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SOME EXTENSIONS OF ADDITIVE PROPERTIES OF GENERAL SEQUENCES

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Let $A = \{a_1, a_2, \ldots\}(a_1 < a_2 < \cdots)$ be an infinite sequence of positive integers. Let $k \geqslant 2$ be a fixed integer and denote by $R_k(n)$ the number of solutions of $n = a_{i_1} + a_{i_2} + \cdots + a_{i_k}$. Erdős, Sárközy and Sós studied the boundness of $|R_2(n+1) - R_2(n)|$ and the monotonicity property of $R_2(n)$. In this paper, we extend some results to $k > 2$.

1. INTRODUCTION

Let $k \geqslant 2$ be a fixed integer and let $A = \{a_1, a_2, \ldots\}(a_1 < a_2 < \cdots)$ be an infinite sequence of positive integers. We write

$$f(z) = \sum_{a \in A} z^a, \quad A(n) = \sum_{\substack{a \in A \\ a \leqslant n}} 1, \quad B(A, n) = \sum_{\substack{a-1 \notin A \\ a \in A,\, a \leqslant n}} 1.$$

For $n = 0, 1, 2, \ldots$ let $R_k(n)$ denote the number of solutions of

$$a_{i_1} + a_{i_2} + \cdots + a_{i_k} = n, \quad a_{i_1} \in A,\ a_{i_2} \in A, \ldots,\ a_{i_k} \in A.$$

Then the generating function of $R_k(n)$ is $f^k(z)$.

Erdős, Sárközy and Sós studied the representation function $R_2(n)$. For examples, in [**2, 3**], they examined the possible order of growth of the function $R_2(n)$ in comparison with that of functions such as $\log n$ or $\log n \log\log n$; in [**4**], they showed that under certain assumptions on A, $|R_2(n+1) - R_2(n)|$ cannot be bounded; in [**5**], they proved that $R_2(n+1) \geqslant R_2(n)$ for all large n if and only if $A(N) = N + O(1)$.

It is natural to extend these results to the case of k summands, that is, to the function $R_k(n)$. In [**6**], Horváth extended the result in [**2**] to $k > 2$. He showed that if $F(n)$ is a monotonic increasing arithmetic function with $F(n) \to +\infty$ and $F(n) = o\big(n(\log n)^{-2}\big)$, then $|R_k(n) - F(n)| = o\Big(\big(F(n)\big)^{1/2}\Big)$ cannot hold. In [**1**], Dombi studied the monotonicity property of $R_k(n)$ for $k > 4$. He proved that there exists an $A \subset \mathbb{N}$ such that $R_k(n)$ is

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increasing for every $k > 4$ and $n > n_0(k)$ and the density of A is equal to 1/2. In this paper, we have the following results:

Theorem 1. *There exist infinitely many integers N such that*

$$\sum_{n=0}^{N}\bigl(R_k(n+1)-R_k(n)\bigr)^2 \geqslant c(k)\bigl(B(A,N)\bigr)^k, \tag{1}$$

where $c(k) = e^{-2k}2^{1-2k}(1+(2k)!)^{-1}$.

Corollary 1. *For large enough N,*

$$\sum_{n=0}^{N}\bigl(R_k(n+1)-R_k(n)\bigr)^2 = o\Bigl(\bigl(B(A,N)\bigr)^k\Bigr)$$

cannot hold.

Corollary 2. *If*

$$\lim_{N\to+\infty}\frac{B(A,N)}{N^{1/k}} = +\infty,$$

then $\bigl|R_k(n+1)-R_k(n)\bigr|$ cannot be bounded.

Theorem 2. *If*

$$A(n) = o\biggl(\Bigl(\frac{n}{\log n}\Bigr)^{2/k}\biggr), \tag{2}$$

then $R_k(n)$ cannot be eventually monotonic increasing.

2. Proofs

Lemma 1. *For $0 < x < 1$ and $m \in \mathbb{N}$, we have*

$$(1-x)^{-m-1} = 1 + \sum_{n=1}^{+\infty}\binom{n+m}{m}x^n > \frac{1}{m!}\sum_{n=1}^{+\infty}n^m x^n.$$

Lemma 2. ([6]) *For large N, we have*

$$\int_0^1 \frac{1}{|1-z|}\,d\alpha \ll \log N,$$

where $z = e^{-1/N}e^{2\pi i\alpha}$, α is a real variable.

Lemma 3. *If $R_k(n+1) \geqslant R_k(n)$ for $n \geqslant n_0$, then*

$$R_k(n) \leqslant \frac{(A(2n))^k}{n} \quad \text{for } n \geqslant n_0. \tag{3}$$

PROOF: For $n \geqslant n_0$, we have

$$(A(2n))^k = \Big(\sum_{\substack{a\in A\\ a\leqslant 2n}} 1\Big)^k \geqslant \sum_{\substack{a_{i_1}+\cdots+a_{i_k}\leqslant 2n\\ a_{i_1},\dots,a_{i_k}\in A}} 1 = \sum_{i=1}^{2n} R_k(i)$$
$$\geqslant \sum_{i=n+1}^{2n} R_k(i) \geqslant \sum_{i=n+1}^{2n} R_k(n) = nR_k(n).$$

Hence

$$R_k(n) \leqslant \frac{(A(2n))^k}{n} \quad \text{for } n \geqslant n_0.$$

This completes the proof of Lemma 3. ∎

LEMMA 4. *If $F(n)$ is a real arithmetic function satisfying $0 \leqslant F(n) \leqslant n$, and $F(n) = 0$ holds only for finitely many integers n, then there exist infinitely many integers N such that*

$$\text{(4)} \qquad \frac{F(N+i)}{F(N)} < \Big(\frac{N+i}{N}\Big)^2 \quad \text{for } i = 1, 2, \dots$$

PROOF: Suppose that (4) holds only for finitely many N. Then there exists an integer N_0 such that

$$F(N) > 0 \ \text{ for } \ N \geqslant N_0.$$

Then there exists an integer $N' = N'(N)$ satisfying $N' > N$ and

$$\frac{F(N')}{F(N)} \geqslant \Big(\frac{N'}{N}\Big)^2.$$

By induction, we get that there exist integers $N_0 < N_1 < N_2 < \cdots < N_j < \cdots$ such that

$$\frac{F(N_{j+1})}{F(N_j)} \geqslant \Big(\frac{N_{j+1}}{N_j}\Big)^2 \quad \text{for } j = 0, 1, 2, \dots .$$

Hence

$$F(N_{l+1}) = F(N_0)\prod_{j=0}^{l} \frac{F(N_{j+1})}{F(N_j)} \geqslant F(N_0)\prod_{j=0}^{l}\Big(\frac{N_{j+1}}{N_j}\Big)^2$$
$$= F(N_0)\Big(\frac{N_{l+1}}{N_0}\Big)^2 > N_{l+1}^{3/2}$$

for large enough l, which contradicts the fact that $F(N_{l+1}) \leqslant N_{l+1}$.

This completes the proof of Lemma 4. ∎

PROOF OF THEOREM 1: If $A = \{1, 2, \dots\}$, then the result is obvious. Now let $A \subset \{1, 2, \dots\}$ be an infinite sequence and let $S(n) = \sum_{j=0}^{n} \big(R_k(j+1) - R_k(j)\big)^2$. Suppose that there are only finitely many integers N satisfying (1). Then there exists an integer N_0 such that for $N \geqslant N_0$, we have

$$S(N) < e^{-2k}2^{1-2k}(1+(2k)!)^{-1}\big(B(A,N)\big)^k. \tag{5}$$

By Lemma 4, there exist infinitely many integers N such that

$$\frac{B(A,N+i)}{B(A,N)} < \Big(\frac{N+i}{N}\Big)^2 \qquad \text{for } i = 1,2,\dots . \tag{6}$$

Let N denote a large integer satisfying (5) and (6). We write $e^{2\pi i\alpha} = e(\alpha)$, and we put $r = e^{-1/N}$, $z = re(\alpha)$, where α is a real variable.

The infinite series

$$f(z) = \sum_{a\in A} z^a \text{ and } f(z)(1-z) = \sum_{n=1}^{+\infty} b_n z^n$$

are absolutely convergent for $|z| < 1$.

Let

$$J_1 = \int_0^1 \big|f(z)(1-z)\big|^k \, d\alpha.$$

Then by Hölder's inequality and Parseval's formula,

$$\begin{aligned}
J_1^{2/k} &= \Big(\int_0^1 \Big|f(z)(1-z)\Big|^k \, d\alpha\Big)^{2/k}\Big(\int_0^1 1\, d\alpha\Big)^{1-2/k} \\
&\geqslant \int_0^1 \Big|f(z)(1-z)\Big|^2 \, d\alpha \\
&= \int_0^1 \Big|\sum_{n=1}^{+\infty} b_n z^n\Big|^2 \, d\alpha \\
&= \sum_{n=1}^{+\infty} b_n^2 r^{2n} \\
&\geqslant r^{2N} \sum_{\substack{n\leqslant N \\ n\in A, n-1\notin A}} b_n^2 \\
&= e^{-2}B(A,N).
\end{aligned}$$

Hence

$$J_1 \geqslant \Big(e^{-2}B(A,N)\Big)^{k/2} = e^{-k}\big(B(A,N)\big)^{k/2}. \tag{7}$$

On the other hand, by Cauchy inequality and Parseval's formula, we have

$$J_1 = \int_0^1 \Big|f^k(z)(1-z)\Big| \cdot |1-z|^{k-1} \, d\alpha$$

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$$
\begin{aligned}
&< 2^{k-1}\int_0^1 \left|f^k(z)(1-z)\right| d\alpha \\
&= 2^{k-1}\int_0^1 \left|\sum_{n=1}^{+\infty} R_k(n)z^n(1-z)\right| d\alpha \\
&= 2^{k-1}\int_0^1 \left|\sum_{n=1}^{+\infty} (R_k(n)-R_k(n-1))z^n\right| d\alpha \\
&\leqslant 2^{k-1}\left(\int_0^1 \left|\sum_{n=1}^{+\infty} (R_k(n)-R_k(n-1))z^n\right|^2 d\alpha\right)^{1/2} \\
&= 2^{k-1}\left(\sum_{n=1}^{+\infty} (R_k(n)-R_k(n-1))^2 r^{2n}\right)^{1/2} \\
&= 2^{k-1}\left((1-r^2)\frac{1}{1-r^2}\sum_{n=1}^{+\infty} (R_k(n)-R_k(n-1))^2 r^{2n}\right)^{1/2} \\
&= 2^{k-1}\left((1-r^2)\sum_{n=1}^{+\infty} S(n-1)r^{2n}\right)^{1/2} \\
&\leqslant 2^{k-1}\left((1-r^2)\sum_{n=1}^{+\infty} S(n)r^{2n}\right)^{1/2} \\
&= 2^{k-1}\left((1-e^{-2/N})\left(\sum_{n=1}^{N} S(n)r^{2n} + \sum_{n=N+1}^{+\infty} S(n)r^{2n}\right)\right)^{1/2}.
\end{aligned}
$$

For $0 < x < 1$, we have $1 - e^{-x} < x$, and in view of (5) and (6), we have

$$
\begin{aligned}
J_1 &< 2^{k-1}\left(\frac{2}{N}\left(\sum_{n=1}^{N} S(N) + \sum_{n=N+1}^{+\infty} S(n)r^{2n}\right)\right)^{1/2} \\
&= 2^{k-1}\frac{1}{e^k 2^{k-1}(1+(2k)!)^{1/2}}\left((B(A,N))^k + N^{-1}\sum_{n=N+1}^{+\infty} (B(A,n))^k r^{2n}\right)^{1/2} \\
&< \frac{1}{e^k(1+(2k)!)^{1/2}}(B(A,N))^{k/2}\left(1 + N^{-2k-1}\sum_{n=N+1}^{+\infty} n^{2k}r^{2n}\right)^{1/2}.
\end{aligned}
$$

Put $x = r^2$ and $m = 2k$ in Lemma 1, and $1 - e^{-x} > x/2$ for $0 < x < 1$, thus

$$
\begin{aligned}
J_1 &< \frac{1}{e^k(1+(2k)!)^{1/2}}(B(A,N))^{k/2}\left(1 + N^{-2k-1}(2k)!(1-r^2)^{-2k-1}\right)^{1/2} \\
&< \frac{1}{e^k(1+(2k)!)^{1/2}}(B(A,N))^{k/2}\left(1 + (2k)!N^{-2k-1}(1/N)^{-2k-1}\right)^{1/2} \qquad (8) \\
&= e^{-k}(B(A,N))^{k/2}.
\end{aligned}
$$

By (7) and (8), we have

$$
e^{-k}(B(A,N))^{k/2} \leqslant J_1 < e^{-k}(B(A,N))^{k/2},
$$

which is impossible, thus the assumption cannot hold.

This completes the proof of Theorem 1. ∎

PROOF OF THEOREM 2: Now suppose that (2) holds and $R_k(n+1) \geqslant R_k(n)$ for $n \geqslant n_0$. By Lemma 4, there exist infinitely many integers N such that

$$\frac{A(N+i)}{A(N)} < \left(\frac{N+i}{N}\right)^2 \qquad \text{for } i = 1, 2, \ldots \tag{9}$$

Let $N(\geqslant n_0)$ denote a large integer satisfying (9). We write $e^{2\pi i\alpha} = e(\alpha)$, and we put $r = e^{-1/N}, z = re(\alpha)$, where α is a real variable. Then the infinite series $f(z) = \sum_{a\in A} z^a$ is absolutely convergent for $|z| < 1$.

Let

$$J_2 = \int_0^1 |f(z)|^k \, d\alpha.$$

Then by Hölder's inequality and Parseval's formula,

$$\begin{aligned} J_2^{2/k} &= \left(\int_0^1 |f(z)|^k \, d\alpha\right)^{2/k} \left(\int_0^1 1 \, d\alpha\right)^{1-2/k} \geqslant \int_0^1 |f(z)|^2 \, d\alpha \\ &= \sum_{a\in A} r^{2a} \geqslant \sum_{\substack{a\in A \\ a\leqslant N}} r^{2N} = e^{-2} \sum_{\substack{a\in A \\ a\leqslant N}} 1 = e^{-2} A(N). \end{aligned}$$

Hence,

$$J_2 \geqslant \left(e^{-2} A(N)\right)^{k/2} = e^{-k} \left(A(N)\right)^{k/2}. \tag{10}$$

On the other hand,

$$\begin{aligned} J_2 &= \int_0^1 |f^k(z)| \, d\alpha \\ &= \int_0^1 \left|\sum_{n=1}^{+\infty} R_k(n) z^n\right| d\alpha \\ &= \int_0^1 \left|(1-z) \sum_{n=1}^{+\infty} R_k(n) z^n\right| |1-z|^{-1} \, d\alpha. \end{aligned}$$

Let

$$T = \left|(1-z) \sum_{n=1}^{+\infty} R_k(n) z^n\right|.$$

Then

$$\begin{aligned} T &= \left|\sum_{n=1}^{+\infty} \left(R_k(n) - R_k(n-1)\right) z^n\right| \\ &\leqslant \sum_{n=1}^{n_0} |R_k(n) - R_k(n-1)| r^n + \sum_{n=n_0+1}^{+\infty} |R_k(n) - R_k(n-1)| r^n \end{aligned}$$

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$$< \sum_{n=1}^{n_0} \left| R_k(n) - R_k(n-1) \right| + \sum_{n=n_0+1}^{+\infty} \left(R_k(n) - R_k(n-1) \right) r^n$$
$$< 2\sum_{n=1}^{n_0} \left| R_k(n) - R_k(n-1) \right| + \sum_{n=1}^{+\infty} \left(R_k(n) - R_k(n-1) \right) r^n$$
$$= c_1 + \sum_{n=1}^{+\infty} R_k(n)(r^n - r^{n+1}) = c_1 + (1-r)\sum_{n=1}^{+\infty} R_k(n) r^n$$
$$< c_1 + \sum_{n=1}^{n_0-1} R_k(n) + (1-r)\sum_{n=n_0}^{+\infty} R_k(n) r^n$$
$$< c_2 + (1-e^{-1/N})\Big(\sum_{n=n_0}^{N} R_k(N) + \sum_{n=N+1}^{+\infty} R_k(n) r^n \Big),$$

where c_1, c_2 are constants.

For $0 < x < 1$, we have $1 - e^{-x} < x$, and in view of (3) and (9), we have

$$T < c_2 + N^{-1}\Big(N \cdot \frac{\big(A(2N)\big)^k}{N} + \sum_{n=N+1}^{+\infty} \frac{\big(A(2n)\big)^k}{n} \cdot r^n \Big)$$
$$< c_2 + N^{-1}\Big(\big(A(N)\big)^k (2N/N)^{2k} + \sum_{n=N+1}^{+\infty} \big(A(N)\big)^k (2n/N)^{2k} n^{-1} r^n \Big)$$
$$< c_2 + N^{-1}\big(A(N)\big)^k \cdot 2^{2k}\Big(1 + N^{-2k} \sum_{n=N+1}^{+\infty} n^{2k-1} r^n \Big).$$

Put $x = r$ and $m = 2k-1$ in Lemma 1, and $1 - e^{-x}(x/2)$ for $0 < x < 1$, thus

$$T < c_2 + N^{-1}\big(A(N)\big)^k \cdot 2^{2k}\Big(1 + N^{-2k} \cdot \frac{1}{(2k-1)!}(1-e^{-1/N})^{-2k} \Big)$$
$$< c_2 + N^{-1}\big(A(N)\big)^k \cdot 2^{2k}\Big(1 + N^{-2k} \frac{1}{(2k-1)!}\Big(\frac{1}{2N}\Big)^{-2k} \Big)$$
$$\ll N^{-1}\big(A(N)\big)^k.$$

By Lemma 2, we have

$$J_2 \ll N^{-1}\big(A(N)\big)^k \log N. \tag{11}$$

By (10) and (11), we have

$$e^{-k}\big(A(N)\big)^{k/2} \leqslant J_2 \ll N^{-1}\big(A(N)\big)^k \log N.$$

Hence

$$\big(A(N)\big)^{k/2} \gg \frac{N}{\log N},$$

which contradicts the assumption that $A(n) = o\big((n/\log n)^{2/k}\big)$.

This completes the proof of Theorem 2. □

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References

[1] G. Dombi, 'Additive properties of certain sets', *Acta Arith.* **103** (2002), 137–146.

[2] P. Erdős and A. Sárközy, 'Problems and results on additive properties of general sequences, I', *Pacific J. Math.* **118** (1985), 347–357.

[3] P. Erdős and A. Sárközy, 'Problems and results on additive properties of general sequences, II', *Acta Math. Hungar.* **48** (1986), 201–211.

[4] P. Erdős, A. Sárközy and V.T. Sós, 'Problems and results on additive properties of general sequences, III', *Studia Sci. Math. Hungar.* **22** (1987), 53–63.

[5] P. Erdős, A. Sárközy and V.T. Sós, 'Problems and results on additive properties of general sequences, IV', in *in: Number theory (Ootacamund, 1984)*, Lecture Notes in Math. **1122** (Springer-Verlag, Berlin, 1985), **pp.** 85–104.

[6] G. Horváth, 'On an additive property of sequences of nonnegative integers', *Period. Math. Hungar.* **45** (2002), 73–80.

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