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Some Remarks on the Algebraic Sum of Ideals and Riesz Subspaces

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Abstract. Following ideas used by Drewnowski and Wilansky we prove that if I is an infinite dimensional and infinite codimensional closed ideal in a complete metrizable locally solid Riesz space and I does not contain any order copy of $\mathbb{R}^{\mathbb{N}}$ then there exists a closed, separable, discrete Riesz subspace G such that the topology induced on G is Lebesgue, $I \cap G = \{0\}$, and $I + G$ is not closed.

1 Introduction

It is easy to observe that every Hausdorff topological vector space contains two closed sets A, B whose algebraic sum $A + B$ is not closed. A similar situation holds for closed (linear) subspaces: *if X is a closed non minimal subspace of infinite codimension in an F-space (= complete metrizable topological vector space) E, then there exists a closed non minimal subspace Y of E such that $X \cap Y = \{0\}$ and $X + Y$ is not closed* (see [3, Thm 3.3]). Let us recall that a topological vector space (E, τ) is said to be non minimal if it admits a Hausdorff vector topology strictly coarser than τ. The cartesian product $\mathbb{R}^{\Gamma}$ of real lines is the unique minimal locally convex topological vector space. The search for an example of a minimal non locally convex space took a long time. Finally, a suitable, somewhat exotic, space was constructed in [4]. The reader interested in the question about closedness of $X+Y$ for closed linear subspaces X, Y is referred to [2] and especially to [3] where a discussion of this problem and related topics are accompanied by many historical remarks. It is important that $X + Y$ is always closed whenever Y is an arbitrary closed subspace, X is minimal, and $X \cap Y = \{0\}$ (see [2, Prop. 2.3]). The previous statement implies that $X+Y$ is closed if Y is closed and X is finite dimensional, or X is closed and finite codimensional.

The situation becomes more subtle if we consider Riesz subspaces or ideals in locally solid Riesz spaces. For terminology related to these types of spaces we refer to [1, 8]. Below we recall, for the reader's convenience, several basic definitions and facts needed in our considerations. A real vector space E equipped with a relation of partial ordering $\leqslant$ is called a Riesz space (= a vector lattice) if the relation has the following three properties:

(A) $\forall x, y, z \in E,\ x \leqslant y \Rightarrow x + z \leqslant y + z$,
(M) $\forall x, y \in E, \forall r \in \mathbb{R}_+,\ x \leqslant y \Rightarrow rx \leqslant ry$,

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(S) every subset of E consisting of two elements x, y has a least upper bound $x \vee y$.

The definition implies that an arbitrary set $\{x, y\} \subset E$ has the greatest lower bound $x \wedge y$ associated with a supremum by the equality $x \wedge y = -((-x) \vee (-y))$. Moreover every $x \in E$ has its modulus $|x| = x \vee (-x)$. Many classical spaces, such as $C(K)$ (continuous real functions on a compact set K) and $L^p(\mu)$ $(0 \leqslant p \leqslant \infty)$, admit a natural partial order: $f \leqslant g$ if and only if $f(s) \leqslant g(s)$ for all s (or for μ-almost all s in the case $f, g \in L^p(\mu)$).

If F is a linear subspace of a Riesz space E and F is closed under a modulus, *i.e.,* $x \in F \Rightarrow |x| \in F$, then F is said to be a Riesz subspace (a vector sublattice). We should also mention the notion of solidness: a set $S \subset E$ is solid whenever $|x| \leqslant |s|$ for some $s \in S$ implies $x \in S$. Solid linear subspaces are called ideals, and an ideal $F \subset E$ is a band when $\sup A \in F$ for every subset $A \subset F$ having a supremum in E. The orthogonal complement X^{d} of a set $X \subset E$ defined by $X^{\mathrm{d}} = \bigcap_{x \in X}\{y \in E : |y| \wedge |x| = 0\}$ is an important example of a band. We will say that a set $A \subset E$ consists of pairwise disjoint elements whenever $|a| \wedge |b| = 0$ for all $a, b \in A$. Similarly, two subspaces F, G are disjoint if $|f| \wedge |g| = 0$ for all $f \in F$ and $g \in G$.

Let us note the algebraic sum of Riesz subspaces cannot be a Riesz subspace; the subspace of affine functions fails to be a Riesz subspace of $C[0, 1]$ but it is the sum of two (one dimensional!) Riesz subspaces. On the other hand if I is an ideal and F is a Riesz subspace, then $I + F$ is a Riesz subspace–for $x \in I, f \in F$ there holds $|x + f| = (|x + f| - |f|) + |f| \in I + F$. It is natural to ask if the reverse implication is true; *i.e.,* is a Riesz subspace I an ideal whenever $I + F$ is a Riesz subspace for an arbitrary Riesz subspace F?. Below we show a negative answer.

Let E be a Riesz subspace in $\mathbb{R}^{\mathbb{N}}$ containing all unit vectors e_i. Consider a Riesz subspace $F = \{x = (x_n) \in E : x_1 = x_2\}$. Clearly F is not an ideal, because $F \ni e_1 + e_2 \geqslant e_1 \notin F$. It is easy to check that $E = F + \mathbb{R}e_1 = F + \mathbb{R}e_2$. Let 1_A stand for the characteristic function of a set A. Fix a Riesz subspace $G \subset E$. We have to show $F + G$ is a Riesz subspace of E. Consider two cases.

1. The subspace G contains a sequence g such that $g_1 \neq g_2$ and $|g_1| = |g_2|$. Since $-g \in G$ we can assume $g_1 > 0$. We have $2g_1e_2 = (|g| - g) - (|g| - g)1_{\mathbb{N} \setminus \{1,2\}} \in G + F$, and so $G + F = G + F + \mathbb{R}e_2 = E$.
2. Suppose that the previous case does not hold, *i.e.,* for every $g \in G$ either $g_1 = g_2$ or $|g_1| \neq |g_2|$. Fix $g \in G$, $f \in F$. If $g_1 = g_2$ then $g \in F$. Hence $|f + g| \in F \subset F + G$.

Assume $|g_1| \neq |g_2|$ and put $c = \frac{|f_1+g_2|-|f_1+g_1|}{|g_2|-|g_1|}$. It is easy to check $|f_1 + g_1| - c|g_1| = |f_1 + g_2| - c|g_2|$. Let t denote this common value. Remembering that $f_1 = f_2$, we obtain $|f + g| = [(|f + g| - c|g|)1_{\mathbb{N} \setminus \{1,2\}} + t(e_1 + e_2)] + c|g| \in F + G$.

Let us turn our attention to locally solid Riesz spaces, *i.e.,* to Riesz spaces equipped with a linear topology having a base for the neighborhoods of zero consisting of solid sets. The class of locally solid Riesz spaces contains normed lattices (*i.e.,* their topology is determined by a monotone norm q: $|x| \leqslant |y| \Rightarrow q(x) \leqslant q(y)$), and clearly Banach lattices. Our main considerations will concern F-lattices, *i.e.,* complete metrizable locally solid Riesz spaces, which will be denoted by $(E, \|\cdot\|)$ (or shortly by E), where $\|\cdot\|$ is a monotone F-norm generating the topology on E (the functional $\|\cdot\|$ has the same properties as a monotone norm, but the homogeneity of a norm, *i.e.,*

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the condition $q(tx) = |t|q(x)$, is replaced by $\|tx\| \to 0$ as $t \to 0$). It is clear that F-lattices are Archimedean: $\inf_n \frac{1}{n}x = 0$ for every $x \geqslant 0$.

Repeating arguments applied in the proof of [6, Proposition 1.2.2] we obtain the following interesting fact:

(∗) *If I_1, I_2 are closed ideals in an F-lattice, then $I_1 + I_2$ is closed.*

The assumption of a (topological) completeness is crucial. Indeed, consider a Riesz subspace

$$E = \{ f \in C[0,1] : \text{there exist } 0 = x_0 < x_1 < \cdots < x_n = 1 \text{ such that } f \text{ is affine on each interval } [x_{i-1}, x_i] \},$$

equipped with the topology of uniform convergence (see [8, Exercise 1, p. 217]). Clearly E is a normed lattice but it is not a Banach lattice. The ideals

$$I_1 = \bigcap_n \Big\{ f \in E : f\Big(\frac{1}{2n}\Big) = 0 \Big\}, \quad I_2 = \bigcap_n \Big\{ f \in E : f\Big(\frac{1}{2n-1}\Big) = 0 \Big\}$$

are closed, but $I_1 + I_2 \neq \overline{I_1 + I_2}$. On the other hand E has a curious property: $I + I^{\mathrm{d}}$ is closed for every closed ideal I of E (I^{d} denotes the disjoint complement of I). The last property is an immediate consequence of the following result (see [13] or [9, Thm 0.3.8]).

Let K be a compact space and let E be a Riesz subspace of $C(K)$ whose elements separate points from closed sets (*i.e.*, for every nonempty closed set $X \subset K$ and $s \notin X$ there exists $f \in E$ satisfying conditions $f(s) = 1$, $f(X) = \{0\}$). The following statements are equivalent.

(a) If I is a closed ideal, with respect to the topology of uniform convergence, then $I + I^{\mathrm{d}}$ is closed in E.

(b) If $f \in E$ and U is a nonempty open subset of K such that $f(\partial U) = \{0\}$, then $f 1_{\overline{U}} \in E$.

(∂U denotes the boundary of U and $1_{\overline{U}}$ is the characteristic function of the closed set $\overline{U}$.)

On the other hand it may happen that I is closed while $I + I^{\mathrm{d}}$ is not closed. We obtain a suitable example modifying the space E defined above. Let $B[0,1]$ be the space of real valued bounded functions on the unit interval equipped with the sup norm and put

$$F = \{ f \in B[0,1] : \text{there exist } 0 = x_0 < x_1 < \cdots < x_n = 1 \text{ such that } f \text{ is affine on each interval } (x_{i-1}, x_i) \}.$$

It is not very hard to check (see [9, Example (P2), p. 28] or [13]) that if $A = \bigcup_n [\frac{1}{2n+1}, \frac{1}{2n}]$, then $I = \{ f \in F : f(A) = \{0\}\}$ is a closed ideal, but $I + I^{\mathrm{d}} \neq \overline{I + I^{\mathrm{d}}}$. This fact implies also the sum $I^{\mathrm{dd}} + I^{\mathrm{d}}$ of disjoint bands is not closed. On the other hand, if F, G are disjoint closed Riesz subspaces in an F-lattice, then $F + G$ is closed, because sequences $(f_n) \subset F$ and $(g_n) \subset G$ are Cauchy whenever $(f_n + g_n)$ is convergent.

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We will say that a Riesz space E contains an order copy of another Riesz space F if there exists a Riesz isomorphism T mapping F into E; *i.e.,* T is a linear injection and $|T(x)| = T(|x|)$ for all $x \in F$. The following result will be useful in our further considerations.

Lemma 1.1 *For an F-lattice $(E, \|\cdot\|)$ the following statements are equivalent.*

(a) *E does not contain any order copy of $\mathbb{R}^{\mathbb{N}}$.*
(b) *If a sequence (y_n) consists of strictly positive pairwise disjoint elements from E, then $\inf_n \sup_{t>0} \|ty_n\| > 0$.*

Proof (a) $\Rightarrow$ (b) If E contains a sequence (y_n) of positive pairwise disjoint elements such that $\inf_n \sup_{t>0} \|ty_n\| = 0$, then there exists a subsequence (y_{n_k}) having the property that the series $\sum_{k=1}^{\infty} t_k y_{n_k}$ is convergent for arbitrary real numbers t_k. An operator $T\colon \mathbb{R}^{\mathbb{N}} \to E$ defined by $T((t_k)) = \sum_{k=1}^{\infty} t_k y_{n_k}$ is a Riesz isomorphism; *i.e.,* E contains an order copy of $\mathbb{R}^{\mathbb{N}}$.

(b) $\Rightarrow$ (a) Suppose $T\colon \mathbb{R}^{\mathbb{N}} \to E$ is a Riesz isomorphism and let e_n denote the n-th unit vector. The elements $T(e_n)$ are nonzero and pairwise disjoint. Therefore there exist reals t_n such that the numbers $\|t_n T(e_n)\|$ are separated from zero, a contradiction, because $(t_n e_n)$ tends to zero in $\mathbb{R}^{\mathbb{N}}$ and T is continuous (see [1, Thm. 5.19]). ■

Remark 1.2 Since the topology of pointwise convergence in $\mathbb{R}^{\mathbb{N}}$ is minimal, every Riesz isomorphism $T\colon \mathbb{R}^{\mathbb{N}} \to E$ is a homeomorphism, and so it maps $\mathbb{R}^{\mathbb{N}}$ onto a closed Riesz subspace. Lemma 1.1 is closely related to the well-known characterization of F-spaces without copies of $\mathbb{R}^{\mathbb{N}}$: an F-space $(E, \|\cdot\|)$ does not contain any isomorphic (= linearly homeomorphic) copy of $\mathbb{R}^{\mathbb{N}}$ if and only if E does not contain arbitrary short lines, *i.e.,* $\inf_{x\neq 0} \sup_t \|tx\| > 0$. Unfortunately we do not know if there exists an F-lattice with isomorphic, but not order isomorphic, copies of $\mathbb{R}^{\mathbb{N}}$.

The space $\mathbb{R}^{\mathbb{N}}$ belongs to a class of discrete Riesz spaces. Let us recall that a nonzero positive element e is discrete in a Riesz space E whenever $|x| \leqslant e$ implies $x = te$ for some number t (unit vectors are examples of discrete elements in $\mathbb{R}^{\mathbb{N}}$). If every nonzero positive $x \in E$ dominates a discrete element, then E is called discrete.

2 Main Result

We start with a lemma stating that an ideal having a "big" codimension contains many disjoint elements.

Lemma 2.1 *Let I be a closed ideal of infinite codimension in an F-lattice E. The set $E \setminus I$ contains a sequence of nonzero pairwise disjoint elements.*

Proof Since the quotient E/I is an infinite dimensional F-lattice, it contains a sequence of positive nonzero pairwise disjoint elements because E/I is Archimedean (see [5, Thm 26.10] or [1, Exercise 13, p.46]). The elements are of the form $Q(x_n)$, $n \in \mathbb{N}$, where $Q\colon E \to E/I$ is the canonical quotient map, which is a Riesz homomorphism, *i.e.,* $|Q(x)| = Q(|x|)$ or, equivalently, $Q(x \wedge y) = Q(x) \wedge Q(y)$ ([1, Thm.

1.31 and Thm. 1.34]). We can assume $0 < x_n \notin I$ because $0 < Q(x_n) = |Q(x_n)| = Q(|x_n|)$.

Choosing numbers $c_n > 0$ such that $\sum_{n=1}^{\infty} \|c_n x_n\| < \infty$ and putting $y_n = x_n - x_n \wedge \sum_{k\neq n} {c_n}^{-1} c_k x_k$, we obtain the required elements. Indeed, there holds $0 \leqslant y_n$ and

$$0 \leqslant y_m \wedge y_j \leqslant \Big(x_m - x_m \wedge \frac{c_j}{c_m} x_j\Big) \wedge \Big(x_j - x_j \wedge \frac{c_m}{c_j} x_m\Big)$$
$$\leqslant \max({c_m}^{-1}, {c_j}^{-1})\big[(c_m x_m - c_m x_m \wedge c_j x_j) \wedge (c_j x_j - c_j x_j \wedge c_m x_m)\big] = 0,$$

i.e., $y_m \wedge y_j = 0$ for $m \neq j$. The continuity of Q implies $Q(y_n) = Q(x_n) - Q(x_n) \wedge \sum_{k\neq n} {c_n}^{-1} c_k Q(x_k)$. Moreover, $Q(x_k) \in \{Q(x_n)\}^{\mathrm{d}}$ for $k \neq n$, and so $Q(x_n) \wedge \sum_{k\neq n} {c_n}^{-1} c_k Q(x_k) = 0$ by the closedness of the band $\{Q(x_n)\}^{\mathrm{d}}$. We have just proved $Q(y_n) = Q(x_n)$, hence $y_n \notin I$. ■

Now we formulate a result that is the opposite of the statement $(*)$ quoted in the introduction. Our proof of the result uses an idea of a construction of a suitable basic sequence applied by the authors of [3, 10]. We also apply a theorem saying that a separable σ-Dedekind complete F-lattice $(E, \|\cdot\|)$ (*i.e.*, every order bounded from above countable subset of E has a supremum) satisfies the Lebesgue property (equivalently the topology on E is a Lebesgue topology): for every net (x_α) decreasing to zero there holds $\|x_\alpha\| \to 0$ (see [1, Thm. 3.29]).

Theorem 2.2 *Let I be a closed infinite dimensional and infinite codimensional ideal in an F-lattice $E = (E, \|\cdot\|)$. If I does not contain any order copy of $\mathbb{R}^{\mathbb{N}}$, then there exists a closed separable discrete Riesz subspace G such that the induced topology is Lebesgue, $I \cap G = \{0\}$ and $I + G$ is not closed.*

Proof Consider three cases.

1. Suppose $I^{\mathrm{d}} = \{0\}$, *i.e.*, I is order dense in E. According to Lemma 2.1 there are pairwise disjoint elements $x_n \in E_+ \setminus I$. Multiplying, if necessary, every x_n by sufficiently small but strictly positive numbers, we can assume $\|x_n\| \to 0$. Choose $0 < y_n' \in [0, x_n] \cap I$. Since $\mathbb{R}^{\mathbb{N}}$ is not order embeddable in I, we are able to find numbers $t_n > 0$ such that $\inf_n \|t_n y_n'\| > 0$ (see Lemma 1.1). Let $y_n = t_n y_n'$ and define $z_n = y_n + x_n$. Clearly $z_k \wedge z_j = 0$ for distinct indices, and so $G = \overline{\mathrm{span}}\{z_n : n \in \mathbb{N}\}$ is a closed, separable, discrete Riesz subspace. It is easy to check that G is σ-Dedekind complete. By virtue of [1, Thm 3.29], $(G, \|\cdot\|)$ satisfies the Lebesgue property. If $g \in G \setminus \{0\}$ then $g = \sum_{n=1}^{\infty} a_n z_n$ with $a_n \neq 0$ for at least one n. Hence $|g|$ dominates a nonzero element $|a_n| x_n \notin I$. Therefore, $g \notin I$ and we obtain $I \cap G = \{0\}$. The closedness of I and G implies that the natural projection $P\colon I + G \to I$ has the closed graph. If $I + G$ were closed, then P would be continuous by the closed graph theorem, but $\|z_n - y_n\| \to 0$ while $\|P(z_n - y_n)\| = \|y_n\| \nrightarrow 0$.

2. Assume $0 < \dim I^{\mathrm{d}} < \infty$. The ideal $I + I^{\mathrm{d}}$ is order dense (see [1, Thm 1.25]) and its dimension, as well as codimension, is infinite. Moreover $I + I^{\mathrm{d}}$ is closed (see [6, Prop. 1.2.2]). According to the first case there exists a closed separable discrete Riesz subspace G satisfying the Lebesgue property such that $G \cap (I + I^{\mathrm{d}}) = \{0\}$, $I + I^{\mathrm{d}} + G \neq \overline{I + I^{\mathrm{d}} + G}$. Since I^{d} is a finite dimensional band, $I^{\mathrm{d}} + G$ is a closed Riesz

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subspace satisfying all desired conditions (let us note that also $I \cap G = \{0\}$ and $I + G$ cannot be closed because in the opposite case $I + G + I^{\mathrm{d}}$ would be closed as the sum of a closed and finite dimensional subspaces).

3. If $\dim I^{\mathrm{d}} = \infty$, fix two sequences (y_n), (x_n) of strictly positive pairwise disjoint elements in I and I^{d} respectively. Multiplying y_n and x_n by suitable large or small positive numbers we can assume all y_n's lie outside some neighborhood of zero and (x_n) tends to zero. Repeating arguments from the part 1 we prove the Riesz subspace $G = \overline{\operatorname{span}}\{y_n + x_n : n \in \mathbb{N}\}$ is as required. ∎

A "locally solid" analogue of minimal topologies is considered in the theory of locally solid Riesz spaces. Namely a Hausdorff locally solid Riesz space (E, τ) is called minimal if E does not admit a Hausdorff locally solid topology essentially coarser than τ (see [1, Section 7.5] for more details). Clearly (E, τ) is minimal if and only if every continuous injective Riesz homomorphism on E is a homeomorphism. The above equivalence implies that if B is a projection band in a minimal (E, τ), then $(B, \tau|_B)$ is minimal. Indeed, every continuous injective Riesz homomorphism $T\colon B \to F$ generates an operator of the same type on E, *i.e.,* the operator $\widehat{T}\colon E \to F \times B^{\mathrm{d}}$ defined by $\widehat{T}(x) = (T(Px), x - Px)$ where P is the band projection onto B. The map $\widehat{T}$ is a homeomorphism by the minimality of (E, τ), and so T is a homeomorphism too.

In contrast to the general case of F-spaces we have a very nice example of an F-lattice whose topology is minimal in the family of Hausdorff locally solid topologies. If (S, Σ, μ) is a σ-finite measure space then the Riesz space $L^0(\mu)$ of all equivalence classes of Σ-measurable real valued functions on S equipped with the topology of convergence in measure on sets of finite measure is a minimal F-lattice; see [1, p. 210] (on the other hand, for an atomless measure μ the space $L^0(\mu)$ admits a Hausdorff vector topology coarser than the topology of μ-convergence–see [7]). It is well known that minimal linear topologies are complete ([2, Prop. 2.2]), while there exist incomplete topologies minimal in the class of locally solid topologies (the topology of the μ-convergence is minimal in $L^p(\mu)$–see [1, Thm. 7.74]).

A minimal F-lattice (E, τ) has very strong order and topological properties. For instance τ is unique Hausdorff locally solid topology admitted by E (see [1, Thm. 5.20]). Applying [1, Cor. 7.68] we obtain E coincides with its universal completion E^u, and so E is laterally complete. Moreover by [1, Thm. 7.56] and [1, Thm. 7.2] E is super Dedekind complete and E contains a weak unit. According to [1, Thm. 7.67] τ is a Lebesgue topology. The remark finishing [11] implies τ is also a Levi topology (*i.e.,* every increasing τ-bounded net of positive elements has a supremum in E).

Besides minimal spaces, there were considered quotient-minimal topological vector spaces X (*i.e.,* X/Y is a minimal space for every closed subspace $Y \subset X$); see [2]. Quotient Riesz spaces E/F have their own peculiarity; one must always assume that F is an ideal (for an explanation why this assumption is natural and important; see [12, Section 2]). We formulate a locally solid analogue of the quotient-minimality: a locally solid Riesz space (E, τ) is quotient-by-ideals-minimal whenever E/F is minimal for every closed ideal F in E, we do not obtain any new class of F-lattices, *i.e.,* an F-lattice (E, τ) is quotient-by-ideals-minimal if and only if E is minimal. Indeed, if E

is minimal, then the topology τ is Lebesgue, and so closed ideals F in E are projection bands ([1, Thm. 3.7]). Therefore E/F is order isomorphic to the projection band F^{d}, which is, as we have already noticed, a minimal locally solid Riesz space.

The following problems seem to be natural in connection with the fact that $X+Y$ is closed for a minimal subspace X and a closed subspace Y satisfying the condition $X \cap Y = \{0\}$:

1. Is $I + G$ closed whenever G is a closed Riesz subspace, I is a closed ideal, and I (with the topology induced from E) is a minimal locally solid Riesz space?
2. Does the Theorem 2.2 remain true if we assume that I is not a minimal locally solid Riesz space instead of the (stronger) requirement that no order copy of $\mathbb{R}^{\mathbb{N}}$ is included in I?

Modifying the proof of [2, Proposition 2.2] we easily obtain an answer to a question similar to question 1.

Proposition 2.3 *Let I be a closed ideal in an F-lattice $E = (E, \tau)$. If $F \subset E$ is a closed Riesz subspace and τ restricted to F is minimal, then $I + F$ is closed.*

Proof Suppose first that $I \cap F = \{0\}$. The quotient map $Q\colon E \to E/I$ restricted to F is a continuous injective Riesz homomorphism, and so it is a homeomorphism. Hence $Q(F)$ is closed in E/I because $Q(F)$ is complete. The continuity of Q implies that $I + F = Q^{-1}(Q(F))$ is closed.

Let $I \cap F = F_1 \neq \{0\}$. The subspace F_1 is a closed ideal in a Dedekind complete Riesz space F, and τ restricted to F is a Lebesgue topology. According to [1, Thm. 3.7] F_1 is a projection band in F. If F_1^{d} denotes the disjoint complement of F_1 in F, then F_1^{d} is minimal and $I \cap F_1^{\mathrm{d}} = \{0\}$. The first part of the proof shows $I + F_1^{\mathrm{d}} = I + F$ is closed. ∎

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