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POWERFUL NUMBERS IN SHORT INTERVALS

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Let $\kappa \geqslant 2$ be an integer. We show that there exist infinitely many positive integers N such that the number of κ-full integers in the interval $\left(N^\kappa, (N+1)^\kappa\right)$ is at least $(\log N)^{1/3+o(1)}$. We also show that the ABC-conjecture implies that for any fixed $\delta > 0$ and sufficiently large N, the interval $(N, N + N^{1-(2+\delta)/\kappa})$ contains at most one κ-full number.

1. Introduction

Let $\kappa > 1$ be an integer. An integer $m \geqslant 1$ is called *κ-full* if $p^\kappa \mid m$ holds for all prime factors p of m. For example, κ-powers, that is, numbers m of the form n^κ, are κ-full. Usually, for $\kappa = 2$ such numbers are called *squarefull*.

It is clear that for any integer N, the open interval $\left(N^\kappa, (N+1)^\kappa\right)$ does not contain any κ-powers. In this paper, we show that the intervals of the above form can contain arbitrarily many κ-full numbers. This result extends some of the results obtained in [**3**], where squarefull numbers have been investigated. It is useful to recall that the counting function of the κ-full numbers up to x is of the same order of magnitude as the counting function of the κ-powers (see [**1**]), thus this difference in their behaviour is based on purely arithmetic reasons.

The proof is an extension of that given in [**3**] but uses the *Roth Theorem* instead of a result on continued fractions of some quadratic irrationalities. Alternatively, one can use the fully effective *Liouville Theorem* (which leads to a marginally weaker but more uniform statement). In fact, even in the case $\kappa = 2$ both these approaches lead to a slightly better constant than that of [**3**].

We recall that the Roth Theorem asserts that for any irrational root α of a monic irreducible polynomial $f(X) \in \mathbb{Q}[X]$ and any $\delta > 0$, there exists a constant $C(\alpha, \delta) > 0$ such that for any integers r and $s > 0$, we have

$$\left|\alpha - \frac{r}{s}\right| > \frac{C(\alpha,\delta)}{s^{2+\delta}}.$$

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However, this result is not effective in a sense that no explicit expression for $C(\alpha, \delta)$ is known (see [**5**, Theorem 2A of Chapter 5]).

We also recall the *Dirichlet Theorem* which asserts that, for any real numbers $\alpha_1, \ldots, \alpha_m$ and integer $Q > 1$, there exist integers $r_1, \ldots, r_m$ and $0 < s \leqslant Q$ such that

$$\left|\alpha_j - \frac{r_j}{s}\right| \leqslant \frac{1}{sQ^{1/m}},$$

(see [**5**, Theorem 1A of Chapter 2]).

We also show that the *ABC-conjecture* implies that for any $\delta > 0$ and sufficiently large L, the shorter interval $(L, L + L^{1-(2+\delta)/\kappa})$ contains at most one κ-full number.

We also obtain an unconditional (but much weaker) upper bound on the number of integers in short intervals $(L, L + K)$, which are κ-full numbers for at least one $\kappa \geqslant 2$. This complements the result obtained in [**4**], where the upper bound $\exp\bigl(40(\log\log L \log\log\log L)^{1/2}\bigr)$ on the number of perfect powers in the interval $(L, L + L^{1/2})$ (provided $L \geqslant 16$) is established.

Throughout this paper, we use Vinogradov symbols $\gg$ and $\ll$ as well as Landau symbols O and o with their regular meanings. We recall that the notations $A \ll B$, $B \gg A$ and $A = O(B)$ are all equivalent to the fact that $|A| \leqslant cB$ holds with positive constant c.

2. Short Intervals with Many κ-full Numbers

Theorem 1. *For any integer $\kappa \geqslant 2$, there are infinitely many N, such that the open interval $\bigl(N^\kappa, (N+1)^\kappa\bigr)$ contains at least*

$$M \geqslant \left(\left(\frac{3}{8} + o(1)\right)\frac{\log N}{\log\log N}\right)^{1/3}$$

κ-full integers.

Proof: Let $1 < d_1 < \ldots < d_{2\ell}$ be the first 2ℓ squarefree integers greater than 1, that is, $d_j = \pi^2 j/6 + o(j)$. We also denote

$$D = \prod_{j=1}^{2\ell} d_j,$$

and remark that $D \leqslant (4\ell)^{2\ell}$, provided that ℓ is large enough.

Let $\alpha_j = d_j^{-1/\kappa}$, $j = 1, \ldots, 2\ell$.

We define

$$R = \bigl(\kappa 2^{\kappa-1}(4\ell)^{2\ell+1/\kappa}\bigr)^{2\ell}$$

and let q be the smallest integer

$$q \geqslant R$$

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for which, for some integers r_j,

(1) $$\left|\alpha_j - \frac{r_j}{q}\right| \leqslant \frac{1}{q^{1+1/2\ell}}, \qquad j = 1, \dots, 2\ell.$$

We see that

$$q \leqslant Q,$$

where $Q = R^{2\ell(1+\delta)}C(\alpha_1, \delta)^{-2\ell}$. Indeed, otherwise applying the Dirichlet Theorem, we see that

$$\left|\alpha_j - \frac{r_j}{s}\right| \leqslant \frac{1}{sQ^{1/2\ell}}, \qquad j = 1, \dots, 2\ell,$$

for some positive integer $s \leqslant Q$. Due to the minimality condition on q, we have $s \leqslant R$. On the other hand, by the Roth Theorem, we have

$$\frac{C(\alpha_1, \delta)}{s^{2+\delta}} < \left|\alpha_1 - \frac{r_1}{s}\right| \leqslant \frac{1}{sQ^{1/2\ell}}.$$

Therefore

$$s > \left(C(\alpha_1, \delta)Q^{1/2\ell}\right)^{1/(1+\delta)} = R,$$

which is impossible.

We see from (1) that, for $j = 1, \dots, 2\ell$,

$$|q - d_j^{1/\kappa} r_j| \leqslant \frac{d_j^{1/\kappa}}{q^{1/2\ell}} \leqslant \frac{(4\ell)^{1/\kappa}}{R^{1/2\ell}} \leqslant 1.$$

Therefore,

$$d_j^{1/\kappa} r_j = \alpha_j^{-1} r_j \leqslant q + 1, \qquad j = 1, \dots, 2\ell.$$

We now derive,

$$\begin{aligned} |q^\kappa - d_j r_j^\kappa| = |q - d_j^{1/\kappa} r_j| \sum_{\nu=0}^{\kappa-1} q^{\kappa-1-\nu}(d_j r_j)^\nu &\leqslant \kappa(q+1)^{\kappa-1}|q - d_j^{1/\kappa} r_j| \\ &\leqslant \frac{\kappa(4\ell)^{1/\kappa}(q+1)^{\kappa-1}}{R^{1/2\ell}}. \end{aligned}$$

Putting $n = Dq$ we derive

$$\begin{aligned} |n^\kappa - d_j D^\kappa r_j^\kappa| &\leqslant \frac{\kappa(4\ell)^{1/\kappa} D^\kappa (q+1)^{\kappa-1}}{R^{1/2\ell}} \\ &\leqslant D^{\kappa-1} q^{\kappa-1} \frac{\kappa(4\ell)^{2\ell+1/\kappa}(1+1/q)^{\kappa-1}}{R^{1/2\ell}} \\ &= n^{\kappa-1} \frac{\kappa(4\ell)^{2\ell+1/\kappa}(1+1/q)^{\kappa-1}}{R^{1/2\ell}} \\ &\leqslant n^{\kappa-1} \frac{\kappa 2^{\kappa-1}(4\ell)^{2\ell+1/\kappa}}{R^{1/2\ell}} = n^{\kappa-1}. \end{aligned}$$

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Therefore one of the intervals $\big((n-1)^\kappa, n^\kappa\big)$ or $\big(n^\kappa, (n+1)^\kappa\big)$ contains at least $M \geqslant \ell$ of the integers $d_j D^\kappa r_j^\kappa$, $j = 1, \dots, 2\ell$, which are obviously pairwise distinct (because d_j is squarefree for all $j = 1, \dots, 2\ell$), and κ-full. We now have

$$\begin{aligned} n = Dq \leqslant DQ &= DR^{2\ell(1+\delta)} C(\alpha_1, \delta)^{-2\ell} \\ &\leqslant \big(4\ell / C(\alpha_1, \delta)\big)^{2\ell} (\kappa 2^{\kappa-1} (4\ell)^{2\ell + 1/\kappa})^{4\ell^2(1+\delta)} \\ &= \exp((8(1+\delta) + o(1))\ell^3 \log \ell). \end{aligned}$$

Hence, since κ is fixed,

$$\ell^3 \log \ell \geqslant \Big(\frac{1}{8(1+\delta)} + o(1)\Big) \log n,$$

which implies that

$$M \geqslant \bigg(\Big(\frac{3}{8(1+\delta)}\Big)^{1/3} + o(1)\bigg) \Big(\frac{\log n}{\log\log n}\Big)^{1/3}.$$

Recalling that δ is arbitrary, the proof is complete. ∎

3. Upper Bounds

We first recall the statement of the *ABC-conjecture*. For any nonzero integer m let

$$\gamma(m) = \prod_{p \mid m} p.$$

Conjecture 1. *For every $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that for any integers a, b, c with $c = a + b$ and $\gcd(a, b) = 1$, the bound*

$$\max\{|a|, |b|, |c|\} \leqslant C(\varepsilon) \gamma(abc)^{1+\varepsilon}$$

holds.

Theorem 2. *The ABC-conjecture implies that if κ and $\delta > 0$ are fixed, then there exists L_0 such that the interval $(L, L + L^{1-(2+\delta)/\kappa})$ contains at most one κ-full number for $L > L_0$.*

Proof: Let $\varepsilon = \delta/\kappa$. Assume that the above interval contains at least two κ-full numbers, say $a < b$. Then $\gamma(a)$, $\gamma(b) \leqslant (2L)^{1/\kappa}$ and $c = b - a < L^{1-(2+\delta)/\kappa}$. Applying the ABC-conjecture to the equation $c = b - a$, we get

$$L < b \leqslant C(\varepsilon)(2^{2/\kappa} L^{1-\delta/\kappa})^{1+\varepsilon} = C(\delta/\kappa) 2^{2/\kappa + 2\delta/\kappa^2} L^{1-\delta^2/\kappa^2}.$$

Hence

$$L < \big(C(\delta/\kappa) 2^{2/\kappa + 2\delta/\kappa^2}\big)^{\kappa^2/\delta^2},$$

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which completes the proof. $\square$

We remark that, unfortunately, the best known results towards the ABC-conjecture (see [**6**]), are not strong enough to produce any nontrivial estimates for κ-full numbers in short intervals.

We now obtain a much weaker but unconditional bound.

THEOREM 3. *For any positive integers L and K the interval $(L, L+K)$ contains at most $O(K \log\log K/\log K)$ squarefull numbers.*

PROOF: Let
$$w = \frac{\log K}{\log\log K}.$$
We separate the squarefull numbers of the interval $(L, L+K)$ into two nonintersecting subsets. The set $\mathcal{S}_1$ consists of the squarefull numbers which have a prime divisor p with $w \leqslant p \leqslant K$ (and thus are divisible by p^2). The set $\mathcal{S}_2$ consists of all other squarefull numbers in this interval. Clearly,

$$\#\mathcal{S}_1 \leqslant \sum_{w \leqslant p < K} \left(\frac{K}{p^2} + 1\right) \ll \frac{K}{w \log w} + \frac{K}{\log K} \ll \frac{K}{\log K}.$$

Using the Brun sieve (see [**2**, Theorem 2.2]), we also derive

$$\#\mathcal{S}_2 \ll K \prod_{w \leqslant p \leqslant K} \left(1 - \frac{1}{p}\right) \ll \frac{K \log\log K}{\log K},$$

which completes the proof. $\square$

4. REMARKS

As we have mentioned, the Roth Theorem is not effective. However, our arguments can be used with the completely explicit Liouville Theorem which asserts that for any irrational root α of a monic irreducible polynomial $f(X) \in \mathbb{Q}[X]$ of degree $\deg f = k \geqslant 2$ and any integers r and $s > 0$,
$$\left|\alpha - \frac{r}{s}\right| \geqslant \frac{c(\alpha)}{s^k},$$
where $c(\alpha) > 0$ depends only on α (see [**5**, Theorem 1A of Chapter 5]). It is easy to see from any standard proof of this inequality that the constant $c(\alpha)$ can be taken to be

$$c(\alpha) = \left(\Delta \max_{t \in [\alpha-1, \alpha+1]} \left\{1, \left|f'(t)\right|\right\}\right)^{-1},$$

where Δ is the least common multiple of all the denominators of the coefficients of f. For example, when $f(X) = X^\kappa - 1/d$ with some positive integers $\kappa \geqslant 2$ and d and $\alpha = d^{-1/\kappa}$, we can take $c(\alpha) = 1/(d\kappa 2^{\kappa-1})$.

Using this result leads to a uniform and explicit version of Theorem 1 with respect to κ (the constant $(3/8)^{1/3}$ becomes $\left(3/8(\kappa-1)\right)^{1/3}$). In particular, there are infinitely many N, such that the number of $\kappa(N)$-full integers in $\left(N^{\kappa(N)},(N+1)^{\kappa(N)}\right)$ is at least $(\log N)^{1/3+o(1)}$, where $\kappa(N)$ is any function of N satisfying $\kappa(N)=(\log N)^{o(1)}$, for example. Moreover, the above interval can contain arbitrarily many $\kappa(N)$-full integers, where $\kappa(N)=o\left((\log N)^{1/2}\right)$.

References

[1] P. Bateman and E. Grosswald, 'On a theorem of Erdös and Szekeres', *Illinois J. Math.* **2** (1958), 88–98.

[2] H. Halberstam and H.-E. Richert, *Sieve methods* (Academic Press, London, 1974).

[3] J.-M. De Koninck and F. Luca, 'Sur la proximité des nombres puissants', *Acta Arith.* **114** (2004), 149–157.

[4] J.H. Loxton, 'Some problems involving powers of integers', *Acta Arith.* **46** (1986), 113–123.

[5] W.M. Schmidt, *Diophantine approximation* (Springer-Verlag, Berlin, 1980).

[6] C.L. Stewart and K.R. Yu, 'On the *abc* conjecture, II', *Duke Math. J.* **108** (2001), 169–181.

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