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ASYMPTOTIC BEHAVIOUR OF THE TIME-FRACTIONAL TELEGRAPH EQUATION

VICENTE VERGARA,* *Universidad de Tarapacá*

Abstract

We obtain the long-time behaviour to the variance of the distribution process associated with the solution of the telegraph equation. To this end, we use a version of the Karamata–Feller Tauberian theorem.

Keywords: Telegraph equation; fractional derivative; Mittag–Leffler function; Tauberian theorem

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1. Introduction and preliminaries

Let $\alpha \in (0, 1]$ and $\mu, \nu > 0$ be constants. We consider the time-fractional telegraph equation

$$\partial_t^{2\alpha}(u - u_0 - tu_1) + \mu\partial_t^{\alpha}(u - u_0) - \nu\partial_x^2 u = 0, \qquad t > 0,\ x \in \mathbb{R}, \tag{1.1}$$

where $u_0 = u(0, x)$ plays the role of the initial datum for u, and $u_1 = u_t(0, x)$ means the initial condition of the derivative of u whenever it exists for $\frac{1}{2} < \alpha \le 1$. In general, (1.1) can be solved using e.g. the abstract theory of Volterra equations; see the monograph of Prüss [6].

Equation (1.1) is introduced in [4] subject to the conditions that $u_0 = \delta(x)$ and $u_1 = 0$; we adopt these conditions as well. The solution u_α of (1.1) exhibits interesting properties, one of them being that u_α can be viewed as the probability density function whose distribution process, denoted by X_α, coincides with u_α at time t (cf. [4]). Furthermore, they show that, for $\alpha = \frac{1}{2}$, the variance of $X_{1/2}$ increases like $t^{1/2}$ as $t \to \infty$, which is more slowly than the variance of X_1 ($\alpha = 1$), which increases like t as $t \to \infty$. In this paper we aim to establish this property for all $\alpha \in (0, 1]$. Moreover, we prove that the variance of X_γ increases more slowly than the variance of X_α if and only if $0 < \gamma < \alpha \le 1$. See Theorem 2.1 below.

The term $\partial_t^{\beta} v$ denotes the classical Riemann–Liouville fractional derivative of the (sufficiently smooth) function v of order $\beta > 0$, which is defined by

$$\partial_t^{\beta} v = \frac{\mathrm{d}}{\mathrm{d}t}(g_{1-\beta} * v),$$

where g_α denotes the standard kernel

$$g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \qquad t > 0,\ \alpha > 0.$$

Here $\Gamma(\alpha)$ stands for the gamma function; $g_\alpha * v$ denotes the convolution on the positive half-line $\mathbb{R}_+ := [0, \infty)$ with respect to the time variable, that is, $(g_\alpha * v)(t) = \int_0^t g_\alpha(t - s)v(s)\,\mathrm{d}s,\ t \ge 0$.

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* Postal address: Universidad de Tarapacá, Instituto de Alta Investigación, Antofagasta N. 1520, Arica, Chile.
Email address: vvergaraa@uta.cl

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Let us recall some properties of the standard kernel: $(g_\alpha * g_\beta)(t) = g_{\alpha+\beta}(t)$, $\alpha, \beta > 0$, for all $t \geq 0$, and $\partial_t^\beta g_\beta(t) = 0$ for all $t > 0$; see, e.g. [5].

The solution of the scalar equation (1.1) can be computed explicitly in terms of the Mittag–Leffler function

$$E_{\alpha,\beta}(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \qquad \alpha, \beta > 0, \ x \in \mathbb{C}.$$

For a general presentation of fractional calculus and applications, we refer the reader to [3], [5], and [7].

In particular, in [4, Section 5] the variance of X_α is given by $\mathbb{E}X_\alpha^2$ since the mean value of the processes $X_\alpha(t)$, $t > 0$ is 0. The variance $\mathbb{E}X_\alpha^2$ is explicitly obtained, that is,

$$\mathbb{E}X_\alpha^2(t) = 2\nu t^{2\alpha} E_{\alpha,2\alpha+1}(-\mu t^\alpha). \tag{1.2}$$

We use (1.2) to obtain our results.

The following result is a version of the Karamata–Feller Tauberian theorem (cf. [8]), which establishes that the asymptotic behaviour of a function $w(t)$ as $t \to \infty$ can be determined, under suitable conditions, by looking at the behaviour of its Laplace transform $\hat{w}(z)$ as $z \to 0$, and vice versa. See the monograph [1] for a more general version and proofs.

Theorem 1.1. *Let* $L\colon (0,\infty) \to (0,\infty)$ *be a function that is* slowly varying at ∞, *that is, for every fixed* $x > 0$, *we have* $L(tx)/L(t) \to 1$ *as* $t \to \infty$. *Let* $\beta > 0$, *and let* $w\colon (0,\infty) \to \mathbb{R}$ *be a monotone function whose Laplace transform* $\hat{w}(z)$ *exists for all* $z \in \mathbb{C}_+ := \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda > 0\}$. *Then*

$$\hat{w}(z) \sim \frac{1}{z^\beta} L\left(\frac{1}{z}\right) \quad \text{as } z \to 0 \quad \text{if and only if} \quad w(t) \sim \frac{t^{\beta-1}}{\Gamma(\beta)} L(t) \quad \text{as } t \to \infty.$$

Here the approaches are on the positive real axis and the notation $f(t) \sim g(t)$ *as* $t \to t_*$ *means that* $\lim_{t \to t_*} f(t)/g(t) = 1$.

2. Main result and proof

Theorem 2.1. (i) *Let* $\alpha \in (0,1]$. *Then*

$$\mathbb{E}X_\alpha^2(t) \sim \frac{2\nu}{\mu} \frac{t^\alpha}{\Gamma(\alpha+1)} \quad \text{as } t \to \infty.$$

(ii) *Let* $\alpha, \gamma \in (0,1]$. *Then there exists* $M > 0$ *such that, for all* $t > M$, *we have*

$$\mathbb{E}X_\gamma^2(t) < \mathbb{E}X_\alpha^2(t) \quad \text{if and only if} \quad \gamma < \alpha.$$

Proof. Define $v(t) = 2\nu t^{2\alpha} E_{\alpha,2\alpha+1}(-\mu t^\alpha)$ on $\mathbb{R}_+$. Note that the Laplace transform of $v(t)$ is given by

$$\hat{v}(z) = \frac{2\nu}{z^{2\alpha+1} + \mu z^{\alpha+1}};$$

see, e.g. [3]. This in turn implies that v is the unique solution of

$$\partial_t^{2\alpha} v + \mu \partial_t^\alpha v = 2\nu, \qquad t > 0, \ v(0) = 0. \tag{2.1}$$

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Equation (2.1) can be written as a Volterra equation by convolving (2.1) with the kernel $g_{2\alpha}$, that is,

$$v(t) + \mu(g_\alpha * v)(t) = 2\nu g_{2\alpha+1}(t), \qquad t > 0. \tag{2.2}$$

Now, in order to write the solution of (2.2) by means of the variation of parameters formula for Volterra equations (cf. [2] and [6]), let us introduce the relaxation function s_μ on $\mathbb{R}_+$ as the solution of the Volterra equation

$$s_\mu(t) + \mu(g_\alpha * s_\mu)(t) = 1, \qquad t \geq 0. \tag{2.3}$$

Observe that the unique solution of (2.3) is given by the Mittag–Leffler function, that is,

$$s_\mu(t) = \sum_{k=0}^{\infty} \frac{(-\mu t^\alpha)^k}{\Gamma(\alpha k + 1)} = E_{\alpha,1}(-\mu t^\alpha).$$

It is well known that s_μ is strictly positive and decreasing on $(0, \infty)$ (cf. [3] and [7]).

The solution of (2.2) can now be represented as

$$v(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t s_\mu(t-s) 2\nu g_{2\alpha+1}(s)\,\mathrm{d}s = 2\nu(s_\mu * g_{2\alpha})(t), \qquad t \geq 0.$$

The second term in this equality is the variation of parameters formula for (2.2).

Since s_μ depends on α, set $s_{\mu,\alpha}(t) = s_\mu(t)$.

(i) Observe that $v(t)$ is strictly positive on $\mathbb{R}_+$. For $\alpha \geq \frac{1}{2}$, we have $\dot{v}(t) = 2\nu(s_{\mu,\alpha} * g_{2\alpha-1})(t) > 0$ for all $t > 0$, meanwhile, for $\alpha < \frac{1}{2}$, we obtain $\dot{v}(t) = 2\nu(\dot{s}_{\mu,\alpha} * g_{2\alpha})(t) + 2\nu g_{2\alpha}(t)$. From (2.3) we obtain

$$\begin{aligned} \dot{v}(t) &= 2\nu(\dot{s}_{\mu,\alpha} * g_{2\alpha})(t) + 2\nu g_{2\alpha}(t) \\ &= \frac{2\nu}{\mu}([-\dot{s}_{\mu,\alpha}] * g_\alpha)(t) \\ &> 0, \qquad t > 0. \end{aligned}$$

Therefore, v is a monotone increasing function on $(0, \infty)$ for all $\alpha \in (0, 1]$. On the other hand, $\hat{v}(z) \sim 2\nu/(\mu z^{\alpha+1})$ as $z \to 0$. Next, define $L(t) = 1$ for all $t > 0$. Hence, the statement follows from Theorem 1.1.

(ii) Define $w(t) = 2\nu t^{2\gamma} E_{\gamma,2\gamma+1}(-\mu t^\gamma)$. Then $w(t)$ can be written, by means of the variation of parameters formula for Volterra equations, for the corresponding solution of (2.2) as follows:

$$w(t) = 2\nu(s_{\mu,\gamma} * g_{2\gamma})(t), \qquad t \geq 0.$$

Since v and w are strictly positive and increasing functions on $(0, \infty)$, it follows from (i) that there exists an $M > 0$ such that

$$w(t) \leq \frac{2\nu}{\mu} \frac{t^\gamma}{\Gamma(\gamma+1)} < v(t)$$

holds for all $t > M$.

This completes the proof.

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References

[1] FELLER, W. (1971). *An Introduction to Probability Theory and its Applications*, Vol. II, 2nd edn. John Wiley, New York.
[2] GRIPENBERG, G., LONDEN, S.-O. AND STAFFANS, O. (1990). *Volterra Integral and Functional Equations* (Encyclopedia Math. Appl. **34**). Cambridge University Press.
[3] KILBAS, A. A., SRIVASTAVA, H. M. AND TRUJILLO, J. J. (2006). *Theory and Applications of Fractional Differential Equations* (North-Holland Math. Studies **204**). Elsevier, Amsterdam.
[4] ORSINGHER, E. AND BEGHIN, L. (2004). Time-fractional telegraph equations and telegraph processes with Brownian time. *Prob. Theory Relat. Fields* **128,** 141–160.
[5] PODLUBNY, I. (1999). *Fractional Differential Equations* (Math. Sci. Eng. **198**). Academic Press, San Diego, CA.
[6] PRÜSS, J. (1993). *Evolutionary Integral Equations and Applications*. Birkhäuser, Basel.
[7] SAMKO, S.-G., KILBAS, A. A. AND MARICHEV, O. I. (1993). *Fractional Integrals and Derivatives*. Gordon and Breach, Yverdon.
[8] VERGARA, V. AND ZACHER, R. (2013). Optimal decay estimates for time-fractional and other non-local subdiffusion equations via energy methods. Submitted.