alpha^2}$. Thus $\alpha \mathcal{R} \alpha^2$. Also, since $\dim \mathbf{N}_\alpha = \dim \mathbf{N}_{\alpha^2}$, we have $\dim \mathbf{R}_\alpha = \dim \mathbf{R}_{\alpha^2}$. But $\mathbf{R}_{\alpha^2} \subseteq \mathbf{R}_\alpha$ and so $\mathbf{R}_\alpha = \mathbf{R}_{\alpha^2}$. Thus $\alpha \mathcal{L} \alpha^2$. Hence $\alpha \mathcal{H} \alpha^2$. So (by [5, Theorem II.2.5.]) H_α is a group and so contains an idempotent. Since $H_\alpha = [\mathbf{n}:\mathbf{r}]$, the result is proved.

Lemma 1.10. *Let α and β be elements of PF_{n-1}^0 in $[\mathbf{n}:\mathbf{r}]$ and $[\mathbf{m}:\mathbf{s}]$ respectively. Then $\alpha\beta \neq 0$ if and only if $\langle \mathbf{m} | \mathbf{r} \rangle \neq 0$.*

Proof. Suppose first that $\alpha\beta \neq 0$. Then $\alpha\beta$, α and β all have the same rank. So, by Lemma 1.2 and Lemma 1.3, $\alpha\beta \in R_\alpha \cap L_\beta$. By [1, Theorem 2.17.], $R_\beta \cap L_\alpha$ contains an idempotent, i.e. $\langle \mathbf{m} | \mathbf{r} \rangle \neq 0$.

Now suppose that $\langle \mathbf{m} | \mathbf{r} \rangle \neq 0$. Then $R_\beta \cap L_\alpha$ contains an idempotent. So, again by [1, Theorem 2.17.], $\alpha\beta \in R_\alpha \cap L_\beta$. Thus $\alpha\beta$ has rank $n-1$, and so $\alpha\beta \neq 0$.

2. The necessary conditions

In this section necessary conditions are found for a subset E' of E to generate Sing_n. Throughout this section there are no restrictions on the field F over which the vector space $\mathbf{V}$ is defined.

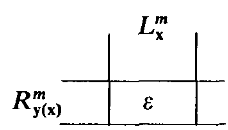
Definition 2.1. Let E' be a subset of E. We shall say that E' *covers* [*sparsely covers*] PF_{n-1}^0 if E' has non-empty intersection with [intersects in exactly one element] each non-zero $\mathcal{L}$-class and each non-zero $\mathcal{R}$-class of PF_{n-1}^0. We shall also say that E' covers PF_{n-1}.

Lemma 2.2. *There exists a sparse covering set E' for PF^0_{n-1}.*

Proof. The proof is by induction on the dimension n of the vector space $\mathbf{V}$. For clarity we shall denote the m-dimensional vector space by $\mathbf{V}_m$.

We now define a set of representatives $\mathbf{V}'_m$ of the one-dimensional subspaces of $\mathbf{V}_m$. So, for all non-zero $\mathbf{x}$ in $\mathbf{V}_m$ there exists a unique $\mathbf{y}$ in $\mathbf{V}'_m$ such that $\langle \mathbf{x}\rangle = \langle \mathbf{y}\rangle$. We shall denote by $L^m_{\mathbf{x}}[R^m_{\mathbf{x}}]$ the $\mathscr{L}$-class [$\mathscr{R}$-class] of PF^0_{m-1} containing those elements with range perpendicular to [null-space equal to] $\langle \mathbf{x}\rangle$.

Now suppose, as the induction hypothesis, that there exists a sparse covering set E'_m of PF^0_{m-1}. Then there exists exactly one element ε in $L^m_{\mathbf{x}} \cap E'_m$ for each $\mathbf{x} \in \mathbf{V}'_m$. All the elements in $\mathbf{R}_\varepsilon$ have the same null-space, generated by a particular element of $\mathbf{V}'_m$. If we denote this element by $\mathbf{y}(\mathbf{x})$, we have, in fact, defined a mapping $\mathbf{V}'_m \to \mathbf{V}'_m$ by $\mathbf{x} \mapsto \mathbf{y}(\mathbf{x})$. This mapping is characterised by $L^m_{\mathbf{x}} \cap R^m_{\mathbf{y}(\mathbf{x})} \cap E'_m$ being non-empty.



This mapping is clearly a bijection. Notice that there exists an idempotent, namely ε, with null-space $\langle \mathbf{y}(\mathbf{x})\rangle$ and range $\langle \mathbf{x}\rangle^\perp$. Thus $[\mathbf{y}(\mathbf{x}):\mathbf{x}]$ is a group $\mathscr{H}$-class, and so $\langle \mathbf{y}(\mathbf{x}) \,|\, \mathbf{x}\rangle \neq 0$.

If $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ is an element of $\mathbf{V}'_m$ and $a \in F$, then denote by $(\mathbf{x}, a)$ the element of $\mathbf{V}'_{m+1}$ that generates the space $\langle (x_1, x_2, \ldots, x_m, a)\rangle$. We shall denote by $(\mathbf{0}, 1)$ the element of $\mathbf{V}'_{m+1}$ that generates the space $\langle (0, 0, \ldots, 0, 1)\rangle$. Clearly, these are all distinct, and every element of $\mathbf{V}'_{m+1}$ may be denoted in this way. Notice that if $\mathbf{y} = (y_1, y_2, \ldots, y_{m+1})$, then for some $\mathbf{x} \in \mathbf{V}'_m \cup \{\mathbf{0}\}$ and some λ, $a \in F$ we have $(y_1, y_2, \ldots, y_m) = \lambda \mathbf{x}$ and $y_{m+1} = \lambda a$.

We shall now set up a bijection $\bar{\mathbf{y}}: V'_{m+1} \to V'_{m+1}$ such that $L^{m+1}_{(\mathbf{x},a)} \cap R^{m+1}_{\mathbf{y}(\mathbf{x},a)}$ is a group $\mathscr{H}$-class of PF^0_m for all $\mathbf{x}$ in $\mathbf{V}'_m$ and all a in F, and also such that $L^{m+1}_{(0,1)} \cap R^{m+1}_{(0,1)}$ is a group $\mathscr{H}$-class of PF^0_m. We shall construct $\bar{\mathbf{y}}$ so that for $\mathbf{x}$ in $\mathbf{V}'_m$ and a in F we have $\bar{\mathbf{y}}(\mathbf{x}, a) = (\mathbf{y}(\mathbf{x}), z)$ for some z in F. We need to have $\langle \bar{\mathbf{y}}(\mathbf{x}, a) \,|\, (\mathbf{x}, a)\rangle \neq 0$, and so we must have $\langle \mathbf{y}(\mathbf{x}) \,|\, \mathbf{x}\rangle + (z\xi)(a\chi) \neq 0$. Now, by the definition of $\mathbf{y}(\mathbf{x})$, we know that $\langle \mathbf{y}(\mathbf{x}) \,|\, \mathbf{x}\rangle \neq 0$. Thus, if $a \neq 0$, we need $z\xi \neq -\langle \mathbf{y}(\mathbf{x}) \,|\, \mathbf{x}\rangle/(a\chi)$ and, if $a = 0$, z may take any value we choose. We know that F contains the elements 0 and 1. Thus if $a \neq 0$, we may put $z\xi = 1 - \langle \mathbf{y}(\mathbf{x}) | \mathbf{x}\rangle/(a\chi)$. The only value that this may not take is 1 since $\langle \mathbf{y}(\mathbf{x}) | \mathbf{x}\rangle \neq 0$. So if $a = 0$ we shall set $z = 1$. Thus

$$\bar{\mathbf{y}}(\mathbf{x}, a) = \begin{cases} (\mathbf{y}(\mathbf{x}), b(\mathbf{x}, a)) & \text{if} \quad \mathbf{x} \in \mathbf{V}'_m \\ (\mathbf{0}, 1) & \text{if} \quad \mathbf{x} = \mathbf{0} \quad \text{and} \quad a = 1 \end{cases},$$

where

$$b(\mathbf{x}, a) = \begin{cases} [1 - \langle \mathbf{y}(\mathbf{x}) \,|\, \mathbf{x}\rangle/(a\chi)]\xi^{-1} & \text{if} \quad a \neq 0 \\ 1\xi^{-1} = 1 & \text{if} \quad a = 0 \end{cases}.$$

It is easy to check that $\bar{\mathbf{y}}$ is a bijection.

From the definition of $\bar{\mathbf{y}}$ we have that for all $(\mathbf{x},a)$ in $\mathbf{V}'_{m+1}$, $\langle \bar{\mathbf{y}}(\mathbf{x},a)|(\mathbf{x},a)\rangle \neq 0$. Thus $L^{m+1}_{(\mathbf{x},a)} \cap R^{m}_{\bar{\mathbf{y}}(\mathbf{x},a)}$ contains an idempotent. Hence the set $E'_{m+1} = \{(\bar{\mathbf{y}}(\mathbf{x},a):(\mathbf{x},a)):(\mathbf{x},a)\in \mathbf{V}'_{m+1}\}$ is a sparse cover for PF^0_m.

It remains to show that we may anchor the induction at $m=2$. Since every one-dimensional subspace of V_2 may be generated by a vector of the form $(1,a)$ or by the vector $(0,1)$, it is easy to see that the set

$$\{((1,(1-(a\chi)^{-1})\xi^{-1}):(1,a)):a\in F\backslash\{0\}\} \cup [((1,1):(1,0)),\ ((0,1):(0,1))\}$$

forms a sparse cover for PF^0_1.

Definition 2.3. Let E' be a subset of E and ϕ, $\gamma \in E'$. Then the relation $\pi(E')$ is defined by: $(\phi,\gamma)\in\pi(E')$ if there exist elements $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_p$ in E' such that $\phi\varepsilon_1\varepsilon_2\dots\varepsilon_p\gamma\in PF_{n-1}$.

Lemma 2.4. *If E' is a subset of E and E' generates* Sing$_n$, *then E' covers PF_{n-1} and $\pi(E')$ is the universal relation on E'.*

Proof. Let β be any element of PF_{n-1}. Since E' generates Sing$_n$, it certainly generates PF_{n-1}. Thus there exist elements $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_p$ in E' such that $\beta=\varepsilon_1\varepsilon_2\dots\varepsilon_p$. Now, since rank $\beta=$rank ε_i $(i=1,2,\dots,p)$, we have that $\mathbf{N}_\beta=\mathbf{N}_{\varepsilon_1}$ and $\mathbf{R}_\beta=\mathbf{R}_{\varepsilon_p}$. Thus $\beta\mathscr{R}\varepsilon_1$ and $\beta\mathscr{L}\varepsilon_p$. Hence both $R_\beta\cap E'$ and $L_\beta\cap E'$ are non-empty. Since β was chosen arbitrarily, it follows that E' covers PF_{n-1}.

Now let ϕ, $\gamma\in E'$, and let $\alpha\in R_\phi\cap L_\gamma$. Since E' generates α we have that $\alpha=\varepsilon_1\varepsilon_2\dots\varepsilon_p$ for some $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_p$ in E'. But $\varepsilon_1\mathscr{R}\alpha$ and $\varepsilon_p\mathscr{L}\alpha$. Thus $\phi\mathscr{R}\varepsilon_1$ and $\gamma\mathscr{L}\varepsilon_p$. Hence $\phi\varepsilon_1=\varepsilon_1$ and $\varepsilon_p\gamma=\varepsilon_p$. So $\alpha=\phi\varepsilon_1\varepsilon_2\dots\varepsilon_p\gamma$, i.e. $\phi\varepsilon_1\varepsilon_2\dots\varepsilon_p\gamma\in PF_{n-1}$. Since $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_p\in E'$, we have that $(\phi,\gamma)\in\pi(E')$. Since ϕ and γ were chosen arbitrarily, it follows that $\pi(E')$ is the universal relation on E'.

3. Sufficient conditions and minimum order generating sets when F is a finite field

Throughout this section we shall take F to be the finite field with q elements.

Theorem 3.1. *Let* $\mathbf{V}$ *be an n-dimensional vector space over a finite field F. Let* Sing$_n$ *be the semigroup of singular endomorphisms of* $\mathbf{V}$ *and let PF_{n-1} be the set of elements in* Sing$_n$ *with rank $n-1$. Let E' be a subset of the idempotents of PF_{n-1}. Then E' generates* Sing$_n$ *if and only if $\pi(E')$ is the universal relation on E' and E' covers PF_{n-1}.*

Proof. We have already shown that if E' generates Sing$_n$ then $\pi(E')$ is universal on E' and E' covers PF_{n-1}.

To show the converse it will suffice to show that E' generates E, the set of all idempotents in PF_{n-1}, for (by [3]) we have that E generates Sing$_n$.

Let $\alpha\in E$. Since E' covers PF_{n-1}, there exist ϕ,γ in E' such that $\phi\mathscr{R}\varepsilon$ and $\gamma\mathscr{L}\varepsilon$. Since $\pi(E')$ is universal on E', we have that $(\phi,\gamma)\in\pi(E')$. Hence there exist $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_p$ in E' such that $\alpha=\phi\varepsilon_1\varepsilon_2\dots\varepsilon_p\gamma$ has rank $n-1$. Now, $\mathbf{N}_\alpha=\mathbf{N}_\phi$ and $\mathbf{R}_\alpha=\mathbf{R}_\gamma$. Thus $\alpha\mathscr{R}\phi$ and $\alpha\mathscr{L}\gamma$. Hence $\alpha\mathscr{R}\varepsilon$ and $\alpha\mathscr{L}\varepsilon$, i.e. $\alpha\mathscr{H}\varepsilon$. Now, since F is finite, Sing$_n$ is finite and so

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certainly H_ε is finite. So α belongs to a finite group. Thus, for some integer $k \geqq 1$, α^k is the identity of that group, i.e. $\alpha^k = \varepsilon$. Since α is a product of elements of E', we have that E' generates ε. But this holds for all elements of E and so E' generates E as required.

The next three lemmas will be used in the proof of Theorem 3.5.

Lemma 3.2. *If* $|F| = q$, *then the number of non-zero* $\mathscr{L}$*-classes* [$\mathscr{R}$*-classes*] *in* PF_{n-1}^0 *is* $(q^n - 1)/(q-1)$.

Proof. By Lemma 2.2, we know that there is a bijection between the elements of a sparse cover of PF_{n-1}^0 and the $\mathscr{L}$-classes [$\mathscr{R}$-classes] of PF_{n-1}^0. Thus there is a bijection between the $\mathscr{L}$-classes and $\mathscr{R}$-classes of PF_{n-1}^0. Since F is finite it follows that PF_{n-1}^0 is finite and so there are only finitely many $\mathscr{L}$-classes [$\mathscr{R}$-classes] in PF_{n-1}^0. Consequently there are the same number of $\mathscr{L}$-classes as $\mathscr{R}$-classes in PF_{n-1}^0.

From the comments of Notation 1.8, we know that there is a bijection between the one-dimensional subspaces of $\mathbf{V}$ and the non-zero $\mathscr{L}$-classes of PF_{n-1}^0. Now, the number of non-zero vectors in $\mathbf{V}$ is $q^n - 1$. However, for each $\mathbf{x}$ in $\mathbf{V}$ and for all non-zero scalars λ in F, we have $\langle \mathbf{x} \rangle = \langle \lambda \mathbf{x} \rangle$. Hence there are $(q^n - 1)/(q-1)$ one-dimensional subspaces in $\mathbf{V}$.

Lemma 3.3. *If* $|F| = q$, *then the number of idempotents in any non-zero* $\mathscr{L}$*-class* [$\mathscr{R}$*-class*] *of* PF_{n-1}^0 *is* q^{n-1}.

Proof. The number of idempotents in a given $\mathscr{L}$-class L is the number of $\mathscr{R}$-classes containing an idempotent in L, i.e. the number of $\mathscr{R}$-classes which intersect L in a group. If the elements in L have range $\langle \mathbf{r} \rangle$ then this is just $Q = |\{\langle \mathbf{n} \rangle : \langle \mathbf{n} | \mathbf{r} \rangle \neq 0\}|$. Since the number of one-dimensional subspaces of $\mathbf{V}$ is $(q^n - 1)/(q-1)$, we have that

$$Q = (q^n - 1)/(q-1) - |\{\langle \mathbf{n} \rangle : \langle \mathbf{n} | \mathbf{r} \rangle = 0\}|.$$

But $\{\langle \mathbf{n} \rangle : \langle \mathbf{n} | \mathbf{r} \rangle = 0\} = \{\langle \mathbf{n} \rangle : \mathbf{n} \in \langle \mathbf{r} \rangle^\perp\}$. Since $\dim \langle \mathbf{r} \rangle^\perp = n-1$. we have that $|\{\langle \mathbf{n} \rangle : \mathbf{n} \in \langle \mathbf{r} \rangle^\perp\}| = (q^{n-1} - 1)/(q-1)$. Thus $Q = (q^n - 1)/(q-1) - (q^{n-1} - 1)/(q-1) = q^{n-1}$ as required.

Lemma 3.4. *If* F *is a finite field and* E' *is a sparse cover for* PF_{n-1}^0, *then* $\pi(E')$ *is the universal relation on* E'.

Proof. Let ϕ, γ be any two elements of E' and suppose that $\phi\pi(E') \cap \gamma[\pi(E')]^{-1}$ is empty. Since each $\mathscr{L}$-class of PF_{n-1}^0 contains q^{n-1} idempotents and E' is a sparse cover of PF_{n-1}^0, we know that there are exactly q^{n-1} elements ε_i of E' such that $\phi\varepsilon_i \neq 0$ in PF_{n-1}^0 (by Lemma 1.9 and Lemma 1.10). Hence $|\phi\pi(E')| \geqq q^{n-1}$. Similarly, since each $\mathscr{R}$-class of PF_{n-1}^0 contains q^{n-1} idempotents, we have that there exist exactly q^{n-1} elements ε_i' of E' such that $\varepsilon_i'\gamma \neq 0$ in PF_{n-1}^0. Thus $|\gamma[\pi(E')]^{-1}| \geqq q^{n-1}$. Now, since we have assumed that $\phi\pi(E') \cap \gamma[\pi(E')]^{-1}$ is empty, we have

$$\begin{aligned}(q^n - 1)/(q-1) = |E'| &\geqq |\phi\pi(E') \cup \gamma[\pi(E')]^{-1}| \\ &= |\phi\pi(E')| + |\gamma[\pi(E')]^{-1}| \geqq q^{n-1} + q^{n-1} = 2q^{n-1}.\end{aligned}$$

Thus $q^{n-1}(q-2) \leqq -1$, which is impossible since $q \geqq 2$. Consequently, $\phi\pi(E') \cap \gamma[\pi(E')]^{-1}$ contains some element, ε say. Thus $(\phi, \varepsilon) \in \pi(E')$ and $(\varepsilon, \gamma) \in \pi(E')$. Since $\pi(E')$ is transitive, it follows that $(\phi, \gamma) \in \pi(E')$.

We now have:

Theorem 3.5. *Let* $\mathbf{V}$ *be an n-dimensional vector space over a finite field F. Let* Sing_n *denote the semigroup of singular endomorphisms of* $\mathbf{V}$ *and let* PF_{n-1} *be the set of elements of* Sing_n *with rank* $n-1$. *Then there exists a subset* E' *of the idempotents of* PF_{n-1} *such that* E' *is a sparse cover for* PF_{n-1} *and* E' *generates* Sing_n. *Further, any sparse cover for* PF_{n-1} *generates* Sing_n.

Proof. This is immediate from Lemma 2.2, Theorem 3.1 and Lemma 3.4.

(If F is an arbitrary field, the above theorem no longer holds. A counter-example to a generalisation of Theorem 3.5 may be found in [2], as may a proof of the following weaker result.

Theorem 3.6. *Let* $\mathbf{V}$ *be an n-dimensional vector space over an arbitrary field F. Let* Sing_n *denote the semigroup of singular endomorphisms of* $\mathbf{V}$ *and let* PF_{n-1} *be the set of elements of* Sing_n *with rank* $n-1$. *Then there exists a subset* E' *of the idempotents of* PF_{n-1} *such that* E' *is a sparse cover for* PF_{n-1} *and* E' *generates* Sing_n.)

Corollary 3.7. *Let* $\mathbf{V}$ *be an n-dimensional vector space over a finite field F of order q. Let* Sing_n *be the semigroup of singular endomorphisms of* $\mathbf{V}$ *and let E be the idempotents of* Sing_n *of rank* $n-1$. *Then*

$$\min\{|E'| : E' \subseteq E, \langle E' \rangle = \mathrm{Sing}_n\} = (q^n - 1)/(q-1).$$

Proof. This is immediate from Lemma 2.4, Lemma 3.2 and Theorem 3.5.

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